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Positive Almost Periodic Solutions for an Epidemic Model with Saturated Treatment

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Abstract. This paper is concerned with a non-autonomous SIR epidemic model, which involves almost periodic incidence rate and saturated treatment function. By using the differential inequality technique and Lyapunov functional method, we obtain the existence and global exponential stability of almost periodic solutions for the addressed SIR model, which improve and supplement existing ones. Also, an example and its numerical simulations are given to demonstrate our theoretical results.

1. Introduction

In the study of epidemic dynamics, the effects of a periodically varying environment are important for the evolutionary theory, as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Hence, the effects of the periodic environment on epidemic models have been the object of intensive analysis by numerous authors, some of the results can be found in [1, 2, 4–6, 8, 10, 13] and the references are therein. Recently, the following epidemic model with saturated treatment:

$$
\begin{cases}
S'(t) = A(t) - \boldsymbol{d}(t)S(t) - \dfrac{\lambda(t)S(t)I(t)}{1+\beta(t)I(t)}, \\
I'(t) = \dfrac{\lambda(t)S(t)I(t)}{1+\beta(t)I(t)} - [\boldsymbol{d}(t)+\nu(t)+\mu(t)]I(t) - \dfrac{\gamma(t)I(t)}{1+\alpha(t)I(t)}, \\
R'(t) = \mu(t)I(t) + \dfrac{\gamma(t)I(t)}{1+\alpha(t)I(t)} - \boldsymbol{d}(t)R(t),
\end{cases} \tag{1.1}
$$

was produced in [9], where S, I and R are the susceptible class, the infectious class and the recovered class, respectively. Positive functions A, $\boldsymbol{d}$, μ, ν are the recruitment rate

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of the population, the natural death rate of the population, the natural recovery rate of the infective individuals and the disease-related death rate, respectively. While contacting with infected individuals, the susceptible individuals become infected at a saturated incidence rate $\lambda SI/(1+\beta I)$. Through treatment, the infected individuals recover at a saturated treatment function $\gamma I/(1+\alpha I)$. Assume that all the coefficient functions are periodic, the author in [9] derived a criterion on the global exponential stability of positive periodic solutions for this model.

On the other hand, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the authors in [7] used continuous theorem to establish some sufficient conditions for the existence and multiplicity of positive almost periodic solutions of SIR model with saturated incidence rate and constant removal rate. Unfortunately, we found that the mapping N of Lemma 3.4 in [7] is not guaranteed to be L-compact. For more details, we refer to [11,12,18], where the authors declared that the continuous theorem of coincidence degree is not suitable to solve almost periodic problem, for the reason that almost periodic function family does not meet the compact condition of coincidence degree theory. Therefore, the existence of almost periodic solutions for (1.1) is incomplete to hold in [7]. Moreover, to the best of our knowledge, there are few papers published on positive almost periodic solutions of epidemic model with saturated treatment. Motivated by the above discussions, in this paper, we aim to employ a novel argument to establish the existence and global exponential stability of positive of almost periodic solutions for system (1.1).

For convenience, it will be assumed that $A, \boldsymbol{d}, \beta, \alpha, \nu, \mu, \gamma\colon \mathbb{R} \to (0,+\infty)$ and $\lambda\colon \mathbb{R} \to [0,+\infty)$ are almost periodic functions. We denote by $\mathbb{R}^n$ ($\mathbb{R} = \mathbb{R}^1$) the set of all n-dimensional real vectors (real numbers). For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$ and define $\|x\| = \max_{i\in\{1,2,\ldots,n\}} |x_i|$. A matrix or vector $A \geq 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A_1 and A_2, $A_1 \geq A_2$ (resp. $A_1 > A_2$) means that $A_1 - A_2 \geq 0$ (resp. $A_1 - A_2 > 0$).

The initial conditions associated with (1.1) are defined as follows:

$$S(t_0) > 0, \quad I(t_0) > 0, \quad R(t_0) \geq 0. \tag{1.2}$$

For the sake of simplicity of notations, for a bounded continuous function g defined on $\mathbb{R}$, we denote

$$g^+ = \sup_{t\in\mathbb{R}} |g(t)|, \quad g^- = \inf_{t\in\mathbb{R}} |g(t)|.$$

Throughout this paper, we make the following assumptions for (1.1):

$$A^- > 0,\ d^- > 0,\ \beta^- > 0,\ \mu^- > 0,\ \beta(t) \le \alpha(t) \quad \text{for all } t \in \mathbb{R}. \tag{1.3}$$

The main purpose of this paper is to establish some sufficient conditions on the existence and exponential stability of almost periodic solutions for (1.1). To the best of our knowledge, this has not been done before. The remaining of this paper is organized as follows. In Section 2, we give some basic definitions and lemmas, which play an important role in Section 3 to establish the existence of the almost periodic solutions of (1.1). Here we also study the global exponential stability of almost periodic solutions. The paper concludes with an example to illustrate the effectiveness of the obtained results by numerical simulations.

2. Preliminary results

In this section, we shall first recall some basic definitions and lemmas which are used in what follows.

Definition 2.1. [3, 14] Let $u\colon \mathbb{R} \to \mathbb{R}$ be continuous in t. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\delta : |u(t+\delta) - u(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \varepsilon$ for all $t \in \mathbb{R}$.

Lemma 2.2. [9, Lemma 2.1] *Every solution $(S(t), I(t), R(t))$ of* (1.1) *with initial value conditions* (1.2) *is positive and bounded on $(t_0, +\infty)$.*

Lemma 2.3. [9, Lemma 2.2] *Let*

$$L^S = \sup_{t\in\mathbb{R}} \frac{A(t)}{\boldsymbol{d}(t)} \ge l^S = \inf_{t\in\mathbb{R}} \frac{A(t)}{\boldsymbol{d}(t) + \frac{\lambda(t)}{\beta(t)}} > 0,$$

$$l^I = \inf_{t\in\mathbb{R}} \frac{1}{\beta(t)} \left[\frac{\lambda(t) \inf_{t\in\mathbb{R}} \frac{A(t)}{\boldsymbol{d}(t)+\frac{\lambda(t)}{\beta(t)}} - \gamma(t)}{\boldsymbol{d}(t) + \nu(t) + \mu(t)} - 1 \right] > 0,$$

and $(S(t), I(t), R(t))$ be a solution of system (1.1) *with initial values condition* (1.2)*. Then,*

$$l^S \le \liminf_{t\to+\infty} S(t) \le \limsup_{t\to+\infty} S(t) \le L^S, \quad \liminf_{t\to+\infty} I(t) \ge l^I \quad \text{and} \quad \liminf_{t\to+\infty} R(t) > 0.$$

Lemma 2.4. *Assume that*

$$\sup_{t\in\mathbb{R}} \left\{ -\boldsymbol{d}(t) + \frac{\lambda(t) L^S}{(1+\beta(t)l^I)(1+\beta(t)l^I)} \right\} < 0, \tag{2.1}$$

$$\sup_{t\in\mathbb{R}} \left\{ -\left[\boldsymbol{d}(t) + \nu(t) + \mu(t)\right] + \frac{\lambda(t)}{\beta(t)} + \frac{\lambda(t) L^S}{(1+\beta(t)l^I)(1+\beta(t)l^I)} \right\} < 0, \tag{2.2}$$

and the assumptions of Lemma 2.3 hold. Moreover, assume that $x(t) = (S(t), I(t), R(t))$ *is a solution of equation* (1.1) *with initial condition* (1.2). *Then, for any* $\varepsilon > 0$, *there exists* $l = l(\varepsilon) > 0$ *such that every interval* $[\tau, \tau + l]$ *contains at least one number* δ *for which there exists* $N > 0$ *satisfying*

$$|S(t+\delta) - S(t)| < \varepsilon, \quad |I(t+\delta) - I(t)| < \varepsilon, \quad |R(t+\delta) - R(t)| < \varepsilon \quad \textit{for all } t \geq N.$$

Proof. Observe (2.1) and (2.2). Let

$$\limsup_{t\to+\infty} I(t) = L^I, \quad \limsup_{t\to+\infty} R(t) = L^R \quad \text{and} \quad \liminf_{t\to+\infty} R(t) = l^R.$$

Without loss of generality, we assume that $0 < \varepsilon < \min\left\{l^I, l^S, l^R\right\}$,

$$\sup_{t\in\mathbb{R}} \left\{ -\boldsymbol{d}(t) + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} < 0$$

and

$$\sup_{t\in\mathbb{R}} \left\{ -[\boldsymbol{d}(t) + \nu(t) + \mu(t)] + \frac{\lambda(t)}{\beta(t)} + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} < 0.$$

Consequently, we can choose two positive constants ζ and η such that

$$\eta < \boldsymbol{d}^-, \quad \sup_{t\in\mathbb{R}} \left\{ \zeta - \boldsymbol{d}(t) + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} < -\eta < 0, \tag{2.3}$$

$$\sup_{t\in\mathbb{R}} \left\{ \zeta - [\boldsymbol{d}(t) + \nu(t) + \mu(t)] + \frac{\lambda(t)}{\beta(t)} + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} < -\eta < 0. \tag{2.4}$$

From Lemmas 2.2 and 2.3, there exists $\widehat{t_0} \geq t_0$ such that

$$l^S - \varepsilon \leq S(t) \leq L^S + \varepsilon,\ l^I - \varepsilon \leq I(t) \leq L^I + \varepsilon,\ l^R - \varepsilon \leq R(t) \leq L^R + \varepsilon \quad \text{for all } t \geq \widehat{t_0}.$$

For simplicity of notations, we denote

$$\begin{aligned}
\epsilon_1(\delta, t) = {} & [A(t+\delta) - A(t)] - [\boldsymbol{d}(t+\delta) - \boldsymbol{d}(t)]S(t+\delta) \\
& - \frac{I(t+\delta)S(t+\delta)}{1 + \beta(t+\delta)I(t+\delta)}[\lambda(t+\delta) - \lambda(t)] \\
& - \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1 + \beta(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1 + \beta(t)I(t+\delta)}\right], \\
\epsilon_2(\delta, t) = {} & \{[\boldsymbol{d}(t) - \boldsymbol{d}(t+\delta)] + [\nu(t) - \nu(t+\delta)] + [\mu(t) - \mu(t+\delta)]\} I(t+\delta) \\
& + \frac{I(t+\delta)S(t+\delta)}{1 + \beta(t+\delta)I(t+\delta)}[\lambda(t+\delta) - \lambda(t)] \\
& + \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1 + \beta(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1 + \beta(t)I(t+\delta)}\right]
\end{aligned}$$

$$- \frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)}[\gamma(t+\delta)-\gamma(t)]$$
$$- \gamma(t)\left[\frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\alpha(t)I(t+\delta)}\right]$$

and

$$\begin{aligned}\epsilon_3(\delta,t) = & -[\boldsymbol{d}(t+\delta)-\boldsymbol{d}(t)]R(t+\delta) + [\mu(t+\delta)-\mu(t)]I(t+\delta) \\ & + \frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)}[\gamma(t+\delta)-\gamma(t)] \\ & + \gamma(t)\left[\frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\alpha(t)I(t+\delta)}\right].\end{aligned}$$

From the theory of uniformly almost periodic family in [14], we know that for any $\overline{\varepsilon} > 0$, it is possible to find a real number $l = l(\overline{\varepsilon}) > 0$, for any interval with length $l(\overline{\varepsilon})$, there exists a number $\delta = \delta(\overline{\varepsilon})$ in this interval such that

$$\begin{gathered}|A(t+\delta)-A(t)| < \overline{\varepsilon}, \quad |\boldsymbol{d}(t+\delta)-\boldsymbol{d}(t)| < \overline{\varepsilon}, \quad |\lambda(t+\delta)-\lambda(t)| < \overline{\varepsilon}, \quad |\beta(t+\delta)-\beta(t)| < \overline{\varepsilon}, \\ |\nu(t+\delta)-\nu(t)| < \overline{\varepsilon}, \quad |\mu(t+\delta)-\mu(t)| < \overline{\varepsilon}, \quad |\alpha(t+\delta)-\alpha(t)| < \overline{\varepsilon}, \quad |\gamma(t+\delta)-\gamma(t)| < \overline{\varepsilon}.\end{gathered}$$

Here, we choose ε is sufficiently small such that

$$|\epsilon_i(\delta,t)| \le \frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\eta\varepsilon \quad \text{for all } t \in \mathbb{R},\ i = 1,2,3, \tag{2.5}$$

and

$$\Lambda = \sup_{t\in\mathbb{R}}\left[\mu(t) + \frac{\gamma(t)}{(1+\alpha(t)l^I)(1+\alpha(t)l^I)}\right].$$

Pick $N_0 \ge \max\left\{t_0-\delta, \widehat{t_0}, \widehat{t_0}-\delta\right\}$. For $t \in \mathbb{R}$, denote

$$(x_1(t), x_2(t), x_3(t)) = (S(t+\delta)-S(t), I(t+\delta)-I(t), R(t+\delta)-R(t)).$$

Then, for all $t > N_0$, we get

$$\begin{aligned}x_1'(t) = & [A(t+\delta)-A(t)] - \boldsymbol{d}(t)[S(t+\delta)-S(t)] - [\boldsymbol{d}(t+\delta)-\boldsymbol{d}(t)]S(t+\delta) \\ & - \frac{I(t+\delta)S(t+\delta)}{1+\beta(t+\delta)I(t+\delta)}[\lambda(t+\delta)-\lambda(t)] \\ & - \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1+\beta(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\beta(t)I(t+\delta)}\right] \\ & - \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1+\beta(t)I(t+\delta)} - \frac{I(t)}{1+\beta(t)I(t)}\right] - \frac{\lambda(t)I(t)}{1+\beta(t)I(t)}[S(t+\delta)-S(t)] \\ = & -\boldsymbol{d}(t)[S(t+\delta)-S(t)] - \frac{\lambda(t)I(t)}{1+\beta(t)I(t)}[S(t+\delta)-S(t)] \\ & - \frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t))(1+\beta(t)I(t+\delta))}[I(t+\delta)-I(t)]\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&+ [A(t+\delta) - A(t)] - [\boldsymbol{d}(t+\delta) - \boldsymbol{d}(t)]S(t+\delta) \\
&- \frac{I(t+\delta)S(t+\delta)}{1+\beta(t+\delta)I(t+\delta)}[\lambda(t+\delta) - \lambda(t)] \\
&- \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1+\beta(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\beta(t)I(t+\delta)}\right] \\
= &- \left[\boldsymbol{d}(t) + \frac{\lambda(t)I(t)}{1+\beta(t)I(t)}\right]x_1(t) - \frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t))(1+\beta(t)I(t+\delta))}x_2(t) + \epsilon_1(\delta,t),
\end{aligned}
$$

$$
\begin{aligned}
x_2'(t) = &-[\boldsymbol{d}(t) + \nu(t) + \mu(t)][I(t+\delta) - I(t)] \\
&- \frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)I(t+\delta))}[I(t+\delta) - I(t)] + \frac{\lambda(t)I(t)}{1+\beta(t)I(t)}[S(t+\delta) - S(t)] \\
&+ \{[d(t) - d(t+\delta)] + [\nu(t) - \nu(t+\delta)] + [\mu(t) - \mu(t+\delta)]\}\, I(t+\delta) \\
&+ \frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t+\delta))(1+\beta(t)I(t))}[I(t+\delta) - I(t)] \\
&+ \frac{I(t+\delta)S(t+\delta)}{1+\beta(t+\delta)I(t+\delta)}[\lambda(t+\delta) - \lambda(t)] \\
&+ \lambda(t)S(t+\delta)\left[\frac{I(t+\delta)}{1+\beta(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\beta(t)I(t+\delta)}\right] \\
&- \frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)}[\gamma(t+\delta) - \gamma(t)] \\
&- \gamma(t)\left[\frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\alpha(t)I(t+\delta)}\right] \\
= &- \left\{[\boldsymbol{d}(t) + \nu(t) + \mu(t)] + \frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)I(t+\delta))}\right. \\
&\left. - \frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t+\delta))(1+\beta(t)I(t))}\right\}x_2(t) + \frac{\lambda(t)I(t)}{1+\beta(t)I(t)}x_1(t) + \epsilon_2(\delta,t)
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
x_3'(t) = &-\boldsymbol{d}(t)x_3(t) + \left[\mu(t) + \frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)I(t+\delta))}\right]x_2(t) \\
&- [\boldsymbol{d}(t+\delta) - \boldsymbol{d}(t)]R(t+\delta) + [\mu(t+\delta) - \mu(t)]I(t+\delta) \\
&+ \frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)}[\gamma(t+\delta) - \gamma(t)] \\
&+ \gamma(t)\left[\frac{I(t+\delta)}{1+\alpha(t+\delta)I(t+\delta)} - \frac{I(t+\delta)}{1+\alpha(t)I(t+\delta)}\right] \\
= &-\boldsymbol{d}(t)x_3(t) + \left[\mu(t) + \frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)I(t+\delta))}\right]x_2(t) + \epsilon_3(\delta,t).
\end{aligned} \tag{2.8}
$$

We trivially extend $(S(t), I(t), R(t))$ to $\mathbb{R}$ by letting $(S(t), I(t), R(t)) = (S(t_0), I(t_0), R(t_0))$ for $t \in (-\infty, t_0]$. Set $u(t) = (x_1(t), x_2(t))$ and

$$U(t) = \sup_{s\in(-\infty,t]}\left\{e^{\zeta s}\|u(s)\|\right\}.$$

It is obvious that $e^{\zeta t}\|u(t)\| \le U(t)$ and $U(t)$ is non-decreasing. Calculating the upper left derivative of $e^{\zeta t}|x_1(t)|$ and $e^{\zeta t}|x_2(t)|$, in view of (2.6) and (2.7), we have

$$\begin{aligned} D^-\left\{e^{\zeta t}|x_1(t)|\right\} &\leq -\left\{\left[\boldsymbol{d}(t)+\frac{\lambda(t)I(t)}{1+\beta(t)I(t)}\right]-\zeta\right\}e^{\zeta t}|x_1(t)| \\ &\quad+\frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t))(1+\beta(t)I(t+\delta))}|x_2(t)|e^{\zeta t}+e^{\zeta t}|\epsilon_1(\delta,t)| \\ &\leq -[\boldsymbol{d}(t)-\zeta]e^{\zeta t}|x_1(t)|+\frac{\lambda(t)(L^S+\varepsilon)}{[1+\beta(t)(l^I-\varepsilon)][1+\beta(t)(l^I-\varepsilon)]}|x_2(t)|e^{\zeta t} \\ &\quad+e^{\zeta t}|\epsilon_1(\delta,t)| \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &D^-\left\{e^{\zeta t}|x_2(t)|\right\} \\ &\leq -\left\{[\boldsymbol{d}(t)+\nu(t)+\mu(t)]+\frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)I(t+\delta))}\right. \\ &\quad\left.-\frac{\lambda(t)S(t+\delta)}{(1+\beta(t)I(t+\delta))(1+\beta(t)I(t+\delta))}-\zeta\right\}e^{\zeta t}|x_2(t)| \\ &\quad+\frac{\lambda(t)I(t)}{1+\beta(t)I(t)}|x_1(t)|e^{\zeta t}+e^{\zeta t}|\epsilon_2(\delta,t)| \\ &\leq\left\{\zeta-[\boldsymbol{d}(t)+\nu(t)+\mu(t)]+\frac{\lambda(t)(L^S+\varepsilon)}{[1+\beta(t)(l^I-\varepsilon)][1+\beta(t)(l^I-\varepsilon)]}\right\}e^{\zeta t}|x_2(t)| \\ &\quad+\frac{\lambda(t)}{\beta(t)}|x_1(t)|e^{\zeta t}+e^{\zeta t}|\epsilon_2(\delta,t)|, \end{aligned} \tag{2.10}$$

where $t > N_0$.

Now, we distinguish two cases to prove that

$$\|u(t)\| \leq \frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\varepsilon \quad \text{for sufficiently large } t.$$

Case 1: $U(t) > e^{\zeta t}\|u(t)\|$ for all $t \geq N_0$. We claim that

$$U(t) \equiv U(N_0) \quad \text{for all } t \geq N_0.$$

Assume, by way of contradiction, that there exists $t_1 > N_0$ such that $U(t_1) > U(N_0)$. Since

$$e^{\zeta t_1}\|u(t_1)\| < U(t_1) \quad \text{and} \quad e^{\zeta t}\|u(t)\| \leq U(N_0) \quad \text{for all } t \leq N_0,$$

there must exist $\xi \in (N_0, t_1)$ such that $e^{\zeta\xi}\|u(\xi)\| = U(t_1) \geq U(\xi)$, which is a clear contradiction of the fact that $U(\xi) > e^{\zeta\xi}\|u(\xi)\|$. This proves the claim. Then there exists $t_2 > N_0$ such that

$$\|u(t)\| \leq e^{-\zeta t}U(t) = e^{-\zeta t}U(N_0) < \frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\varepsilon < \varepsilon \quad \text{for all } t \geq t_2,$$

where t_2 satisfies $\varepsilon^{-1}U(N_0)(2+3\frac{\Lambda}{\boldsymbol{d}^-}) < e^{\zeta t_2}$.

Case 2: There is a $t^* \geq N_0$ such that $U(t^*) = e^{\zeta t^*} \|u(t^*)\|$.

If $U(t^*) = e^{\zeta t^*} \|u(t^*)\| = e^{\zeta t^*} |x_1(t^*)|$, then (2.9) implies that

$$\begin{aligned}
0 &\leq D^- \left\{ e^{\zeta t} |x_1(t)| \right\}\Big|_{t=t^*} \\
&\leq -[\boldsymbol{d}(t^*) - \zeta] |x_1(t^*)| e^{\zeta t^*} + \frac{\lambda(t^*)(L^S + \varepsilon)}{[1 + \beta(t^*)(l^I - \varepsilon)][1 + \beta(t^*)(l^I - \varepsilon)]} |x_2(t^*)| e^{\zeta t^*} \\
&\quad + |\epsilon_1(\delta, t^*)| e^{\zeta t^*} \\
&\leq \sup_{t \in \mathbb{R}} \left\{ -[\boldsymbol{d}(t) - \zeta] + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} U(t^*) + e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \eta \varepsilon \\
&\leq -\eta U(t^*) + e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \eta \varepsilon
\end{aligned}$$

which yields

$$e^{\zeta t^*} |x_1(t^*)| = U(t^*) \leq e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon \quad \text{and} \quad \|u(t^*)\| \leq \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon. \tag{2.11}$$

On the other hand, if $U(t^*) = e^{\zeta t^*} \|u(t^*)\| = e^{\zeta t^*} |x_2(t^*)|$, then (2.10) gives us

$$\begin{aligned}
0 &\leq D^- \left\{ e^{\zeta t} |x_2(t)| \right\}\Big|_{t=t^*} \\
&\leq \left\{ \zeta - [\boldsymbol{d}(t^*) + \nu(t^*) + \mu(t^*)] + \frac{\lambda(t^*)(L^S + \varepsilon)}{[1 + \beta(t^*)(l^I - \varepsilon)][1 + \beta(t^*)(l^I - \varepsilon)]} \right\} e^{\zeta t^*} |x_2(t^*)| \\
&\quad + \frac{\lambda(t^*)}{\beta(t^*)} |x_1(t^*)| e^{\zeta t^*} + e^{\zeta t^*} |\epsilon_2(\delta, t^*)| \\
&\leq \sup_{t \in \mathbb{R}} \left\{ \zeta - [\boldsymbol{d}(t) + \nu(t) + \mu(t)] + \frac{\lambda(t)}{\beta(t)} + \frac{\lambda(t)(L^S + \varepsilon)}{[1 + \beta(t)(l^I - \varepsilon)][1 + \beta(t)(l^I - \varepsilon)]} \right\} U(t^*) \\
&\quad + e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \eta \varepsilon \\
&\leq -\eta U(t^*) + e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \eta \varepsilon,
\end{aligned}$$

which implies

$$e^{\zeta t^*} |x_2(t^*)| = U(t^*) \leq e^{\zeta t^*} \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon \quad \text{and} \quad \|u(t^*)\| \leq \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon. \tag{2.12}$$

For any $t > t^*$, with the same approach as in deriving (2.11) and (2.12), we can show

$$e^{\zeta t} \|u(t)\| \leq \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon e^{\zeta t} \quad \text{and} \quad \|u(t)\| \leq \frac{1}{2 + 3\frac{\Lambda}{\boldsymbol{d}^-}} \varepsilon \quad \text{if } U(t) = e^{\zeta t} \|u(t)\|. \tag{2.13}$$

On the other hand, if $U(t) > e^{\zeta t} \|u(t)\|$ and $t > t^*$, then we can choose $t^* \leq \widetilde{t} < t$ such that

$$U(\widetilde{t}) = e^{\zeta \widetilde{t}} \left\|u(\widetilde{t})\right\| \quad \text{and} \quad U(s) > e^{\zeta s} \|u(s)\| \quad \text{for all } s \in (\widetilde{t}, t].$$

This, together with (2.13), leads to $\left\|u(\widetilde{t})\right\| \le \left(2+3\frac{\Lambda}{\boldsymbol{d}^-}\right)^{-1}\varepsilon$. Using a similar argument to that in the proof of Case 1, we can show that $U(s) \equiv U(\widetilde{t})$ for all $s \in (\widetilde{t}, t]$, which implies

$$\|u(t)\| < e^{-\zeta t}U(t) = e^{-\zeta t}U(\widetilde{t}) = \left\|u(\widetilde{t})\right\| e^{-\zeta(t-\widetilde{t})} < \frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\varepsilon.$$

Therefore, there must exist $\widetilde{t}_0 > \max\{N_0, t^*\}$ such that $\|u(t)\| \le \left(2+3\frac{\Lambda}{\boldsymbol{d}^-}\right)^{-1}\varepsilon$ for all $t > \widetilde{t}_0$.

Next, we show that $|x_3(t)| < \varepsilon$ for sufficiently large t. From (2.8), we have

$$\begin{aligned}
|x_3(t)| = &\left| e^{-\int_{\widetilde{t}_0}^{t}\boldsymbol{d}(\theta)\,d\theta}x_3(\widetilde{t}_0) + \int_{\widetilde{t}_0}^{t} e^{-\int_v^t \boldsymbol{d}(\theta)\,d\theta} \right.\\
&\left.\times\left[\mu(v)x_2(v) + \frac{\gamma(v)x_2(v)}{(1+\alpha(v)I(v))(1+\alpha(v)I(v))} + \epsilon_3(\delta, v)\right] dv\right|\\
&\le e^{\widetilde{t}_0\boldsymbol{d}^-}\left|x_3(\widetilde{t}_0)\right| e^{-\boldsymbol{d}^- t} + e^{-\boldsymbol{d}^- t}\int_{\widetilde{t}_0}^{t} e^{v\boldsymbol{d}^-}\sup_{t\in\mathbb{R}}\left[\mu(t) + \frac{\gamma(t)}{(1+\alpha(t)l^I)(1+\alpha(t)l^I)}\right] dv\\
&\times \frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\varepsilon + e^{-\boldsymbol{d}^- t}\int_{\widetilde{t}_0}^{t} e^{v\boldsymbol{d}^-}\frac{1}{2}\eta\varepsilon\,dv\\
&\le e^{\widetilde{t}_0\boldsymbol{d}^-}\left|x_3(\widetilde{t}_0)\right| e^{-\boldsymbol{d}^- t} + \frac{\Lambda}{\boldsymbol{d}^-}\frac{1}{2+3\frac{\Lambda}{\boldsymbol{d}^-}}\varepsilon + \frac{1}{\boldsymbol{d}^-}\frac{1}{2}\eta\varepsilon
\end{aligned}$$

for all $t \ge \widetilde{t}_0$, which implies that there exists $N > \widetilde{t}_0$ such that $|x_3(t)| < \varepsilon$ for all $t > N$.

In summary, for all $t > N$, we obtain

$$\begin{aligned}
|x_1(t)| &= |S(t+\delta) - S(t)| < \varepsilon,\\
|x_2(t)| &= |I(t+\delta) - I(t)| < \varepsilon,\\
|x_3(t)| &= |R(t+\delta) - R(t)| < \varepsilon.
\end{aligned}$$

This ends the proof. □

3. Main results

In this section, we establish sufficient conditions for the existence and global exponential stability of almost periodic solutions of (1.1).

Theorem 3.1. *Under the assumptions of Lemma* 2.4*, equation* (1.1) *has one positive almost periodic solution* $(S^*(t), I^*(t), R^*(t))$*, which is globally exponentially stable.*

Proof. Given a solution $(S(t), I(t), R(t))$ of equation (1.1) with initial conditions satisfying

$$S(t_0) > 0, \quad I(t_0) > 0, \quad R(t_0) > 0.$$

We also trivially extend $(S(t), I(t), R(t))$ to $\mathbb{R}$ as before. Set

$$\begin{aligned}
\widetilde{\epsilon}_1(k,t) = {} & [A(t+t_k) - A(t)] - [\boldsymbol{d}(t+t_k) - \boldsymbol{d}(t)]S(t+t_k) \\
& - \left[\frac{\lambda(t+t_k)S(t+t_k)I(t+t_k)}{1+\beta(t+t_k)I(t+t_k)} - \frac{\lambda(t)S(t+t_k)I(t+t_k)}{1+\beta(t)I(t+t_k)}\right], \\
\widetilde{\epsilon}_2(k,t) = {} & -\{[\boldsymbol{d}(t+t_k) - \boldsymbol{d}(t)] + [\nu(t+t_k) - \nu(t)] + [\mu(t+t_k) - \mu(t)]\} I(t+t_k) \\
& + \left[\frac{\lambda(t+t_k)S(t+t_k)I(t+t_k)}{1+\beta(t+t_k)I(t+t_k)} - \frac{\lambda(t)S(t+t_k)I(t+t_k)}{1+\beta(t)I(t+t_k)}\right] \\
& - \left[\frac{\gamma(t+t_k)I(t+t_k)}{1+\alpha(t+t_k)I(t+t_k)} - \frac{\gamma(t)I(t+t_k)}{1+\alpha(t)I(t+t_k)}\right]
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\epsilon}_3(k,t) = {} & [\mu(t+t_k) - \mu(t)]I(t+t_k) - [\boldsymbol{d}(t+t_k) - \boldsymbol{d}(t)]R(t+t_k) \\
& + \left[\frac{\gamma(t+t_k)I(t+t_k)}{1+\alpha(t+t_k)I(t+t_k)} - \frac{\gamma(t)I(t+t_k)}{1+\alpha(t)I(t+t_k)}\right],
\end{aligned}$$

where $\{t_k\}$ is any sequence of real numbers.

It follows from Lemma 2.2 and Lemma 2.3 that the solution $(S(t), I(t), R(t))$ is bounded and positive, and there exist positive constants M and m such that

$$(m, m, m) \leq (S(t), I(t), R(t)) \leq (M, M, M) \quad \text{for all } t \in \mathbb{R},$$

which implies that the right-hand side of (1.1) is also bounded and $(S'(t), I'(t), R'(t))$ is a bounded function on $[t_0, +\infty)$. Thus, since $(S(t), I(t), R(t)) \equiv (S(t_0), I(t_0), R(t_0))$ for $(-\infty, t_0]$, we see that $(S(t), I(t), R(t))$ is uniformly continuous on $\mathbb{R}$. Then, from the almost periodicity of A, d, λ, α, β, ν and μ, we can select a sequence $t_k \to \infty$ such that

$$\begin{aligned}
& |A(t+t_k) - A(t)| < \frac{1}{k}, \quad |\boldsymbol{d}(t+t_k) - \boldsymbol{d}(t)| < \frac{1}{k}, \quad |\lambda(t+t_k) - \lambda(t)| < \frac{1}{k}, \\
& |\beta(t+t_k) - \beta(t)| < \frac{1}{k}, \quad |\nu(t+t_k) - \nu(t)| < \frac{1}{k}, \quad |\mu(t+t_k) - \mu(t)| < \frac{1}{k}, \\
& |\alpha(t+t_k) - \alpha(t)| < \frac{1}{k}, \quad |\gamma(t+t_k) - \gamma(t)| < \frac{1}{k}, \qquad |\widetilde{\epsilon}_i(k,t)| \leq \frac{1}{k}, \quad i = 1, 2, 3,
\end{aligned} \tag{3.1}$$

for all $t \in \mathbb{R}$. Since $\{(S(t+t_k), I(t+t_k), R(t+t_k))\}$ is uniformly bounded and equicontinuous, by the Ascoli-Arzelà theorem and the diagonal selection principle, we can choose a subsequence of $\{t_k\}$ (not relabelled) such that $\{(S(t+t_k), I(t+t_k), R(t+t_k))\}$ uniformly converges to a continuous function $(S^*(t), I^*(t), R^*(t))$ on any compact subset of $\mathbb{R}$ and

$$(m, m, m) \leq (S^*(t), I^*(t), R^*(t)) \leq (M, M, M) \quad \text{for all } t \in \mathbb{R}.$$

To complete the proof, we first prove that $(S^*(t), I^*(t), R^*(t))$ is a solution of (1.1). In fact, for any $t \geq t_0$ and $\triangle t \in \mathbb{R}$, from (3.1), we have

$$\begin{aligned}
&S^*(t+\triangle t) - S^*(t)\\
&= \lim_{k\to\infty}[S(t+\triangle t+t_k) - S(t+t_k)]\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[A(s+t_k) - \boldsymbol{d}(s+t_k)S(s+t_k) - \frac{\lambda(s+t_k)S(s+t_k)I(s+t_k)}{1+\beta(s+t_k)I(s+t_k)}\right]ds\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[A(s) - \boldsymbol{d}(s)S(s+t_k) - \frac{\lambda(s)S(s+t_k)I(s+t_k)}{1+\beta(s)I(s+t_k)}\right]ds\\
&\quad + \lim_{k\to\infty}\int_t^{t+\triangle t}\widetilde{\epsilon_1}(k,s)\,ds\\
&= \int_t^{t+\triangle t}\left[A(s) - \boldsymbol{d}(s)S^*(s) - \frac{\lambda(s)S^*(s)I^*(s)}{1+\beta(s)I^*(s)}\right]ds,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&I^*(t+\triangle t) - I^*(t)\\
&= \lim_{k\to\infty}[I(t+\triangle t+t_k) - I(t+t_k)]\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[\frac{\lambda(s+t_k)S(s+t_k)I(s+t_k)}{1+\beta(s+t_k)I(s+t_k)} - (\boldsymbol{d}(s+t_k)+\nu(s+t_k)+\mu(s+t_k))I(s+t_k)\right.\\
&\qquad\left. - \frac{\gamma(s+t_k)I(s+t_k)}{1+\alpha(s+t_k)I(s+t_k)}\right]ds\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[\frac{\lambda(s)S(s+t_k)I(s+t_k)}{1+\beta(s)I(s+t_k)} - (d(s)+\nu(s)+\mu(s))I(s+t_k)\right.\\
&\qquad\left. - \frac{\gamma(s)I(s+t_k)}{1+\alpha(s)I(s+t_k)}\right]ds + \lim_{k\to\infty}\int_t^{t+\triangle t}\widetilde{\epsilon_2}(k,s)\,ds\\
&= \int_t^{t+\triangle t}\left[\frac{\lambda(s)S^*(s)I^*(s)}{1+\beta(s)I^*(s)} - (\boldsymbol{d}(s)+\nu(s)+\mu(s))I^*(s) - \frac{\gamma(s)I^*(s)}{1+\alpha(s)I^*(s)}\right]ds
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&R_3^*(t+\triangle t) - R_3^*(t)\\
&= \lim_{k\to\infty}[R_3(t+\triangle t+t_k) - R_3(t+t_k)]\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[\mu(s+t_k)I(s+t_k) + \frac{\gamma(s+t_k)I(s+t_k)}{1+\alpha(s+t_k)I(s+t_k)} - \boldsymbol{d}(s+t_k)R(s+t_k)\right]ds\\
&= \lim_{k\to\infty}\int_t^{t+\triangle t}\left[\mu(s)I(s+t_k) + \frac{\gamma(s)I(s+t_k)}{1+\alpha(s)I(s+t_k)} - \boldsymbol{d}(s)R(s+t_k)\right]ds\\
&\quad + \lim_{k\to\infty}\int_t^{t+\triangle t}\widetilde{\epsilon_3}(k,s)\,ds\\
&= \int_t^{t+\triangle t}\left[\mu(s)I^*(s) + \frac{\gamma(s)I^*(s)}{1+\alpha(s)I^*(s)} - \boldsymbol{d}(s)R^*(s)\right]ds,
\end{aligned} \tag{3.4}$$

where $t+\triangle t \geq t_0$. Consequently, (3.2), (3.3) and (3.4) imply that

$$\begin{aligned}
&\frac{d}{dt}S_1^*(t) = A(t) - \boldsymbol{d}(t)S^*(t) - \frac{\lambda(t)S^*(t)I^*(t)}{1+\beta(t)I^*(t)},\\
&\frac{d}{dt}I^*(t) = \frac{\lambda(t)S^*(t)I^*(t)}{1+\beta(t)I^*(t)} - (\boldsymbol{d}(t)+\nu(t)+\mu(t))I^*(t) - \frac{\gamma(t)I^*(t)}{1+\alpha(t)I^*(t)},
\end{aligned}$$

$$\frac{d}{dt}R^*(t) = \mu(t)I^*(t) + \frac{\gamma(t)I^*(t)}{1+\alpha(t)I^*(t)} - \boldsymbol{d}(t)R^*(t).$$

In other words, $(S^*(t), I^*(t), R^*(t))$ is a solution of (1.1).

Next, we show that $(S^*(t), I^*(t), R^*(t))$ is almost periodic. From Lemma 2.4, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[\tau, \tau + l]$ contains at least one number δ for which there exists $N > 0$ satisfying

$$\|(S(t+\delta) - S(t), I(t+\delta) - I(t), R(t+\delta) - R(t))\| < \varepsilon \quad \text{for all } t > N.$$

Then, for any fixed $s \in \mathbb{R}$, we can find a sufficiently large positive integer $N_1 > N$ such that for any $k > N_1$, $s + t_k > N$ and

$$\|(S(s+t_k+\delta) - S(s+t_k), I(s+t_k+\delta) - I(s+t_k), R(s+t_k+\delta) - R(s+t_k))\| < \varepsilon.$$

Letting $k \to \infty$, we obtain

$$\|(|S^*(s+\delta) - S^*(s)|, |I^*(s+\delta) - I^*(s)|, |R^*(s+\delta) - R^*(s)|)\| \le \varepsilon.$$

This tells us that $(S^*(t), I^*(t), R^*(t))$ is a positive almost periodic solution.

Finally, let $(\widehat{S}(t), \widehat{I}(t), \widehat{R}(t))$ be a solution of system (1.1) with initial value conditions (1.2). Then, arguing as in the proof of Lemma 2.3 in [9], we can show that there exist $T \ge t_0$ and two positive constants σ and K such that

$$\left|\widehat{S}(t) - S^*(t)\right| \le Ke^{-\sigma t}, \quad \left|\widehat{I}(t) - I^*(t)\right| \le Ke^{-\sigma t} \quad \text{for all } t \ge T. \tag{3.5}$$

Moreover, there exist two constants $t_R \ge T$ and $K_R > 0$ such that

$$\left|\widehat{R}(t) - R^*(t)\right| \le K_R e^{-\sigma t} \quad \text{for all } t \ge t_R. \tag{3.6}$$

Hence, (3.5) and (3.6) entail Theorem 3.1. This completes the proof. $\square$

4. An example

In this section, we give an example to demonstrate the results obtained in the previous sections.

Example 4.1. Consider the following SIR model with almost periodic incidence rate and saturated treatment function:

$$\begin{cases} S'(t) = 20 - 0.02S(t) - \dfrac{(2\times 10^{-3} + 2\times 10^{-4}\sin(\frac{\pi t}{3}) + 3\times 10^{-4}\sin t)S(t)I(t)}{1+0.5I(t)}, \\ I'(t) = \dfrac{(2\times 10^{-3} + 2\times 10^{-4}\sin(\frac{\pi t}{3}) + 3\times 10^{-4}\sin t)S(t)I(t)}{1+0.5I(t)} - 0.09I(t) - \dfrac{0.05I(t)}{1+0.5I(t)}, \\ R'(t) = 0.02I(t) + \dfrac{0.05I(t)}{1+0.5I(t)} - 0.02R(t). \end{cases} \tag{4.1}$$

One can easily check that the assumptions of Lemma 2.4 are satisfied. Hence, from Theorem 3.1, system (4.1) has exactly one positive almost periodic solution. Moreover, the almost periodic solution is globally exponentially stable with the exponential convergent rate $\sigma = 0.001$. The fact is verified by the numerical simulation in Figure 4.1.

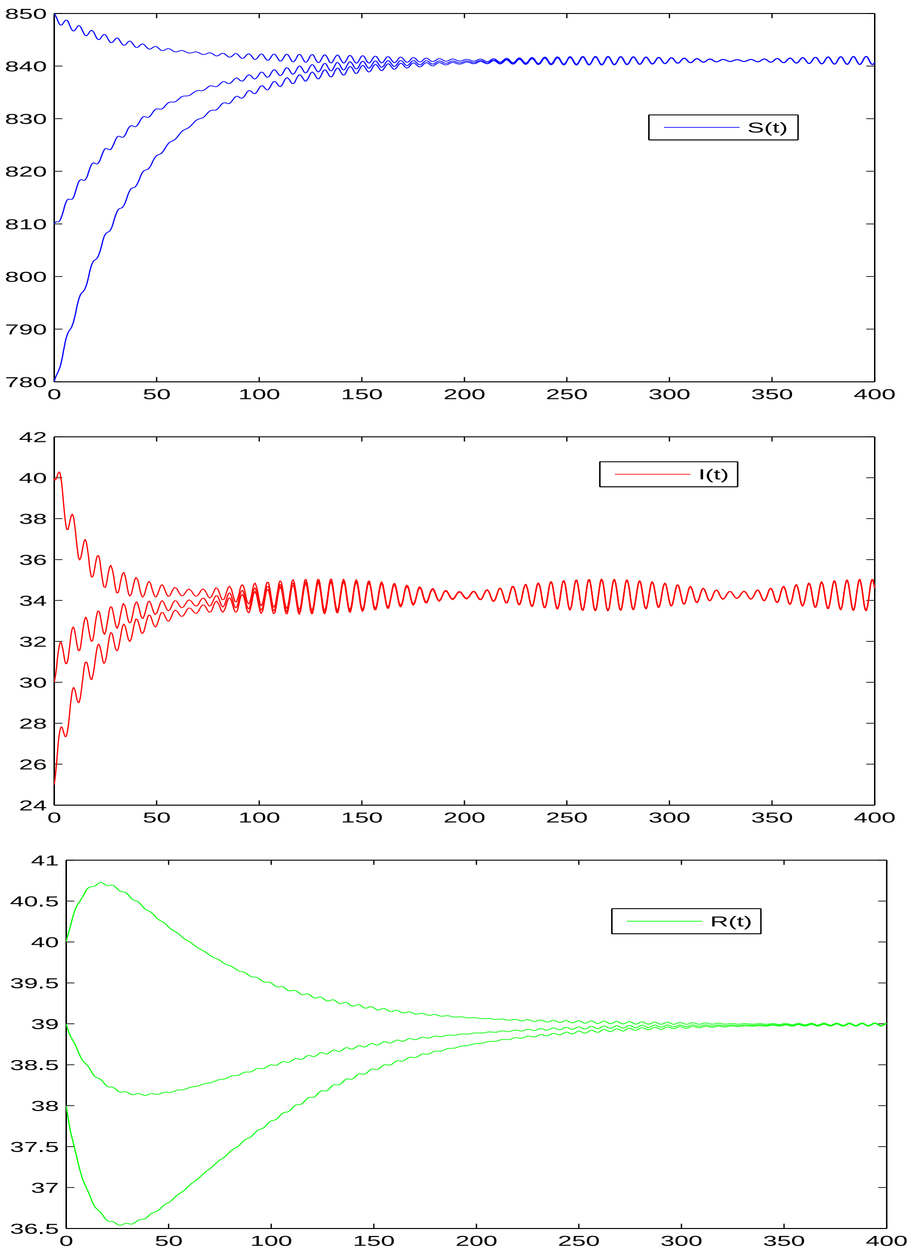

Figure 4.1: Numerical solutions of system (4.1) for initial values $(S, I, R) \equiv (780, 25, 38), (810, 30, 39), (850, 40, 40)$, respectively, where $t \in [0, 400]$.

Remark 4.2. Since the almost periodic functions contain periodic functions, we can find that all the results on periodic solutions of (1.1) in [9] are only special cases of Theorem 3.1. Moreover, the global exponential stability of positive almost periodic solutions has not be mentioned in [1, 2, 4–8, 10–13, 15–18]. This implies that all results in [1, 2, 4–13, 15–18] and the references therein cannot be applied to prove the global exponential stability of positive almost periodic solutions for (4.1). In particular, without using coincidence degree theory, we employ a novel proof to establish some criteria to guarantee the existence and stability of positive almost periodic solutions for non-autonomous SIR epidemic model with saturated treatment. The method used in this paper provides a possible method to study the almost periodic problem of other SIR epidemic models.

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