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FACIAL STRUCTURE OF CONVEX SETS IN BANACH SPACES AND INTEGRAND REPRESENTATION OF CONVEX OPERATORS

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Abstract. Many types of convex operators which take values in some complete lattices can be represented by convex integrands. We consider a certain structure of faces of convex sets, and give a new proof of the representation theorem which is applicable in infinite-dimensional cases. As an application of such representations, we consider the conjugate duality of convex operators.

1. Introduction

Let $(\Omega,\ \mu)$ be a measure space and let $S(\Omega)$ be the space of all measurable functions f on Ω such that $f(t) < \infty \quad (a.e.t \in \Omega)$. Let X be a real Banach space. A mapping $F : X \supset D(F) \longrightarrow S(\Omega)$ is called a convex operator if $D(F)$ is a convex set in X, and for each $x,\, y \in D(F)$ and $0 < \alpha < 1$,

$$F((1-\alpha)x + \alpha y)(t) \leq (1-\alpha)F(x)(t) + \alpha F(y)(t) \qquad (a.e.t \in \Omega).$$

On the other hand, a function $f : X \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is called a convex integrand if for almost all t in Ω the function $f(\cdot, t)$ is convex on X. The convex integrand theory is well known and there are many applications. (See [7] for example.) We say that a convex integrand f represents a convex operator F if

$$f(x,t) = \begin{cases} F(x)(t) & \text{for a.e.} t \in \Omega,\ x \in D(F), \\ \infty, & x \notin D(F). \end{cases} \tag{1.1}$$

In two of the author's previous papers [3, 4], many applications of integrand representations of convex operators were demonstrated. However, the existence of integrand representation is nontrivial, and it is known only in some special cases.

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When X is the d-dimensional Euclidian space $\mathbb{R}^d$, the represenstion theorem has been proved in [3]. In this note, we apply the theory of the faces of convex sets, and give a new method of the proof which has an advantage in extending the reperesentation theorem to infinite-dimensional cases.

2. Faces of Convex Sets

When $x,\, y \in X$ are distinct points, then the set $[x,y] = \{(1-t)x + ty \mid 0 \le t \le 1\}$ is called the closed segment between x and y. Half open segments $(x,y],\ [x,y)$ and open segment (x,y) are defined analogously. Throughout this section, we fix a nonempty closed convex set D in X. A convex subset C of D is called a face of D if

$$\left\{ \begin{array}{r} x, y \in D \\ (x,y) \cap C \neq \emptyset \end{array} \right\} \quad \text{implies} \ \ [x,y] \subset C. \tag{2.1}$$

By $\mathfrak{F}(D)$, we denote the set of all faces of D. For $C \in \mathfrak{F}(D)$, $\dim C$ is defined to be the dimension of $\mathrm{aff}\ C$ (the affine hull of C). It is clear that $x \in D$ is an extreme point of D if and only if $\{x\}$ is a 0-dimensional face of D. For preparation, we will state some fundamental properties of faces in the following propositions whose proofs are given in [1].

Proposition 1. *If $C_\lambda \in \mathfrak{F}(D),\ (\lambda \in \Lambda)$, then $\cap_{\lambda\in\Lambda} C_\lambda \in \mathfrak{F}(D)$, and also there exists the smallest face of D containing $\cup_{\lambda\in\Lambda} C_\lambda$. Hence $(\mathfrak{F}(D), \subset)$ forms a complete lattice.*

Proposition 2. *Let C_1 be a face of D and suppose that $C_2 \subset C_1$. Then $C_2 \in \mathfrak{F}(D)$ if and only if $C_2 \in \mathfrak{F}(C_1)$.*

For a convex set C in X, $\mathring{C}$ denotes the relative interior of C, which means the interior of C with respect to the relative topology of $\mathrm{aff}\ C$. In the case $X = \mathbb{R}^d$, every face of a convex set D is a closed set. Indeed, if x is a point of the closure of a face C and $x_0 \in \mathring{C}$, the convexity of C yields $[x_0, x) \subset \mathring{C} \subset C$. Since C is a face of D, x must be in C.

In the following four propositions, we assume that $X = \mathbb{R}^d$.

Proposition 3. *If $C_1, C_2 \in \mathfrak{F}(D)$, and $C_1 \subsetneqq C_2$, then $C_1 \cap \mathring{C}_2 = \emptyset$.*

Proposition 4. *Let x be a point of D and let C be a face of D. Then C is the smallest face of D containing x if and only if $x \in \mathring{C}$.*

Proposition 5. *Let C_1 be a face of D and let x be a relative boundary point of C_1. If C_2 is the smallest face of D containing x, then C_2 is contained by the relative boundary of C_1.*

From these propositons we obtain the following decomposition of a convex set by its faces.

Proposition 6. *For every closed convex set D in $\mathbb{R}^d$, we can write*

$$D = \cup\{\mathring{C}_\lambda \mid C_\lambda \in \mathfrak{F}(D)\}, \tag{2.2}$$

where the union is disjoint.

In infinite-dimensional cases, a convex set D is said to have a face decomposition if D can be written in the form (2, 2). A collection $\{C_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{F}(D)$ is said to be proper if $\lambda \in \Lambda$ and $C_\lambda \subset C_\mu \in \mathfrak{F}(D)$ imply that C_μ is also a member of $\{C_\lambda\}_{\lambda\in\Lambda}$. Now we define

$$\mathfrak{A} = \{A = \bigcup_{\lambda\in\Lambda} \mathring{C}_\lambda \mid \{C_\lambda\}_{\lambda\in\Lambda} \quad \text{is proper}\}.$$

Since $\{\mathring{D}\}$ is proper and $\mathring{D} \in \mathfrak{A}$, $\mathfrak{A}$ is at least nonempty. It is easy to see that if each A_λ $(\lambda \in \Lambda)$ is a member of $\mathfrak{A}$, then so are $\bigcup_{\lambda\in\Lambda} A_\lambda$ and $\bigcap_{\lambda\in\Lambda} A_\lambda$, and therefore $(\mathfrak{A}, \subset)$ is a complete lattice.

Lemma 1. *If $A \in \mathfrak{A}$, then A is a convex set.*

Proof. We write $A = \bigcup_{\lambda\in\Lambda} \mathring{C}_\lambda$ and let x, y be arbitrary points of A. Then there exist λ and μ such that $x \in \mathring{C}_\lambda$ and $y \in \mathring{C}_\mu$. Let z be an arbitrary point of the open segment (x, y), and let C_ν be the smallest face containing z. Since C_ν is a face, we have $[x, y] \subset C_\nu$. By Proposition 4, C_λ is the smallest face containing x, and it follows that $C_\lambda \subset C_\nu$. Since the collection $\{C_\lambda\}_{\lambda\in\Lambda}$ is proper, we obtain $\mathring{C}_\nu \subset A$. This means that $z \in A$, and thus A is convex.

3. Representation of Convex Operators

In this section, we prove a representation theorem of convex operators. Let $D(F)$ be a convex set in X and let $F : D(F) \longrightarrow S(\Omega)$ be a convex operator. Throughout this section, D denotes the closure of $D(F)$. First we state the main theorem.

Theorem 1. *Let X be a separable Banach space, and let $F : X \supset D(F) \longrightarrow S(\Omega)$ be a convex operator. Suppose that $\overline{D(F)}$ has a face decompositon and F is continuous with respect to the almost everywhere convergence, that is, $x_n \longrightarrow x$ in X implies $(F(x_n))(t) \longrightarrow (F(x))(t)$ for almost every $t \in \Omega$. Then F has at least a representation. In other words, there exists a convex integrand $f : X \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ such that* (1, 1) *holds.*

For $D = \overline{D(F)}$, we define $\mathfrak{A}$ as in the Section 2. For $A \in \mathfrak{A}$, a convex integrand $f : A \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to represent F on A, if

$$f(x,t) = \begin{cases} F(x)(t) \text{ for a.e.} t \in \Omega, & x \in A \cap D(F), \\ \infty, & x \in A \setminus D(F). \end{cases}$$

Definition. For a convex operator F, we define

$$\tilde{\mathfrak{A}} = \{(A, f) \mid A \in \mathfrak{A}, \text{ and } f \text{ represents } F \text{ on } A\}.$$

Moreover, for (A_1, f_1), $(A_2, f_2) \in \tilde{\mathfrak{A}}$, we write $(A_1, f_1) \leq (A_2, f_2)$ when $A_1 \subset A_2$ and f_2 is an extension of f_1 to A_2.

Lemma 2. $(\tilde{\mathfrak{A}}, \leq)$ *is inductively ordered.*

Proof. Let $\{(A_\lambda, f_\lambda)\}_{\lambda\in\Lambda}$ be a totally ordered subset of $\tilde{\mathfrak{A}}$. Then $A = \bigcup_{\lambda\in\Lambda} A_\lambda$ is an element of $\mathfrak{A}$. Moreover we can define a convex integrand f on $A \times \Omega$ satisfying $f = f_\lambda$ on $A_\lambda \times \Omega$ for every $\lambda \in \Lambda$. Clearly, $(A, f) \in \tilde{\mathfrak{A}}$ and it is an upper bound of $\{(A_\lambda, f_\lambda)\}_{\lambda\in\Lambda}$.

Lemma 3. *For $A \in \mathfrak{A}$ such that $A \neq D$, we define $\mathfrak{S}_A = \{C \in \mathfrak{F}(D) \mid C \cap A = \emptyset\}$. Then $(\mathfrak{S}_A, \subset)$ is inductively ordered.*

Proof. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of $\mathfrak{S}_A$. If we put $C = \bigcup_{\lambda\in\Lambda} C_\lambda$, then C is a convex set and $C \cap A \neq \emptyset$. Moreover, $C \in \mathfrak{F}(D)$. Indeed, if we assume $(x, y) \cap C \neq \emptyset$, then there exists $\lambda \in \Lambda$ such that $(x, y) \cap C_\lambda \neq \emptyset$. Hence it follows that $[x, y] \subset C_\lambda \subset C$. Thus $C \in \mathfrak{S}_A$ and it is an upper bound of $\{C_\lambda\}_{\lambda\in\Lambda}$.

Lemma 4. *Let A be an element of $\mathfrak{A}$, and assume that $A \neq D$. Then there exists $C \in \mathfrak{S}_A$ such that $A \cup \mathring{C} \in \mathfrak{A}$.*

Proof. By Lemma 3 and Zorn's lemma, $\mathfrak{S}_A$ has at least a maximal element C. It is sufficient to show that $A \cup \mathring{C} \in \mathfrak{A}$. Put $A = \bigcup_{\lambda\in\Lambda} \mathring{C}_\lambda$, and take $C_1 \in \mathfrak{F}(D)$

such that $C_1 \supset C$. Since C is a maximal element of $\mathfrak{S}_A$, we have $C_1 \notin \mathfrak{S}_A$ and hence $C_1 \cap A \neq \emptyset$. Therefore we can choose $\lambda \in \Lambda$ such that $\mathring{C}_\lambda \cap C_1 \neq \emptyset$. It follows from Proposition 3 that $C_\lambda \subset C_1$ holds. Since the collection $\{C_\lambda\}_{\lambda\in\Lambda}$ is proper, $\mathring{C}_1 \subset A \subset A \cup \mathring{C}$. This shows that the collection $\{C_\lambda\}_{\lambda\in\Lambda} \cup \{C\}$ is also proper, and $A \cup \mathring{C} \in \mathfrak{A}$.

Lemma 5. *$\tilde{\mathfrak{A}}$ is not empty. In other words, there exists $A \in \mathfrak{A}$ such that F has a representation f on A.*

Proof. It is sufficient to show that F has a representation f on $\mathring{D}$. Let E be a countable dense subset of $\mathring{D}$. We can assume that E is midpoint convex, that is, $x, y \in E$ implies $(x+y)/2 \in E$. Let B be the set of all rational numbers of the form $\lambda = n/2^m \in [0,1]$. For each $x, y \in E$ and $\lambda \in B$, $\lambda x + (1-\lambda)y$ belongs to E and by the convexity of F,

$$(F(\lambda x + (1-\lambda)y))(t) \leq \lambda(F(x))(t) + (1-\lambda)(F(y))(t) \tag{3.1}$$

holds for all $t \in \Omega \setminus \Omega_1(x,y,\lambda)$ where $\Omega_1(x,y,\lambda) \subset \Omega$ has $\mu-$measure zero. Take the union of $\Omega_1(x,y,\lambda)$ over all $x, y \in E$ and $\lambda \in B$, and denote it by Ω_2. Then $\mu(\Omega_2) = 0$ and $(3,\ 1)$ holds on $\Omega \setminus \Omega_2$ for all $x, y \in E$ and $\lambda \in B$. Hence if we define $f(x,t)$ on $E \times \Omega$ by $f(x,t) = (F(x))(t)$ for $x, y \in E$ and $t \in \Omega$, then f satisfies

$$f(\lambda x + (1-\lambda)y, t) \leq \lambda f(x,t) + (1-\lambda) f(y,t) \tag{3.2}$$

for all $x, y \in E$, $\lambda \in B$, and $t \in \Omega \setminus \Omega_2$. For every $x \in \mathring{D}$, $t \in \Omega \setminus \Omega_3(x)$, $(\mu(\Omega_3(x)) = 0)$, and $\varepsilon > 0$, there exists $\delta = \delta(x,t,\varepsilon) > 0$ such that $y \in \mathring{D}$ and $\|x - y\| < \delta$ imply $|(F(x))(t) - (F(y))(t)| < \varepsilon$, by the continuity conditon of F. Hence for each $t \in \Omega \setminus (\Omega_2 \cup \Omega_3(x))$,

$$|(F(x))(t) - f(y,t)| < \varepsilon$$

holds for all $y \in E \cap V_\delta(x)$, where $V_\delta(x)$ denotes the δ-neighborhood of x. Hence for each $t \in \Omega \setminus (\Omega_2 \cup \Omega_3(x))$, the function $f(\cdot, t)$ is bounded on $E \cap V_\delta(x)$, and by (3, 2), this implies the uniform continuity of $f(\cdot,t)$ on $E \cap V_\delta(x)$. Thus we can define $f(x,t)$ on $\mathring{D} \times \Omega$ by the usual way of taking limit. Now f is obviously a convex integrand on $X \times \Omega$ by giving the value ∞ outside $\mathring{D}$. Again by the continuity condition of F, we have, for each $x \in \mathring{D}$,

$$\begin{aligned}(F(x))(t) &= \lim_{n\to\infty} (F(x_n))(t) \\ &= \lim_{n\to\infty} f(x_n, t) \\ &= f(x,t),\end{aligned}$$

for almost every $t \in \Omega$, where $\{x_n\}$ is a sequence of E converging to x. Thus f is a representation of F on $\mathring{D}$ and this completes the proof.

Lemma 6. *Suppose that $(A, f) \in \tilde{\mathfrak{A}}$ and $A \neq D$. Let $C \in \mathfrak{S}_A$ be a face such that $A \cup \mathring{C} \in \mathfrak{A}$ as in Lemma 4. Then f has an extension f_1 defined on $(A \cup \mathring{C}) \times \Omega$ such that $(A \cup \mathring{C}, f_1) \in \tilde{\mathfrak{A}}$.*

Proof. Let E be a countable dense subset of $\mathring{C}$. We can assume that E is midpoint convex. Let B be the set defined in the proof of Lemma 5. For each $x, y \in E$ and $\lambda \in B$, $\lambda x + (1-\lambda)y$ belongs to E and by the convexity of F,

$$(F(\lambda x + (1-\lambda)y))(t) \leq \lambda(F(x))(t) + (1-\lambda)(F(y))(t) \tag{3.3}$$

holds for all $t \in \Omega \setminus \Omega_4(x, y, \lambda)$ where $\Omega_4(x, y, \lambda) \subset \Omega$ has $\mu-$measure zero. Next for each $x \in E$, there exists a null set $\Omega_5(x)$ such that

$$(F(x))(t) \geq \sup_{z \in A} \lim_{\eta \to +0} f(x + \eta(z - x), t) \tag{3.4}$$

for all $t \in \Omega \setminus \Omega_5(x)$. Take the union of $\Omega_4(x, y, \lambda)$ and $\Omega_5(x)$ over all $x, y \in E$ and $\lambda \in B$, and denote them by Ω_6 and Ω_7 respectively. Then $\mu(\Omega_6) = \mu(\Omega_7) = 0$ and $(3,\ 3)$, $(3,\ 4)$ holds on $\Omega \setminus (\Omega_6 \cup \Omega_7)$ for all $x, y \in E$ and $\lambda \in B$. For $x \in E$ and $t \in \Omega \setminus (\Omega_6 \cup \Omega_7)$, we define $f_0(x, t) = (F(x))(t)$. Then $f_0(\cdot, t)$ is locally bounded in E by the same reasoning as in the proof of Lemma 5. Hence $f_0(\cdot, t)$ can be extended continuously to $\mathring{C}$. Moreover we define

$$f_0(x, t) = \sup_{z \in A} \lim_{\eta \to +0} f(x + \eta(z - x), t)$$

for every $x \in \mathring{C}$ and $t \in \Omega_6 \cup \Omega_7$. Then the function

$$f_1(x, t) = \begin{cases} f(x, t), & (x, t) \in A \times \Omega, \\ f_0(x, t), & (x, t) \in \mathring{C} \times \Omega \end{cases}$$

is obviously a convex integrand, and we can easily see that f_1 is a representation of F on $A \cup \mathring{C}$.

Proof of Theorem 1. By Lemma 3, Lemma 5 and Zorn's lemma, $\tilde{\mathfrak{A}}$ has at least a maximal element (A_0, f_0). Moreover, Lemma 6 shows that $A_0 = D$, and this means that f_0 represents F on D. Defining $f_0 = \infty$ on $D^c \times \Omega$, we complete the construction of a representation of F.

Corollary 1. *Let X be a separable Banach space, and let $F : X \longrightarrow S(\Omega)$ be a convex operator defined on the whole space X. Suppose that F is continuous with respect to the almost everywhere convergence, then F has a representation.*

Proof. Since X is considered to have a face decomposition, this follows directly from Theorem 1.

By Proposition 6, every convex set in the finite-dimensional space $\mathbb{R}^d$ has a face decomposition. Since every convex function on such a set is continuous on the interior of its domain, we have

Corollary 2. *Every convex operator $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ has at least a representation.*

4. Normal Representation

A convex integrand $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be normal if $f(\cdot, t)$ is lower semicontinuous for every $t \in \Omega$ and there exists a coutable family of measurable functions $\xi_n : \Omega \longrightarrow \mathbb{R}^d$ $(n = 1, 2, \cdots)$ such that

(1) for each n, $f(\xi_n(t), t)$ is measurable in $t \in \Omega$,

(2) for each $t \in \Omega$, $\{\xi_n(t)\}_{n=1}^{\infty}$ is dense in $D(f(\cdot, t))$,

where $D(f(\cdot, t)) = \{x \in \mathbb{R}^d \mid f(x, t) < \infty\}$. If a convex integrand f is normal, then $f(\xi(t), t)$ is measurable in $t \in \Omega$ whenever $\xi : \Omega \longrightarrow \mathbb{R}^d$ is measurable. A convex operator F is said to have a normal representation if there exists a normal convex integrand which represents F. We will find some conditions under which a convex operator has a normal representation. By the conjugate of a convex integrand f, we mean the convex integrand $f^* : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(\xi, t) = \sup_{x \in \mathbb{R}^d} \{\langle x, \xi \rangle - f(x, t)\}.$$

Also the biconjugate $f^{**} : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is given by

$$f^{**}(x, t) = \sup_{\xi \in \mathbb{R}^d} \{\langle x, \xi \rangle - f^*(\xi, t)\}.$$

If a convex integrand f is normal, then so are f^* and f^{**}. We note that if a convex integrand f represents a convex operator F then $D(f(\cdot, t))$ does not depend on $t \in \Omega$.

Lemma 7. *Let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of some convex operator. Then f is normal if and only if $f(\cdot, t)$ is lower semicontinuous, in other words, $f^{**} = f$ on $\mathbb{R}^d \times \Omega$.*

Proof. Let $D = D(f(\cdot, t))$ and take a countable subset $\{a_n\}$ of D. If we put $\xi_n(t) = a_n$ for all $t \in \Omega$ and $n = 1, 2, \cdots$, then the family $\{\xi_n\}$ satisfies the definition of normality.

Remark. If a convex integrand f satisfies

(1) for each $x \in \mathbb{R}^d$, $f(x, \cdot)$ is measurable, and

(2) $\overline{D(\cdot, t))}$ does not depend on $t \in \Omega$,

the conclusion of Lemma 7 is also valid.

Let $L(\mathbb{R}^d, S(\Omega))$ denote the space of all linear mappings from $\mathbb{R}^d$ to $S(\Omega)$. We identify $L(\mathbb{R}^d, S(\Omega))$ with the set $S(\Omega)^d = \{\xi = (\xi_1, \cdots, \xi_d) \mid \xi_i \in S(\Omega), i = 1, \cdots, d\}$ by corresponding $(\xi_1, \cdots, \xi_d) \in S(\Omega)^d$ to the mapping $\varphi : \mathbb{R}^d \ni (x_1, \cdots, x_d) \longrightarrow \langle x, \xi \rangle = x_1\xi_1 + \cdots + x_d\xi_d \in S(\Omega)$. The conjugate operator $F^* : L(\mathbb{R}^d, S(\Omega)) \supset D(F^*) \longrightarrow S(\Omega)$ of F is defined by

$$F^*(\xi) = \bigvee_{x \in D(F^*)} (\langle x, \xi \rangle - F(x)),$$

where $\bigvee$ means the supremum in the space $S(\Omega)$, and $D(F^*)$ is the set of all $\xi \in S(\Omega)^d$ such that the supremum $F^*(\xi)$ exists. The biconjugate operator F^{**} is defined on the space $L(L(\mathbb{R}^d, S(\Omega)), S(\Omega)) = L(S(\Omega)^d, S(\Omega))$, and we regard $S(\Omega)^d$ and $\mathbb{R}^d$ as the subspaces of this by corresponding $\eta \in S(\Omega)^d$ and $x \in \mathbb{R}^d$ to $\langle \eta, \cdot \rangle$ and $\langle x, \cdot \rangle \in L(S(\Omega)^d, S(\Omega))$, respectively. For $x \in \mathbb{R}^d$ and $\eta \in S(\Omega)$, F^{**} is defined by

$$F^{**}(x) = \bigvee_{\xi \in D(F^*)} (\langle x, \xi \rangle - F^*(\xi)),$$

$$F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (\langle \eta, \xi \rangle - F^*(\xi)).$$

They are only defined on the domain $D(F^{**})$ where these suprema exist.

Theoem 2. *Let $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ be a convex operator and let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of F. Then the convex integrand f^* and f^{**} are normal representations of F^* and F^{**} respectively. Moreover, for $\xi \in D(F^*)$ and $\eta \in D(F^{**})$,*

$$(F^*(\xi))(t) = f^*(\xi(t), t),$$

$$(F^{**}(\eta))(t) = f^{**}(\eta(t), t)$$

hold for almost every $t \in \Omega$.

This theorem follows from the following lemma.

Lemma 8. *Let $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ be a convex operator, and let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d \cup \{\infty\}$ be a representation of F. Let U be a convex subset* *of* $D(F)$ *and suppose that* $\inf_{x\in U} f(x,t) > -\infty$ *for almost every* $t \in \Omega$*. Then* $\bigwedge_{x\in U} F(x) \in S(\Omega)$ *exists and*

$$(\bigwedge_{x\in U} F(x))(t) = \inf_{x\in U} f(x,t).$$

Proof. Let E be a countable dense set in U. Then we have

$$\inf_{x\in U} f(x,t) = \inf_{x\in E} f(x,t)$$

for $a.e. t \in \Omega$. Hence $\inf_{x\in U} f(x,t)$ is measurable in t and

$$\begin{aligned}(\bigwedge_{x\in U} F(x))(t) &\le (\bigwedge_{x\in E} F(x))(t) \\ &= \inf_{x\in E} f(x,t) \\ &= \inf_{x\in U} f(x,t) \\ &\le (\bigwedge_{x\in U} F(x))(t)\end{aligned}$$

for $a.e. t \in \Omega$, and the lemma is proved.

Proof of Theorem 2. By Lemma 8 we have

$$\begin{aligned}(F^*(\xi))(t) &= \bigvee_{x\in D(F)} (\langle x,\xi\rangle - F(x))(t) \\ &= \sup_{x\in D(F)} (\langle x,\xi(t)\rangle - f(x,t)) \\ &= f^*(\xi(t),t) \quad (a.e. t \in \Omega)\end{aligned}$$

for every $\xi \in D(F^*) \subset S(\Omega)^d$. The latter statement can be obtained by analogy.

Combining Lemma 7 and Theorem 2, we obtain the following result.

Theorem 3. *A convex operator* $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ *satisfies*

$$F^{**}(x) = F(x)$$

for every $x \in D(F)$ *if and only if* F *has a normal representation.*

We note that the theorems in the Section 4 can be extended to infinite-dimensional cases by a parallel argument under the hypothesis in Theorem 1.

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