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INEQUALITIES FOR L_p CENTROID BODY

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Abstract. In this paper, we first establish the Brunn-Minkowski type inequalities for the volume of the L_p centroid body and its polar body with respect to the normalized L_p radial addition. Furthermore, we prove some properties for operator Γ_{-p} and obtain some inequalities for it.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that consists of convex bodies with the origin in their interiors. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. If $K \in \mathcal{K}^n$, then the support function of K, $h_K = h(K, \cdot) : S^{n-1} \to \mathbb{R}$, is defined by

$$h(K,u) = \max\{u \cdot x : x \in K\}, \qquad u \in S^{n-1} \tag{1.1}$$

where $u \cdot x$ denotes the standard inner product of u and x.

For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, denote by $V(K)$ its n-dimensional volume. The centroid body ΓK of K is the origin-symmetric convex body whose support function is given by(see [13])

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K | u \cdot x | dx, \tag{1.2}$$

where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$.

Centroid body was attributed by Blaschke and Dupin (see [3, 14]), it was defined and investigated by Petty [13]. More results regarding centroid body see [1, 3-4, 13-15].

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Recently, Lutwak, Yang and Zhang based on the classical centroid body, first introduced the notion of L_p centroid body(see [7, 11]) as follows: For each compact star-shaped about the origin K in $\mathbb{R}^n$ and for real number $p \geq 1$, the L_p centroid body of K, $\Gamma_p K$, is the convex body, whose support function is defined by

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx, \tag{1.3}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}},$$

and ω_n denotes the n-dimensional volume of the unit ball B_n in $\mathbb{R}^n$, namely

$$\omega_n = \pi^{\frac{n}{2}} / \Gamma(1 + \frac{n}{2}).$$

For L_p centroid body, Lutwak, Yang and Zhang made a series of studies and had obtained many results(see [7-12]). The aim of this paper is to study it further. For reader's convenience, we try to make the paper self-contained. This paper, except for the introduction, is divided into three sections. In Section 2, we recall some basics about convex bodies, star bodies, L_p mixed volume and L_p dual mixed volume.

In Section 3, we establish the Brunn-Minkowski type inequalities (Theorem 3.1) for the volume of the L_p centroid body and its polar body with respect to the normalized L_p radial addition. Thus, this work may be seen as a complementarity of L_p Brunn-Minkowski theory— often called the Brunn-Minkowski-Firey theory. Furthermore, the isolate forms of L_p Busemann$-$Petty centroid inequality is obtained.

For $K \in \mathcal{K}^n_o$ and real $p > 0$, in [12], Lutwak, Yang and Zhang introduced a new star body $\Gamma_{-p} K$. In Section 4, we establish the monotonicity of this new star body.

2. Notation and Preliminary Works

For a compact subset L of $\mathbb{R}^n$, with the origin in its interior, star-shaped with respect to the origin, the radial function $\rho(L, \cdot) : S^{n-1} \to \mathbb{R}$, is defined by

$$\rho(L, u) = \rho_L(u) = max\{\lambda : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body.

Let φ^n_o denote the set of star bodies in $\mathbb{R}^n$. Two star bodies $K, L \in \varphi^n_o$ are said to be dilatate (of each other) if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$.

For $K \in \mathcal{K}_o^n$, the polar body K^* of K, with respect to the origin, is defined by

$$K^* = \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}. \tag{2.2}$$

If $K \in \mathcal{K}_o^n$, then it follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_{K^*} = 1/\rho_K \qquad \text{and} \qquad \rho_{K^*} = 1/h_K. \tag{2.3}$$

For $p \geq 1$, $K, L \in \mathcal{K}^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L$ is defined as the convex body whose support function is given by(see [5,6])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p. \tag{2.4}$$

For $p \geq 1$, the L_p mixed volume, $V_p(K, L)$, of the K and L is defined by[5]:

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.5}$$

This limit exists was demonstrated in [5]. It was shown that corresponding to each origin-symmetric convex body K, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(v)^p dS_p(K, v). \tag{2.6}$$

for each $Q \in \mathcal{K}^n$. The measure $S_p(K, \cdot)$ called the L_p-surface area measure of K. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}.$$

For $K, L \in \varphi_o^n$, and $\varepsilon > 0$, the L_p-harmonic radial combination $K \widetilde{+}_{-p} \varepsilon \cdot L$ is the star body defined by(see [5])

$$\rho(K \widetilde{+}_{-p} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}. \tag{2.7}$$

While this addition and scalar multiplication are obviously dependent on p. The L_p dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of the K and L is defined by(see [5])

$$\frac{n}{-p} \widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.8}$$

The definition above and the polar coordinate formula for volume give the following integral representation of $\widetilde{V}_{-p}(K, L)$

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_L^{-p}(v) dS(v), \tag{2.9}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.6), it follows immediately that for each $K \in \mathcal{K}^n$,

$$V_p(K,K) = V(K). \tag{2.10}$$

From the formula (2.9), it follows immediately that for each $K \in \varphi_o^n$,

$$\widetilde{V}_{-p}(K,K) = V(K). \tag{2.11}$$

We shall require two basic inequalities regarding the L_p mixed volume and the L_p dual mixed volumes. The L_p analog of the classical Minkowski inequality states that for K, $L \in \mathcal{K}_o^n$ and $p \geq 1$

$$V_p(K,L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{2.12}$$

equality holds when $p = 1$ if and only if K and L are homothetic, when $p > 1$ if and only if K is a dilatate of L. The L_p Minkowski inequality was established in [5] by using the Minkowski inequality. The basic inequality for L_p dual mixed volumes is that for $K, L \in \varphi_o^n$ and $p \geq 1$

$$\widetilde{V}_{-p}(K,L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.13}$$

with equality if and only if K is a dilatate of L. This inequality is an immediate consequence of the Hölder inequality and the integral representation (2.9).

For $K \in \mathcal{K}_o^n$ and real $p > 0$, Lutwak, Yang and Zhang introduced a new star body $\Gamma_{-p}K$, whose radial function, for $u \in S^{n-1}$ is given by[12]:

$$\rho_{\Gamma_{-p}K}^{-p}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v). \tag{2.14}$$

Note for $p \geq 1$ the body $\Gamma_{-p}K$ is a convex body.

3. Inequalities for L_p Centroid Body

Let $K, L \in \varphi_o^n$ and $p \geq 1$. We introduce the normalized L_p radial addition of K and L, $K \bar{+}_p L$. First define $\xi > 0$ by

$$\xi^{1/(n+p)} = \frac{1}{n} \int_{S^{n-1}} [V(K)^{-1}\rho(K,u)^{n+p} + V(L)^{-1}\rho(L,u)^{n+p}]^{n/(n+p)} dS(u). \tag{3.1}$$

The body $K \bar{+}_p L \in \varphi_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(K \bar{+}_p L, \cdot)^{n+p} = V(K)^{-1}\rho(K,\cdot)^{n+p} + V(L)^{-1}\rho(L,\cdot)^{n+p}. \tag{3.2}$$

In this section, we establish the Brunn-Minkowski type inequalities for the volume of the L_p centroid body and its polar body with respect to the normalized L_p radial addition.

Theorem 3.1. *Let $K, L \in \varphi_o^n$ and $p \geq 1$. Then*

$$V(\Gamma_p(K\tilde{+}_pL))^{\frac{p}{n}} \geq V(\Gamma_pK)^{\frac{p}{n}} + V(\Gamma_pL)^{\frac{p}{n}}, \tag{3.3}$$

$$V(\Gamma_p^*(K\tilde{+}_pL))^{-\frac{p}{n}} \geq V(\Gamma_p^*K)^{-\frac{p}{n}} + V(\Gamma_p^*L)^{-\frac{p}{n}}, \tag{3.4}$$

the equality in (3.3) *holds when* $p = 1$ *if and only if* Γ_pK *and* Γ_pL *are homothetic, when* $p > 1$ *if and only if* Γ_pK *is a dilatate of* Γ_pL*. The equality in* (3.4) *holds if and only if* Γ_pK *is a dilatate of* Γ_pL*.*

Proof. By (3.1), (3.2) and the polar coordinate formula for volume, we can get $\xi = V(K\tilde{+}_pL)$. Hence from (3.2), we obtain

$$\frac{\rho(K\tilde{+}_pL,\cdot)^{n+p}}{V(K\tilde{+}_pL)} = \frac{\rho(K,\cdot)^{n+p}}{V(K)} + \frac{\rho(L,\cdot)^{n+p}}{V(L)}. \tag{3.5}$$

Using polar coordinates, (1.3) can be written as an integral over S^{n-1}

$$h^p_{\Gamma_pK}(u) = \frac{1}{(n+p)c_{n,p}V(K)} \int_{S^{n-1}} |u\cdot v|^p \rho_K(v)^{n+p} dS(v). \tag{3.6}$$

Then from (3.5) and (3.6), we have

$$\begin{aligned} h^p_{\Gamma_p(K\tilde{+}_pL)}(u) &= \frac{1}{(n+p)c_{n,p}V(K\tilde{+}_pL)} \int_{S^{n-1}} |u\cdot v|^p \rho_{K\tilde{+}_pL}(v)^{n+p} dS(v) \\ &= h^p_{\Gamma_pK}(u) + h^p_{\Gamma_pL}(u). \end{aligned}$$

Combine with the formula (2.6) and the L_p-Minkowski inequality (2.12), for any $Q \in \mathcal{K}_o^n$, it follows immediately

$$\begin{aligned} V_p(Q, \Gamma_p(K\tilde{+}_pL)) &= V_p(Q,\Gamma_pK) + V_p(Q,\Gamma_pL) \\ &\geq V(Q)^{\frac{n-p}{n}}[V(\Gamma_pK)^{\frac{p}{n}} + V(\Gamma_pL)^{\frac{p}{n}}]. \end{aligned}$$

equality holds when $p = 1$ if and only if Γ_pK and Γ_pL are homothetic, when $p > 1$ if and only if Γ_pK is a dilatate of Γ_pL.

Now letting $Q = \Gamma_p(K\tilde{+}_pL)$ in the above inequality, according to (2.9), then (3.3) follows.

Furthermore, by (2.3) and the Minkowski integral inequality, we get

$$\begin{aligned} V(\Gamma_p^*(K\bar{+}_pL))^{-\frac{p}{n}} &= \left[\frac{1}{n}\int_{S^{n-1}} h^{-n}_{\Gamma_p(K\bar{+}_pL)}(u)du\right]^{-\frac{p}{n}} \\ &= \left[\frac{1}{n}\int_{S^{n-1}} (h^p_{\Gamma_pK}(u)+h^p_{\Gamma_pL}(u))^{-\frac{n}{p}}du\right]^{-\frac{p}{n}} \\ &\geq \left[\frac{1}{n}\int_{S^{n-1}} h^{-n}_{\Gamma_pK}(u)du\right]^{-\frac{p}{n}} + \left[\frac{1}{n}\int_{S^{n-1}} h^{-n}_{\Gamma_pL}(u)du\right]^{-\frac{p}{n}} \\ &= V(\Gamma_p^*K)^{-\frac{p}{n}} + V(\Gamma_p^*L)^{-\frac{p}{n}}. \end{aligned}$$

By the equality condition of Minkowski integral inequality, the equality in (3.4) holds if and only if Γ_pK is a dilatate of Γ_pL.

Remark 1. If $p=1$, $K\bar{+}_1L$ is just the harmonic Blaschke linear combination of K and L, $K\hat{+}L$. Then we have the following corollary.

Corollary 3.2. *Let $K, L \in \varphi_o^n$. Then*

$$V(\Gamma(K\hat{+}L))^{\frac{1}{n}} \geq V(\Gamma K)^{\frac{1}{n}} + V(\Gamma L)^{\frac{1}{n}}, \tag{3.7}$$

$$V(\Gamma^*(K\hat{+}L))^{-\frac{1}{n}} \geq V(\Gamma^*K)^{-\frac{1}{n}} + V(\Gamma^*L)^{-\frac{1}{n}}, \tag{3.8}$$

the equality in (3.7) *holds if and only if* Γ_pK *and* Γ_pL *are homothetic, the equality in* (3.8) *holds if and only if* Γ_pK *is a dilatate of* Γ_pL.

In [11] and [7], Lutwak, Yang and Zhang conjectured and proved the following L_p Busemann-Petty centroid inequality, respectively: *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$. Then*

$$V(\Gamma_pK) \geq V(K), \tag{3.9}$$

with equality if and only if K is an ellipsoid centered at the origin.

The following theorem give an isolate forms of (3.9).

Theorem 3.3. *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$. Then*

$$V(\Gamma_pK) \geq [(n+p)c_{n,p}]^{\frac{n}{p}} V(\Gamma_{-p}\Gamma_pK) \geq V(K). \tag{3.10}$$

Equality on the left-hand side holds if and only if Γ_pK is an ellipsoid centered at the origin and equality on the right-hand side holds if and only if K is a dilatate of $\Gamma_{-p}\Gamma_pK$.

To prove the theorem 3.3, we first introduce the following lemma:

Lemma 3.4. *Let* $K, L \in \mathcal{K}_o^n$ *and* $p \geq 1$. *Then*

$$(n+p)c_{n,p}\frac{V_p(L,\Gamma_p K)}{V(L)} = \frac{\widetilde{V}_{-p}(K,\Gamma_{-p}L)}{V(K)}. \tag{3.11}$$

Proof. From the integral representation (2.9), definition (2.14), Fubini's theorem, definition (1.3), and the integral representation (2.6), it follows that

$$\begin{aligned}
\widetilde{V}_{-p}(L,\Gamma_{-p}K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_{\Gamma_{-p}L}^{-p}(v)dS(v) \\
&= \frac{1}{nV(L)}\int_{S^{n-1}} \rho_K^{n+p}(v)\int_{S^{n-1}} \mid u\cdot v \mid^p dS_p(L,v)dS(v) \\
&= \frac{1}{nV(L)}\int_{S^{n-1}}\int_{S^{n-1}} \mid u\cdot v \mid^p \rho_K^{n+p}(v)dS(v)dS_p(L,v) \\
&= \frac{(n+p)c_{n,p}V(K)}{nV(L)}\int_{S^{n-1}} h_{\Gamma_p K}^p(v)dS_p(L,v) \\
&= \frac{(n+p)c_{n,p}V(K)}{V(L)}V_p(L,\Gamma_p K).
\end{aligned}$$

Remark 2. Identity (3.11) for $p = 2$ can be found in [13].

Proof of Theorem 3.3. Taking $K = \Gamma_{-p}L$ in Lemma 3.4 and using (2.11), (2.12), (3.9), we obtain

$$V(L) \geq [(n+p)c_{n,p}]^{\frac{n}{p}} V(\Gamma_p\Gamma_{-p}L) \geq [(n+p)c_{n,p}]^{\frac{n}{p}} V(\Gamma_{-p}L). \tag{3.12}$$

Equality on the left-hand side holds if and only if L is a dilatate of $\Gamma_p\Gamma_{-p}L$ and equality on the right-hand side holds if and only if $\Gamma_{-p}L$ is an ellipsoid centered at the origin.

Taking $L = \Gamma_p K$ in Lemma 3.4 and using (2.10), (2.13), we obtain

$$V(K) \leq [(n+p)c_{n,p}]^{\frac{n}{p}} V(\Gamma_{-p}\Gamma_p K), \tag{3.13}$$

equality holds if and only if K is a dilatate of $\Gamma_{-p}\Gamma_p K$.

Putting $L = \Gamma_p K$ in (3.12) and combining with (3.13), we get (3.10) and the equality condition of it.

4. The Monotonicity for Operator Γ_{-p}

For $p \geq 1$, let Z^*_{-p} denote the class of centered convex bodies that is the range of the operator Γ^*_{-p} on $\mathcal{K}^n_o$; i.e. $Z^*_{-p} = \{\Gamma^*_{-p}K : K \in \mathcal{K}^n_o\}$. In this section, we establish the monotonicity of operator $\Gamma_{-p}(p \geq 1)$. our main result is the following theorem:

Theorem 4.1. *Let $K, L \in \mathcal{K}^n_o$ and $p \geq 1$. If $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, then*

$$\frac{V_p(K,Q)}{V(K)} \geq \frac{V_p(L,Q)}{V(L)}, \tag{4.1}$$

for all $Q \in \mathcal{Z}^*_{-p}$.

Proof. According to the integral representation (2.6), definition (2.14), (2.3) and Fubini's theorem, we immediately get

$$\frac{V_p(K,\Gamma^*_{-p}L)}{V(K)} = \frac{V_p(L,\Gamma^*_{-p}K)}{V(L)}. \tag{4.2}$$

Since $Q \in \mathcal{Z}^*_{-p}$, then exists a $M \in \mathcal{K}^n_o$, such that $Q = \Gamma^*_{-p}M$. Hence from (4.2), we have

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(K,\Gamma^*_{-p}M)}{V(K)} = \frac{V_p(M,\Gamma^*_{-p}K)}{V(M)}, \tag{4.3}$$

and

$$\frac{V_p(L,Q)}{V(L)} = \frac{V_p(M,\Gamma^*_{-p}L)}{V(M)} \tag{4.4}$$

Since $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, then $\Gamma^*_{-p}K \supseteq \Gamma^*_{-p}L$. That is

$$h_{\Gamma^*_{-p}K}(u) \geq h_{\Gamma^*_{-p}L}(u), \text{ for all } u \in S^{n-1}.$$

According to (2.5), we have that

$$V_p(M,\Gamma^*_{-p}K) \geq V_p(M,\Gamma^*_{-p}L),$$

associated with (4.3) and (4.4), we obtain (4.1). ■

Remark 3. Theorem 4.1 is a dual of the following monotonicity of L_p centroid body, which was proved by Grinberg and Zhang in [2]:

Theorem 4.1*. *Let* $K,\ L \in \varphi_o^n$ *and* $p \geq 1$. *If* $\Gamma_p K \subseteq \Gamma_p L$, *then*

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} \leq \frac{\widetilde{V}_{-p}(L,Q)}{V(L)},$$

for all $Q \in \mathcal{L}_p$.

Theorem 4.2. *Let* $K, L \in \mathcal{K}_o^n$ *and* $p \geq 1$. *If for all* $Q \in \mathcal{K}_o^n$, $V_p(K,Q) \leq V_p(L,Q)$, *then*

(i)

$$\frac{V(\Gamma_{-p}K)^{\frac{p}{n}}}{V(K)} \geq \frac{V(\Gamma_{-p}L)^{\frac{p}{n}}}{V(L)}, \tag{4.5}$$

(ii)

$$\frac{V(\Gamma^*_{-p}K)^{-\frac{p}{n}}}{V(K)} \geq \frac{V(\Gamma^*_{-p}L)^{-\frac{p}{n}}}{V(L)}, \tag{4.6}$$

equality holds when $p = 1$ *if and only if* K *is a translate of* L, *when* $p > 1$ *if and only if* $K = L$.

Proof. (i) Since $p \geq 1$, $V_p(K,Q) \leq V_p(L,Q)$ for all $Q \in \mathcal{K}_o^n$, taking $Q = \Gamma_p M$ for any convex body $M \in \mathcal{K}^n$, we have

$$V_p(K, \Gamma_p M) \leq V_p(L, \Gamma_p M), \tag{4.7}$$

equality holds when $p = 1$ if and only if K is a translate of L, when $p > 1$ if and only if $K = L$.

According to Lemma 3.4, we have

$$V(K)\widetilde{V}_{-p}(M, \Gamma_{-p}K) \leq V(L)\widetilde{V}_{-p}(M, \Gamma_{-p}L). \tag{4.8}$$

Taking $M = \Gamma_{-p}L$ and using (2.11), (2.13), we obtain

$$\frac{V(\Gamma_{-p}K)^{\frac{p}{n}}}{V(K)} \geq \frac{V(\Gamma_{-p}L)^{\frac{p}{n}}}{V(L)}, \tag{4.9}$$

with equality if and only if $\Gamma_{-p}K$ is a dilatate of $\Gamma_{-p}L$.

We know that inequality (4.7) and (4.8) are equivalent by Lemma 3.4, but with equality if and only if K is a translate of L ($p = 1$) and if and only if $K = L(p > 1)$ both implies the equality holds in (4.9). Then we get the equality condition of (4.5).

(ii) Since $V_p(K,Q) \leq V_p(L,Q)$, here taking $Q = \Gamma^*_{-p}M$ for any convex body $M \in \mathcal{K}_o^n$, we have

$$V_p(K, \Gamma^*_{-p}M) \leq V_p(L, \Gamma^*_{-p}M), \tag{4.10}$$

equality holds when $p = 1$ if and only if K is a translate of L, when $p > 1$ if and only if $K = L$.

Associated with inequality (4.10) and equality (4.2), we get that

$$V(K)V_p(M, \Gamma^*_{-p}K) \leq V(L)V_p(M, \Gamma^*_{-p}L).$$

Taking $M = \Gamma^*_{-p}L$ and using (2.12), we obtain that

$$\frac{V(\Gamma^*_{-p}K)^{-\frac{p}{n}}}{V(K)} \geq \frac{V(\Gamma^*_{-p}L)^{-\frac{p}{n}}}{V(L)}, \tag{4.11}$$

with equality if and only if $\Gamma^*_p K$ is a dilatate of $\Gamma^*_p L$.

According to the case of equality holds in (4.10) and (4.11), we know that the equality in (4.6) holds when $p = 1$ if and only if K is a translate of L, when $p > 1$ if and only if $K = L$. ■

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