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Global Existence of Weak Solutions for the Nonlocal Energy-weighted Reaction-diffusion Equations

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Abstract. The reaction-diffusion equations provide a predictable mechanism for pattern formation. These equations have a limited applicability. Refining the reaction-diffusion equations must be a good way for supplying the gap between the mathematical simplicity of the model and the complexity of the real world. In this manuscript, we introduce a modified version of reaction-diffusion equation, which we have named "nonlocal energy-weighted reaction-diffusion equation". For any bounded smooth domain $\Omega \subset \mathbb{R}^n$, we establish the global existence of weak solutions $u \in L^2(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;H^{-1}(\Omega))$ to the initial boundary value problem of the nonlocal energy-weighted reaction-diffusion equation for any initial data $u_0 \in H_0^1(\Omega)$.

1. Introduction

The reaction-diffusion equation proposed by Alan Turing [23] is a mathematical model for understanding biological pattern formation. This model is one of the best-known theoretical models used to explain self-regulated pattern formation in the developing animal embryo. However, this equation has yet to gain wide acceptance among scientists. One reason is the gap between the mathematical simplicity of the model and the complexity of the real world. The logic of pattern formulation can be understood with simple models, and by adapting this logic to complex biological phenomena, it becomes easier to extract the essence of the underlying mechanisms [15]. The reaction-diffusion equations provide a predictable mechanism for pattern formation. These equations have a limited applicability. Refining the reaction-diffusion equations must be a good way for supplying the gap between the mathematical simplicity of the model and the complexity of the real world.

The study of the nonlocal energy-weighted reaction-diffusion equations is strongly inspired by the following concept of gradient flows with non-local sense.

Let $(X, \langle \cdot , \cdot \rangle_X)$ be a real Hilbert space (hence $(X, \|\cdot\|_X)$ is a Banach space with norm $\|u\|_X \equiv \sqrt{\langle u, u\rangle_X}$). Let $E\colon X \mapsto \mathbb{R}$ be a functional defined on X. We say that E is Fréchet

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differentiable at u ($\in X$) if there exists a (unique) bounded linear functional $DE(u)\colon X \mapsto \mathbb{R}$ (i.e., $DE(u) \in X^* \equiv \mathcal{L}(X;\mathbb{R})$ = the space of all bounded linear functionals on X; X^* is called the dual space of X.) such that

$$\lim_{\|h\|_X \to 0} \frac{E(u+h) - E(u) - DE(u)(h)}{\|h\|_X} = 0.$$

Suppose that E is Fréchet differentiable at u. Using the Riesz Representation Theorem, there exists a unique element $\widehat{u} \in X$ such that

$$DE(u)(v) \ (= \langle DE(u), v\rangle_{X^*,X}) = \langle \widehat{u}, v\rangle_X, \quad \forall\, v \in X.$$

Let we write

$$\nabla_X E(u) \equiv \widehat{u}.$$

$\nabla_X E(u)$ is called the gradient of E at u. (Note that $\nabla_X E(u) \in X$.) Let $u \in X$ and $v \in X$ with $v \neq 0$. The first variation of E at u in the direction v, $\delta E(u,v)$, is defined by

$$\delta E(u,v) \equiv \lim_{t\to 0} \frac{E(u+tv) - E(u)}{t} = \frac{d}{dt}\,(E(u+tv))\Big|_{t=0}.$$

Assume that E is Fréchet differentiable at u. Then $\delta E(u,v)$ exists for each $v \in X$ with $v \neq 0$, and

$$\delta E(u,v) = DE(u)(v) = \langle \nabla_X E(u), v\rangle_X \quad \text{for all } v \in X \text{ with } v \neq 0.$$

Let $\gamma\colon \mathbb{R} \mapsto X$ be a differentiable curve in X. Then

$$\frac{d}{dt}(E(\gamma(t)))\Big|_{t=\tau} = \langle \nabla_X E(\gamma(\tau)), \gamma'(\tau)\rangle_X, \quad \forall\, \tau \in \mathbb{R}.$$

Moreover, for any $t_0, t_1 \in \mathbb{R}$, we have

$$E(\gamma(t_1)) - E(\gamma(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}(E(\gamma(t)))\,dt = \int_{t_0}^{t_1} \langle \nabla_X E(\gamma(t)), \gamma'(t)\rangle_X\,dt.$$

Suppose that $\gamma\colon [0,T] \mapsto X$ is a solution of the gradient flow

$$\gamma'(t) = -\nabla_X E(\gamma(t)), \quad \forall\, t \in [0,T].$$

Then we have

$$\begin{aligned} E(\gamma(t)) - E(\gamma(0)) &= \int_0^t \frac{d}{dt}(E(\gamma(\tau)))\,d\tau = \int_0^t \langle \nabla_X E(\gamma(\tau)), -\nabla_X E(\gamma(\tau))\rangle_X\,d\tau \\ &= -\int_0^t \|\nabla_X E(\gamma(\tau))\|_X^2\,d\tau = -\int_0^t \|\gamma'(\tau)\|_X^2\,d\tau \le 0, \quad \forall\, t \in [0,T]. \end{aligned}$$

Suppose that X is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$ and corresponding norm $\|\cdot\|_X$. Let $E \in C^1(X;\mathbb{R})$ and let $\gamma\colon I \subseteq \mathbb{R} \mapsto X$ be a differentiable curve in X with $\gamma(t_0) = x \in X$. Then

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0} E(\gamma(t)) &= \langle \nabla_X E(\gamma(t_0)), \gamma'(t_0)\rangle_X \\ &= \|\nabla_X E(\gamma(t_0))\|_X \cdot \|\gamma'(t_0)\|_X \cdot \cos\theta,\end{aligned}$$

where θ is the angle between $\nabla_X E(\gamma(t_0))$ and $\gamma'(t_0)$ in the sense of geometry. Assume that $u\colon I \mapsto X$ is a solution of the gradient flow [2]

$$\partial_t u(t) = -\nabla_X E(u(t)) \in X, \quad \forall\, t \in I. \tag{1.1}$$

Obviously, the direction of $\nabla_X E(u(t))$ and $\partial_t u(t)$ is opposite. Hence, the curve (solution) u of (1.1) is the best path for decreasing the energy E with steepest descent. Evolution equations with the corresponding gradient flow have been of special interest in analysis and mathematical physics since 2001 [17].

Over the last ten years, a great attention has been focused on the research of non-local operators, both for the pure mathematical study and the wide range of applications. These non-local operators arise in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, and water waves [21]. It's well-known that reaction-diffusion equations are important in many area of applied mathematics involving phase transition. In view of the recent progress in understanding the behavior of non-local operators, it gives us a motivation to study the non-local energy-weighted reaction-diffusion equations:

$$u_t = -E_\varepsilon(u)\nabla_X E_\varepsilon(u),$$

where

$$E_\varepsilon(u) \equiv \int_\Omega \left[\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\right] dx$$

and

$$\nabla_X E_\varepsilon(u) = -\varepsilon\Delta u + \frac{1}{\varepsilon} f(u), \quad X \equiv L^2(\Omega), \quad f = W'.$$

On the other hand, the non-local energy-weighted reaction-diffusion equation is the following gradient flow:

$$u_t = -\nabla_X F_\varepsilon(u),$$

where $F_\varepsilon(u) = E_\varepsilon^2(u)/2$. This is another motivation for studying the reaction-diffusion equations with non-local coefficients.

In general, we assume that the coefficients of partial differential equations which are belong to L^∞ in some sense. Notice that the speed $\|u_t\|$ is in terms of $E_\varepsilon(u)$ and the slope $\|\nabla_X E_\varepsilon(u)\|$ in the sense of physical meaning. Therefore, we use the truncation term $E_\varepsilon^M(u) \equiv \min\{E_\varepsilon(u), M\}$ as our non-local energy-weighted coefficient of reaction-diffusion equation.

Owing to the non-local coefficient $E_\varepsilon^M(u)$ in the equation, the Galerkin approximation method is adapted with more difficulties and complexities.

The reaction-diffusion equation (or Allen-Cahn equation)

$$u_t = \varepsilon \Delta u - \frac{1}{\varepsilon} f(u) \tag{1.2}$$

was introduced by Allen and Cahn [1] to describe the macroscopic motion of phase boundaries derived by surface tension. The small parameter $\varepsilon > 0$ is the thickness of the phase boundaries. The function f is the derivative of an energy potential W with two wells of equal depth at ± 1. The function u indicates the phase state in its domain. There are many articles related to the study of Allen-Cahn equation (1.2). The formal derivation of (1.2) was given by Fife [11], Rubinstein, Sternberg and Keller [20], and others. Suppose that there exists a classical solution of (1.2), the existence result of the limit function of solutions of (1.2) is proved by De Mottoni and Schatzman [16], Chen [5], Chen and Elliott [6] and others. The convergence result of radially symmetric solutions of (1.2) is obtained by Bronsard and Kohn [4]. Evans, Soner and Souganidis [9] claimed that the result of the limit of the level-set solution of (1.2) is contained in the viscosity solution for the mean curvature flow studied by Evans and Spruck [10], Chen, Giga, and Goto [7] and others. The result of the limit is a mean curvature flow in the sense of Brakke [3] is given by Ilmanen [13]. The non-local Allen-Cahn equation [19, 24]

$$u_t = \Delta u - f(u) + \frac{1}{|\Omega|} \int_\Omega f(u(y))\, dy$$

was introduced by Rubinstein and Sternberg [19] as a model for phase separation in a binary mixture.

The study of this manuscript is strongly inspired by the above motivations, and the goal of this manuscript is devoted to the study of the global existence of weak solutions for the nonlocal energy-weighted Allen-Cahn equation with Neumann boundary condition of the type

$$(AC)_M \qquad \begin{cases} u_t = -\min\{E_\varepsilon(u), M\} \cdot \nabla_{L^2(\Omega)} E_\varepsilon(u) & \text{in } \Omega_T, \\ u = u_0 & \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T], \end{cases}$$

under the assumptions $\|f\|_{C^0} < M$ and $\|f\|_{C^1} < M$. Here

$$E_\varepsilon(u) = \int_\Omega \left[\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\right] dx,$$

W is a double-well potential with wells ± 1, $f(u) = W'(u)$, $M > 0$ is a constant,

$$\nabla_{L^2(\Omega)} E_\varepsilon(u) = -\varepsilon\Delta u + \frac{1}{\varepsilon} f(u),$$

$\Omega \subset \mathbb{R}^n$ is a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary $\partial\Omega$, $\Omega_T \equiv \Omega \times (0,T]$ for some fixed time $T > 0$, $\varepsilon > 0$ is a small parameter, and the function $u_0 : \Omega \mapsto \mathbb{R}$ is the initial datum.

In this work, we prove that for any $T > 0$, problem $(AC)_M$ admits a weak solution u_M, for each $M > 0$. It is unclear for us that whether or not $(AC)_\infty$ admits a weak solution u_∞ without the assumptions $\|f\|_{C^0} < +\infty$ and $\|f\|_{C^1} < +\infty$, where $(AC)_\infty$ is defined as

$$(AC)_\infty \qquad \begin{cases} u_t = -E_\varepsilon(u) \cdot \nabla_{L^2(\Omega)} E_\varepsilon(u) & \text{in } \Omega_T, \\ u = 0 & \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Without loss of generality, we assume that $\|f\|_{C^0} < +\infty$ and $\|f\|_{C^1} < +\infty$ in Theorems 1.1–1.4. To the best of our knowledge, this is the first work on non-local energy-weighted reaction-diffusion equations. The non-local energy-weighted gradient flows of E_ε with respect to $H^{-1}(\Omega)$, $H^1(\Omega)$, and $H^s(\Omega)$ respectively are our next work in the near future. Here $0 < s < 1$, and $H^s(\Omega)$ is a fractional Hilbert space.

Now, let we consider the nonlocal energy-weighted Allen-Cahn equation of the form

$$\partial_t u = -E_\varepsilon^M(u) \cdot \nabla_{L^2(\Omega)} E_\varepsilon(u) \tag{1.3}$$

with

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0,T). \tag{1.5}$$

Note that

$$\nabla_{L^2(\Omega)} E_\varepsilon(u) = -\varepsilon\Delta u + \frac{1}{\varepsilon} f(u), \tag{1.6}$$

where

$$E_\varepsilon^M(u) \equiv \min\{E_\varepsilon(u), M\}.$$

Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of eigenvalue with $\lambda_i \mapsto +\infty$ as $i \to +\infty$ such that

$$\begin{cases} 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots , \\ -\Delta \phi_i = \lambda_i \phi_i \quad \text{in } \Omega, \\ \frac{\partial \phi_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \phi_i \in H_0^1(\Omega) \cap C^{\infty}(\Omega). \end{cases} \tag{1.7}$$

For each $N \in \mathbb{N}$, we consider the function u^N defined by the Galerkin ansatz

$$u^N(x,t) = \sum_{i=1}^{N} a_i^N(t) \phi_i(x). \tag{1.8}$$

In this paper, we are interested in the global existence (Theorem 1.4) of weak solutions to (1.3), (1.4), (1.5), and (1.6) for the initial data. We first establish Theorems 1.1, 1.2, and 1.3, and based on these properties, we then establish Theorem 1.4.

Theorem 1.1. *Let $T > 0$, $\|f\|_{C^0} < +\infty$, $\|f\|_{C^1} < +\infty$, and suppose that $u_0 \in H_0^1(\Omega)$. Then*

(i) *For each $N \in \mathbb{N}$, there exists a solution $(a_1^N, a_2^N, \ldots, a_N^N)$ of the initial value problem*

$$\begin{cases} (a_j^N)'(t) = -\varepsilon \cdot E_\varepsilon^M(u^N) \cdot \lambda_j \cdot a_j^N(t) - \frac{1}{\varepsilon} E_\varepsilon^M(u^N) \cdot \int_\Omega f(u^N) \phi_j(x)\, dx, \\ a_j^N(0) = \int_\Omega u_0 \phi_j, \quad j = 1, 2, \ldots, N \end{cases}$$

globally defined on $(0,T)$.

(ii) *The function u^N defined by (1.8) exists globally in $\Omega \times [0,T]$ and satisfies*

$$\int_\Omega u_t^N \phi_i = -\int_\Omega E_\varepsilon^M(u^N) \cdot \varepsilon \nabla u^N \cdot \nabla \phi_i - \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) \cdot \phi_i, \tag{1.9}$$

$$\int_\Omega u^N(x,0) \phi_i = \int_\Omega u_0 \phi_i \tag{1.10}$$

for $i = 1, 2, \ldots, N$ and for all $t \in [0,T]$.

(iii) *The function u^N satisfies equations (2.1) and (2.2) on $\Omega \times [0,T]$, and for each $N \in \mathbb{N}$, we have*

$$\sup_{\tau \in [0,T]} \int_\Omega |\nabla u^N|^2(x,\tau)\, dx \leq e^{2\|f\|_{C^1} MT/\varepsilon} \int_\Omega |\nabla u_0|^2\, dx, \tag{1.11}$$

$$\int_\Omega u_t^N \cdot v = \int_\Omega E_\varepsilon^M(u^N) \left(\varepsilon \Delta u^N - \frac{1}{\varepsilon} f(u^N) \right) \cdot v, \quad \forall\, v \in W^N, \tag{1.12}$$

$$\int_\Omega u^N(x,0) \cdot v = \int_\Omega u_0 v\, dx, \quad \forall\, v \in W^N, \tag{1.13}$$

where $W^N \equiv \operatorname{span}\{\phi_1, \ldots, \phi_N\}$.

(iv)

$$\sup_{t\in[0,T]} \|u^N\|_{H_0^1(\Omega)} \le K^*_{\varepsilon,T} \equiv (C(n,\Omega)+1)\cdot\Big(e^{\|f\|_{C^1}\cdot M\cdot T/\varepsilon}\cdot\|\nabla u_0\|_{L^2(\Omega)}\Big) + |\Omega^{-1/2}|\left(\left|\int_\Omega u_0\,dx\right| + \frac{\|f\|_{C^0}\cdot M\cdot T}{\varepsilon}\right)$$

for each $N\in\mathbb{N}$.

(v) $u^N \in L^2(0,T;H_0^1(\Omega))$ *for each* $N\in\mathbb{N}$.

Theorem 1.2. *Suppose that* $T>0$, $\|f\|_{C^0}<+\infty$ *and* $\|f\|_{C^1}<+\infty$. *Let* u^N *be defined by* (1.8), (1.9) *and* (1.10). *Then*

(i) $\|u_t^N\|_{L^2(0,T;H^{-1}(\Omega))}\le C$ *for each* $N\in\mathbb{N}$,

(ii) $\|u_t^N(t)\|_{H^{-1}(\Omega)}\le K$ *for each* $t\in[0,T]$ *and* $N\in\mathbb{N}$, *and*

(iii) $\|u^N\|_{L^2(0,T;H_0^1(\Omega))}\le\widetilde{C}$ *for each* $N\in\mathbb{N}$,

where constants $C=C(T)$, $K=K(T)$, *and* $\widetilde{C}=\widetilde{C}(T)$ *do not depend on* N, *respectively.*

Theorem 1.3. *Suppose that* $T>0$, $\|f\|_{C^0}<+\infty$ *and* $\|f\|_{C^1}<+\infty$. *Let* u^N *be defined by* (1.8), (1.9) *and* (1.10). *Then at the expense of extracting and relabeling subsequences one may assume there exists a limit function* $u\in L^2(0,T;L^2(\Omega))$ *such that, as* $N\to\infty$,

(i) $u^N\to u$ *a.e. in* $\Omega_T\equiv\Omega\times[0,T]$,

(ii) $u^N\to u$ *strongly in* $L^2(\Omega_T)$,

(iii) $u^N\rightharpoonup u$ *weakly in* $L^2(0,T;H_0^1(\Omega))$,

(iv) $\nabla u^N\rightharpoonup\nabla u$ *weakly in* $L^2(\Omega_T)$,

(v) $\partial_t u^N\rightharpoonup\partial_t u$ *weakly in* $L^2(0,T;H^{-1}(\Omega))$,

(vi) $f(u^N)\to f(u)$ *a.e. in* Ω_T,

(vii) $f(u^N)\to f(u)$ *strongly in* $L^2(\Omega_T)$,

(viii) $u^N(t)\rightharpoonup u(t)$ *in* $H_0^1(\Omega)$ *for a.e.* $t\in[0,T]$,

(ix) $u_t^N(t)\rightharpoonup u_t(t)$ *in* $H^{-1}(\Omega)$ *for a.e.* $t\in[0,T]$.

Theorem 1.4. *Let* $T>0$, $\|f\|_{C^0}<+\infty$, $\|f\|_{C^1}<+\infty$, $\Omega_T\equiv\Omega\times(0,T)$, $\|W\|_{C^0}<+\infty$ *and suppose that* $u_0\in H_0^1(\Omega)$. *There exists* $u\in L^2(0,T;H_0^1(\Omega))$, $u_t\in L^2(0,T;H^{-1}(\Omega))$,

$u(0) = u_0$ in $L^2(\Omega)$, and u satisfies (1.3), (1.4), (1.5), *and* (1.6) *in the following weak sense:*

$$\iint_{\Omega_T} u_t\phi = -\iint_{\Omega_T} E^M_\varepsilon(u)\cdot\varepsilon\nabla u\cdot\nabla\phi - \iint_{\Omega_T} E^M_\varepsilon(u)\frac{1}{\varepsilon}f(u)\phi \tag{1.14}$$

for all $\phi \in L^2(0,T;H^1_0(\Omega))$.

This paper is organized as follows. In Section 2, we will establish some auxiliary results and present a proof of Theorem 1.1. In Section 3, we will prove Theorem 1.2. In Section 4, we will prove Theorem 1.3. Finally in Section 5 we present a proof of Theorem 1.4.

2. Proof of Theorem 1.1

We list some useful well-known identities in the following lemma:

Lemma 2.1. [14, Lemma 2.2]

(i) $\int_\Omega \nabla\phi_i\cdot\nabla\phi_j = \lambda_j\delta_{ij}$. *In particular, if $i = j$, then $\int_\Omega |\nabla\phi_i|^2 = \lambda_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.*

(ii) $\int_\Omega |\nabla u^N|^2 = \sum_{i=1}^N \lambda_i(a_i^N)^2$.

(iii) $\int_\Omega(\Delta u^N)^2 = \sum_{i=1}^N \lambda_i^2(a_i^N)^2$.

(iv) $\int_\Omega |\nabla(\Delta u^N)|^2 = \sum_{i=1}^N \lambda_i^3(a_i^N)^2$.

(v) $\sum_{j=1}^N \lambda_j a_j^N\phi_j = -\Delta u^N$.

(vi) $\sum_{i=1}^N \lambda_i^2 a_i^N\phi_i = \Delta^2 u^N$.

(vii) $\int_\Omega(u^N)^2 = \sum_{i=1}^n (a_i^N)^2$.

(viii) $E_\varepsilon(u^N) = \frac{\varepsilon}{2}\sum_{i=1}^N \lambda_i(a_i^N)^2 + \frac{1}{\varepsilon}\Big(\frac{1}{2}|\Omega| - \sum_{i=1}^N (a_i^N)^2 + \frac{1}{2}\int_\Omega\big(\sum_{i=1}^N a_i^N(t)\phi_i(x)\big)^4\,dx\Big)$.

(ix) $\int_\Omega\big(\sum_{i=1}^N a_i^N(t)\phi_i(x)\big)^4\,dx$ *is a 4th degree polynomial of $a_1^N(t),\ldots,a_N^N(t)$.*

(x) $\int_\Omega \phi_i = 0$ *for $i \geq 2$.*

Proof. (i) By (1.7) and the Green's identity,

$$\begin{aligned}\int_\Omega \nabla\phi_i\cdot\nabla\phi_j &= -\int_\Omega \phi_i\Delta\phi_j + \int_{\partial\Omega}\phi_i\frac{\partial\phi_j}{\partial\nu}\,ds = \int_\Omega \phi_i\cdot\lambda_j\phi_j\,dx \\ &= \lambda_j\langle\phi_i,\phi_j\rangle_{L^2(\Omega)} = \lambda_j\cdot\delta_{ij}.\end{aligned}$$

(ii) By (1.7) and (i),

$$\int_\Omega |\nabla u^N|^2 = \int_\Omega \left|\sum_{i=1}^N a_i^N(t)\nabla\phi_i\right|^2 = \int_\Omega \left(\sum_{i=1}^N a_i^N(t)\nabla\phi_i\right)\cdot\left(\sum_{j=1}^N a_j^N(t)\nabla\phi_j\right)$$
$$= \sum_{i=1}^N\sum_{j=1}^N a_i^N(t)a_j^N(t)\left(\int_\Omega \nabla\phi_i\cdot\nabla\phi_j\right) = \sum_{i=1}^N (a_i^N(t))^2\lambda_i.$$

(iii) Since $-\Delta\phi_i = \lambda_i\phi_i$ and $\langle\phi_i,\phi_j\rangle_{L^2(\Omega)} = \delta_{ij}$,

$$\int_\Omega (\Delta u^N)^2 = \int_\Omega \left(\sum_{i=1}^N a_i^N(t)\Delta\phi_i\right)^2 = \int_\Omega \left(\sum_{i=1}^N a_i^N(t)(-\lambda_i\phi_i)\right)^2$$
$$= \int_\Omega \left(\sum_{i=1}^N a_i^N(t)\lambda_i\phi_i\right)\left(\sum_{j=1}^N a_j^N(t)\lambda_j\phi_j\right)$$
$$= \sum_{j=1}^N a_j^N(t)\lambda_j\left[\sum_{i=1}^N a_i^N(t)\lambda_i\left(\int_\Omega \phi_i\phi_j\,dx\right)\right] = \sum_{j=1}^N \lambda_j^2(a_j^N(t))^2.$$

(iv) Note that $-\Delta\phi_i = \lambda_i\phi_i$, and so

$$\nabla(\Delta u^N) = \nabla\left(\sum_{i=1}^N a_i^N(t)\Delta\phi_i\right) = \sum_{i=1}^N a_i^N(t)\nabla(\Delta\phi_i)$$
$$= \sum_{i=1}^N a_i^N(t)\nabla(-\lambda_i\phi_i) = -\sum_{i=1}^N \lambda_i a_i^N(t)\nabla\phi_i.$$

Thus, by (i), we have

$$\int_\Omega |\nabla(\Delta u^N)|^2 = \int_\Omega \left(-\sum_{i=1}^N \lambda_i a_i^N(t)\nabla\phi_i\right)\cdot\left(-\sum_{j=1}^N \lambda_j a_j^N(t)\nabla\phi_j\right)$$
$$= \sum_{j=1}^N \lambda_j a_j^N(t)\left(\sum_{i=1}^N \lambda_i a_i^N(t)\int_\Omega \nabla\phi_i\cdot\nabla\phi_j\right) = \sum_{j=1}^N \lambda_j^3(a_j^N(t))^2.$$

(v) Since $-\Delta\phi_i = \lambda_i\phi_i$,

$$\sum_{i=1}^N \lambda_i a_i^N(t)\phi_i = \sum_{i=1}^N a_i^N(t)(-\Delta\phi_i) = -\Delta\left(\sum_{i=1}^N a_i^N(t)\phi_i\right) = -\Delta u^N.$$

(vi) Note that $\Delta^2\phi_i = \Delta(\Delta\phi_i) = \Delta(-\lambda_i\phi_i) = -\lambda_i\Delta\phi_i = \lambda_i^2\phi_i$. Hence, using the linearity of Δ^2, we have

$$\sum_{i=1}^N \lambda_i^2 a_i^N(t)\phi_i = \sum_{i=1}^N a_i^N(t)(\lambda_i^2\phi_i) = \sum_{i=1}^N a_i^N(t)\Delta^2\phi_i = \Delta^2\left(\sum_{i=1}^N a_i^N(t)\phi_i\right) = \Delta^2 u^N.$$

(vii)

$$\begin{aligned}\int_\Omega (u^N)^2\,dx &= \int_\Omega \left(\sum_{i=1}^N a_i^N(t)\phi_i\right)\left(\sum_{j=1}^N a_j^N(t)\phi_j\right)\\ &= \sum_{i=1}^N a_i^N(t)\left(\sum_{j=1}^N a_j^N(t)\int_\Omega \phi_i\phi_j\right) = \sum_{i=1}^N (a_i^N(t))^2.\end{aligned}$$

(viii) Using the fact that

$$E_\varepsilon(u^N) = \frac{\varepsilon}{2}\int_\Omega |\nabla u^N|^2 + \frac{1}{\varepsilon}\left(\int_\Omega \frac{1}{2} - (u^N)^2 + \frac{1}{2}(u^N)^4\,dx\right)$$

and combining (ii) and (vii) yields that (viii) holds.

(x) By the divergence theorem and (1.7), (x) holds true. □

The following lemma is useful for the proof of Theorem 1.1 but its proof is quite trivial, so we omit the details here.

Lemma 2.2. *Suppose that f and g are real-valued functions defined on a domain $D \subset \mathbb{R}^n$.*

(i) *If f is a C^1-function on D (i.e., f has continuously first partial derivatives in D), then f is locally Lipschitz on D.*

(ii) *If f and g are locally Lipschitz on D, then $\alpha f + \beta g$ is locally Lipschitz on D for any $\alpha, \beta \in \mathbb{R}$.*

(iii) *If f and g are locally Lipschitz on D, then $f \cdot g$ is locally Lipschitz on D, and $|f-g|$ is locally Lipschitz on D.*

(iv) *If f and g are locally Lipschitz on D, then* $\min\{f, g\}$ *($= (f+g-|f-g|)/2$) is locally Lipschitz on D.*

(v) *Let $\pi_j(v_1, \ldots, v_n) \equiv v_j$. Then π_j is locally Lipschitz on D for $j = 1, 2, \ldots, n$. Every projection function is locally Lipschitz on D.*

Using (1.3), (1.4) and (1.6), we have

$$\int_\Omega u_t^N \cdot \phi_j = \int_\Omega E_\varepsilon^M(u^N)\cdot\left(\varepsilon\Delta u^N - \frac{1}{\varepsilon}f(u^N)\right)\cdot\phi_j, \quad j = 1, 2, \ldots, N, \tag{2.1}$$

$$\int_\Omega u^N(x,0)\cdot\phi_j = \int_\Omega u_0\cdot\phi_j, \quad j = 1, 2, \ldots, N. \tag{2.2}$$

Note that

$$\int_\Omega u_t^N\phi_j = \int_\Omega \left[\sum_{i=1}^N (a_i^N)'(t)\cdot\phi_i\right]\cdot\phi_j = (a_j^N)'(t). \tag{2.3}$$

Setting $b_\varepsilon^N(t) \equiv E_\varepsilon^M(u^N)$ and using Lemma 2.1(v),

$$\begin{aligned}
&\int_\Omega E_\varepsilon^M(u^N)\left(\varepsilon\Delta u^N - \frac{1}{\varepsilon}f(u^N)\right)\phi_j \\
&= b_\varepsilon^N(t)\cdot\left[-\varepsilon\cdot\sum_{i=1}^N \lambda_i a_i^N \int_\Omega \phi_i\cdot\phi_j - \frac{1}{\varepsilon}\int_\Omega f(u^N)\phi_j\right] \\
&= -\varepsilon\cdot b_\varepsilon^N(t)\cdot\lambda_j a_j^N(t) - \frac{1}{\varepsilon}b_\varepsilon^N(t)\cdot\int_\Omega f(u^N)\cdot\phi_j.
\end{aligned}\tag{2.4}$$

Equation (2.2) implies that

$$\begin{aligned}
\int_\Omega u_0\cdot\phi_j &= \int_\Omega\left[\sum_{i=1}^N a_i^N(0)\phi_i(x)\right]\cdot\phi_j(x)\,dx \\
&= \sum_{i=1}^N a_i^N(0)\int_\Omega \phi_i\cdot\phi_j\,dx = a_j^N(0).
\end{aligned}\tag{2.5}$$

Note that if $u^N \in W^N \equiv \operatorname{span}\{\phi_1,\ldots,\phi_N\}$ satisfies (2.1) and (2.2), then direct calculations imply that u^N satisfies

$$\int_\Omega u_t^N\cdot v = \int_\Omega E_\varepsilon^M(u^N)\left(\varepsilon\Delta u^N - \frac{1}{\varepsilon}f(u^N)\right)\cdot v, \quad \forall\, v\in W^N,$$
$$\int_\Omega u^N(x,0)\cdot v = \int_\Omega u_0 v\,dx, \quad \forall\, v\in W^N.$$

Such a function u^N is called a weak solution of (1.3) with (1.4) on finite dimensional space W^N.

Equations (2.1)–(2.5) yield the following initial value problem for $a_j^N(t)$, $j = 1,2,\ldots,N$:

$$\begin{cases}(a_j^N)'(t) = -\varepsilon\cdot b_\varepsilon^N(t)\cdot\lambda_j\cdot a_j^N(t) - \frac{1}{\varepsilon}b_\varepsilon^N(t)\cdot\int_\Omega f(u^N)\phi_j(x)\,dx,\\ a_j^N(0) = \int_\Omega u_0\phi_j.\end{cases}\tag{2.6}$$

Using Lemma 2.1(viii) and (ix), we see that $E_\varepsilon(u^N)$ is a polynomial function of variables $a_1^N, a_2^N, \ldots, a_N^N$ with degree 4. Let $P_4(a_1^N,\ldots,a_N^N) = E_\varepsilon(u^N)$. Note that the term

$$\int_\Omega f\left(\sum_{i=1}^N a_i^N(t)\phi_i(x)\right)\cdot\phi_j(x)\,dx$$

is a 3rd-degree polynomial function of $a_1^N, a_2^N, \ldots, a_N^N$. Let

$$P_3^j(a_1^N,\ldots,a_N^N) = \int_\Omega f\left(\sum_{i=1}^N a_i^N(t)\phi_i(x)\right)\cdot\phi_j(x)\,dx.$$

Note that b_ε^N can be expressed as

$$b_\varepsilon^N = \min\{P_4(a_1^N,a_2^N,\ldots,a_N^N), M\} \equiv Q(a_1^N,a_2^N,\ldots,a_N^N).$$

Let π_j denote the projection function defined by

$$\pi_j(v_1, v_2, \ldots, v_n) = v_j, \quad j = 1, 2, \ldots, n.$$

Now, (2.6) can be expressed as

$$\begin{cases} v_j'(t) = -\varepsilon \cdot Q(v) \cdot \lambda_j \cdot \pi_j(v) - \frac{1}{\varepsilon} Q(v) \cdot P_3^j(v), \\ v_j(0) = \int_\Omega u_0 \cdot \phi_j, \quad j = 1, 2, \ldots, N, \end{cases}$$

where $v = (v_1, \ldots, v_N)$ and $v_j = a_j^N$. Define $F_j(t, v)$ by

$$F_j(t, v) = -\varepsilon \cdot Q(v) \cdot \lambda_j \cdot \pi_j(v) - \frac{1}{\varepsilon} Q(v) \cdot P_3^j(v).$$

The function $F_j(t, v)$ is continuous in its domain. Using Lemma 2.2, we see that $F_j(t, v)$ is locally Lipschitz with respect to v in its domain. Thus the function $F \equiv (F_1, \ldots, F_N)$ is continuous with respect to t and v and locally Lipschitz with respect to v.

By Picard-Lindelöf Existence and Uniqueness Theorem, for any (t_0, v_0) in D, there exists a unique solution $v = v(t, t_0, v_0)$ with $v(t_0, t_0, v_0) = v_0$ of

$$v'(t) = F(t, v(t))$$

passing through (t_0, v_0). Furthermore, the domain E in $\mathbb{R}^{N+2}$ of definition of function $v(t, t_0, v_0)$ is open and $v(t, t_0, v_0)$ is continuous in E. Therefore the initial value problem (1.12) and (1.13) has a local solution $u^N(t, x) = \sum_{j=1}^N a_j^N(t)\phi_j(x)$ for $t \in (0, \delta)$ for some $\delta > 0$ and for all $x \in \Omega$, where $(a_1^N(t), \ldots, a_N^N(t))$ are local solution of IVP (2.6) on $(0, \delta)$.

Now we want to show that a global solution $(a_1^N, \ldots, a_N^N)$ exists on $(0, T)$ for any $T > 0$. Hence it follows that $u^N(t, x)$ is a global solution of IVP (1.12) and (1.13) on $(0, T)$ for any $T > 0$.

Multiplying (2.6) by $-\lambda_j a_j^N$ and adding over $j = 1, 2, \ldots, N$, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\left[\sum_{j=1}^N \lambda_j (a_j^N)^2(t)\right] &= -\sum_{j=1}^N \lambda_j a_j^N(t)(a_j^N)'(t) \\ &= \varepsilon \cdot b_\varepsilon^N(t) \cdot \sum_{j=1}^N \lambda_j^2 (a_j^N)^2(t) \\ &\quad + \frac{1}{\varepsilon} b_\varepsilon^N(t) \cdot \int_\Omega \left[\sum_{j=1}^N \lambda_j a_j^N(t)\phi_j(x)\right] f(u^N)\, dx. \end{aligned} \tag{2.7}$$

Using Lemma 2.1(ii), (iii) and (v), we get (follow from (2.7))

$$-\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |\nabla u^N|^2\right] = \varepsilon b_\varepsilon^N(t) \int_\Omega (\Delta u^N)^2 + \frac{1}{\varepsilon} b_\varepsilon^N(t) \int_\Omega (-\Delta u^N) f(u^N). \tag{2.8}$$

By Green's identity and (1.7) ($\frac{\partial \phi_i}{\partial \nu} = 0$ on $\partial\Omega$), we have

$$\begin{aligned}
\int_\Omega (\Delta u^N) f(u^N) &= -\int_\Omega \nabla u^N \cdot \nabla(f(u^N)) + \int_{\partial\Omega} \underbrace{\frac{\partial u^N}{\partial \nu}} \cdot (f(u^N)) \\
&= -\int_\Omega \nabla u^N \cdot (f'(u^N)\nabla u^N) \quad \left(= \sum_{i=1}^N a_i^N(t) \frac{\partial \phi_i}{\partial \nu} = 0 \text{ on } \Omega \right) \\
&= -\int_\Omega f'(u^N)|\nabla u^N|^2.
\end{aligned} \tag{2.9}$$

Following (2.8) and (2.9), we get

$$-\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |\nabla u^N|^2\right] = \varepsilon b_\varepsilon^N(t) \int_\Omega (\Delta u^N)^2 + \frac{1}{\varepsilon} b_\varepsilon^N(t) \int_\Omega f'(u^N)|\nabla u^N|^2. \tag{2.10}$$

Hence we have the following inequality:

$$\begin{aligned}
\frac{d}{dt}\left[\int_\Omega |\nabla u^N|^2(x,t)\,dx\right] &= -2\varepsilon b_\varepsilon^N(t) \int_\Omega (\Delta u^N)^2 - \frac{2}{\varepsilon} b_\varepsilon^N(t) \int_\Omega f'(u^N)|\nabla u^N|^2 \\
&\le \frac{2}{\varepsilon} b_\varepsilon^N(t) \int_\Omega (-f'(u^N))|\nabla u^N|^2 \\
&\le \frac{2\|f\|_{C^1}}{\varepsilon} M \int_\Omega |\nabla u^N|^2.
\end{aligned} \tag{2.11}$$

The above last inequality is obtained by $(-f')$ is bounded above by $\|f\|_{C^1}$ and b_ε^N is less than M. By Gronwall's inequality (differential form) [8, p. 624] and (2.11), we have

$$\int_\Omega |\nabla u^N|^2(x,\tau)\,dx \le \left(\int_\Omega |\nabla u^N|^2(x,0)\,dx\right) e^{2\|f\|_{C^1} M\tau/\varepsilon}$$

for all $\tau \in [0,T]$, for each $N \in \mathbb{N}$. Moreover, we have for each $N \in \mathbb{N}$,

$$\sup_{\tau\in[0,T]} \int_\Omega |\nabla u^N|^2(x,\tau)\,dx \le \left(\int_\Omega |\nabla u^N|^2(x,0)\,dx\right) e^{2\|f\|_{C^1} MT/\varepsilon}.$$

By Lemma 2.1(ii) and (2.5),

$$\begin{aligned}
\int_\Omega |\nabla u^N|^2(x,0)\,dx &= \sum_{i=1}^N \lambda_i (a_i^N(0))^2 = \sum_{i=1}^N \lambda_i \left(\int_\Omega u_0 \phi_i\,dx\right)^2 \\
&\le \sum_{i=1}^\infty \lambda_i \left(\int_\Omega u_0 \phi_i\,dx\right)^2 = \int_\Omega |\nabla u_0|^2.
\end{aligned}$$

The last equality is obtained by the following:

$$
\begin{aligned}
\int_\Omega |\nabla u_0|^2 &= \int_\Omega \nabla u_0 \cdot \nabla u_0 \\
&= \int_\Omega \left(\sum_{k=1}^\infty \langle u_0, \phi_k \rangle_{L^2(\Omega)} \nabla \phi_k \right) \cdot \left(\sum_{j=1}^\infty \langle u_0, \phi_j \rangle_{L^2(\Omega)} \nabla \phi_j \right) dx \\
&= \sum_{k=1}^\infty \langle u_0, \phi_k \rangle_{L^2(\Omega)} \left(\sum_{j=1}^\infty \langle u_0, \phi_j \rangle_{L^2(\Omega)} \int_\Omega \nabla \phi_k \cdot \nabla \phi_j \right) \\
&= \sum_{k=1}^\infty \langle u_0, \phi_k \rangle_{L^2(\Omega)} (\langle u_0, \phi_k \rangle_{L^2(\Omega)} \lambda_k) \qquad \text{(By Lemma 2.1(i))} \\
&= \sum_{k=1}^\infty \lambda_k (\langle u_0, \phi_k \rangle_{L^2(\Omega)})^2 \\
&= \sum_{k=1}^\infty \lambda_k \left(\int_\Omega u_0 \phi_k \, dx \right)^2 .
\end{aligned}
$$

Therefore we have for each $N \in \mathbb{N}$,

$$\sup_{\tau \in [0,T]} \int_\Omega |\nabla u^N|^2(x,\tau)\, dx \le e^{2\|f\|_{C^1} MT/\varepsilon} \int_\Omega |\nabla u_0|^2\, dx.$$

We notice that $\lambda_1 = 0$ (λ_1 has multiplicity one), and $-\Delta \phi_1 = 0 \cdot \phi_1 = 0$ in Ω, hence $-\Delta \phi_1 = 0$ in Ω. Choosing $\phi_1 \equiv |\Omega|^{-1/2}$. Then $\|\phi_1\|_{L^2(\Omega)} = \left(\int_\Omega \phi_1^2\, dx\right)^{1/2} = \left(|\Omega|^{-1} \cdot |\Omega|\right)^{1/2} = 1$. The initial value problem (2.6) for $a_1^N(t)$ takes the form

$$
\begin{cases}
(a_1^N)'(t) = -\frac{|\Omega|^{-1/2}}{\varepsilon} b_\varepsilon^N(t) \int_\Omega f(u^N)\, dx, \\
a_1^N(0) = |\Omega|^{-1/2} \int_\Omega u_0(x)\, dx.
\end{cases}
$$

Moreover, by assumption $\|f\|_{C^0} < +\infty$,

$$|(a_1^N)'(t)| \le \|f\|_{C^0} |\Omega| \frac{|\Omega|^{-1/2}}{\varepsilon} M = \|f\|_{C^0} \frac{|\Omega|^{1/2}}{\varepsilon} M \equiv C_1 \quad \text{for all } t \in [0,T].$$

By Mean Value Theorem, given any $t \in [0,T]$, $a_1^N(t) - a_1^N(0) = (a_1^N)'(\tau) \cdot (t-0)$ for some $\tau \in (0,t)$,

$$|a_1^N(t)| \le |a_1^N(0)| + |(a_1^N)'(\tau)| \cdot |t-0| \quad \le |\Omega|^{-1/2} \left| \int_\Omega u_0(x)\, dx \right| + C_1 \cdot T, \quad \forall\, t \in [0,T].$$

Hence we have

$$\|a_1^N\|_{L^\infty(0,T)} \le |\Omega|^{-1/2} \left| \int_\Omega u_0(x)\, dx \right| + C_1 \cdot T. \tag{2.12}$$

Owing to

$$\begin{aligned}\int_\Omega u^N \cdot \phi_1\,dx &= \int_\Omega \left[\sum_{i=1}^N a_i^N(t)\phi_i(x)\right]\cdot \phi_1(x)\,dx\\ &= a_1^N(t)\left[\int_\Omega \phi_1^2\,dx\right] \qquad (\text{note } \langle \phi_i,\phi_j\rangle_{L^2(\Omega)} = \delta_{ij})\\ &= a_1^N(t),\end{aligned}$$

by $\phi_1 = |\Omega|^{-1/2}$, we then obtain

$$\int_\Omega u^N(x,t)\,dx = a_1^N(t)\cdot |\Omega|^{1/2}.$$

Using (2.12), we have

$$\left|\int_\Omega u^N\right| = |a_1^N(t)|\cdot|\Omega|^{1/2} \le \left(|\Omega|^{-1/2}\left|\int_\Omega u_0\,dx\right| + C_1\cdot T\right)\cdot |\Omega|^{1/2}. \tag{2.13}$$

By Poincaré inequality, there exists a constant $C(n,\Omega)$, depending only on n and Ω, such that

$$\|u^N - (u^N)_\Omega\|_{L^2(\Omega)} \le C(n,\Omega)\|\nabla u^N\|_{L^2(\Omega)}. \tag{2.14}$$

By Minkowski's inequality, (2.14) and (2.13), we have that

$$\begin{aligned}\|u^N\|_{L^2(\Omega)} &= \|(u^N - (u^N)_\Omega) + (u^N)_\Omega\|_{L^2(\Omega)}\\ &\le \|u^N - (u^N)_\Omega\|_{L^2(\Omega)} + \|(u^N)_\Omega\|_{L^2(\Omega)}\\ &\le C(n,\Omega)\|\nabla u^N\|_{L^2(\Omega)} + \|(u^N)_\Omega\|_{L^2(\Omega)},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\|(u^N)_\Omega\|_{L^2(\Omega)} &= \left[\int_\Omega |(u^N)_\Omega|^2\,dx\right]^{1/2} = \left[|(u^N)_\Omega|^2\cdot|\Omega|\right]^{1/2}\\ &= \left|\frac{1}{|\Omega|}\int_\Omega u^N\,dx\right|\cdot|\Omega|^{1/2} \le |\Omega|^{-1/2}\cdot\left|\int_\Omega u^N\,dx\right|\\ &\le \left(|\Omega|^{-1/2}\cdot\left|\int_\Omega u_0\,dx\right| + C_1\cdot T\right).\end{aligned} \tag{2.16}$$

In view of (2.15) and (2.16), we have

$$\|u^N\|_{L^2(\Omega)} \le C(n,\Omega)\|\nabla u^N\|_{L^2(\Omega)} + |\Omega|^{-1/2}\cdot\left|\int_\Omega u_0\,dx\right| + C_1\cdot T. \tag{2.17}$$

It follows from (2.17) and (1.11) that

$$\begin{aligned}\|u^N\|_{H_0^1(\Omega)} &= \left(\|u^N\|^2_{L^2(\Omega)} + \|\nabla u^N\|^2_{L^2(\Omega)}\right)^{1/2}\\ &\le \|u^N\|_{L^2(\Omega)} + \|\nabla u^N\|_{L^2(\Omega)}\\ &\le (C(n,\Omega)+1)\,\|\nabla u^N\|_{L^2(\Omega)} + |\Omega|^{-1/2}\cdot\left|\int_\Omega u_0\,dx\right| + C_1\cdot T,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\|\nabla u^N\|^2_{L^2(\Omega)} &= \left[\int_\Omega |\nabla u^N|^2(x,t)\,dx\right]\\ &\le \sup_{\tau\in[0,T]}\int_\Omega |\nabla u^N|^2(x,\tau)\,dx\\ &\le e^{2\|f\|_{C^1}MT/\varepsilon}\int_\Omega |\nabla u_0|^2\,dx \quad \text{for all } t\in[0,T],\end{aligned}$$

$$\|\nabla u^N\|_{L^2(\Omega)} \le e^{\|f\|_{C^1}MT/\varepsilon}\cdot\|\nabla u_0\|_{L^2(\Omega)} \quad \text{for all } t\in[0,T]. \tag{2.19}$$

Following (2.18) and (2.19), we conclude that

$$\begin{aligned}\|u^N\|_{H_0^1(\Omega)} &\le (C(n,\Omega)+1)\left(e^{\|f\|_{C^1}MT/\varepsilon}\|\nabla u_0\|_{L^2(\Omega)}\right)\\ &\quad + |\Omega|^{-1/2}\left|\int_\Omega u_0\,dx\right| + C_1\cdot T\end{aligned}$$

for all $t\in[0,T]$. Moreover, we get

$$\begin{aligned}\sup_{t\in[0,T]}\|u^N\|_{H_0^1(\Omega)} &\le (C(n,\Omega)+1)\left(e^{\|f\|_{C^1}MT/\varepsilon}\|\nabla u_0\|_{L^2(\Omega)}\right)\\ &\quad + |\Omega|^{-1/2}\left|\int_\Omega u_0\,dx\right| + \frac{\|f\|_{C^0}\cdot|\Omega|^{1/2}\cdot M}{\varepsilon}\cdot T\\ &\equiv K^*_{\varepsilon,T}.\end{aligned}$$

Then we have

$$\sup_{t\in[0,T]}\|u^N\|_{H_0^1(\Omega)} \le K^*_{\varepsilon,T}, \tag{2.20}$$

where $K^*_{\varepsilon,T}$ is independent of N. In view of Lemma 2.1(ii), we conclude that

$$\begin{aligned}\sqrt{\sum_{i=1}^N (a_i^N)^2\cdot\lambda_i} &= \|\nabla u^N\|_{L^2(\Omega)} \le \|u^N\|_{H_0^1(\Omega)}\\ &\le \sup_{t\in[0,T]}\|u^N\|_{H_0^1(\Omega)} \le K^*_{\varepsilon,T}\end{aligned}$$

for all $t\in[0,T]$. Hence we have

$$\|a_i^N\|_{L^\infty(0,T)} \le \lambda_i^{-1/2}\cdot K^*_{\varepsilon,T} \le \lambda_2^{-1/2}\cdot K^*_{\varepsilon,T} \tag{2.21}$$

for $i = 2,\dots,N$. Here $\lambda_2^{-1/2}\cdot K^*_{\varepsilon,T}$ is a constant which is independent of N. Combining (2.12) and (2.21), $a_i^N(t)$ exists globally on $[0,T]$, for $i = 1,2,\dots,N$, and for each given $T>0$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let $\Pi_N \colon L^2(\Omega) \mapsto W^N \equiv \text{span}\{\phi_1, \ldots, \phi_N\}$ be the projection operator, i.e., $\Pi_N(\psi) = \sum_{i=1}^N \langle \psi, \phi_i \rangle_{L^2(\Omega)} \phi_i$ for each $\psi \in L^2(\Omega)$. By (1.3), for each $j = 1, 2, \ldots, N$, we have

$$\int_\Omega u_t^N \phi_j = \int_\Omega E_\varepsilon^M(u^N) \cdot \varepsilon \Delta u^N \cdot \phi_j - \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) \phi_j.$$

By the Integration by Parts,

$$\int_\Omega E_\varepsilon^M(u^N) \cdot \varepsilon(\Delta u^N) \phi_j = - \int_\Omega E_\varepsilon^M(u^N) \cdot \varepsilon \nabla u^N \cdot \nabla \phi_j + \int_{\partial\Omega} E_\varepsilon^M(u^N) \varepsilon \phi_j \frac{\partial u^N}{\partial \nu} \, ds.$$

By (1.7), $\frac{\partial u^N}{\partial \nu} = 0$ on $\partial\Omega \times [0, T]$. Hence we have

$$\int_\Omega E_\varepsilon^M(u^N) \varepsilon(\Delta u^N) \phi_j = - \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla \phi_j.$$

Therefore, we obtain that

$$\int_\Omega u_t^N \phi_j = - \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla \phi_j - \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) \phi_j, \quad j = 1, 2, \ldots, N. \tag{3.1}$$

Since

$$u^N(x, t) = \sum_{i=1}^N a_i^N(t) \phi_i(x),$$

we have

$$u_t^N = \sum_{i=1}^N (a_i^N)'(t) \phi_i(x) \quad \text{and} \quad \nabla u^N = \sum_{i=1}^N a_i^N(t) \nabla \phi_i(x).$$

Note that given any $\phi \in L^2(0, T; H_0^1(\Omega))$, we have $\phi(t) \in H_0^1(\Omega)$ for each $t \in [0, T]$. Using the Fourier Series Theorem, we have

$$\phi(t) = \sum_{i=1}^\infty \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \phi_i, \quad \forall\, t \in [0, T].$$

Then

$$\begin{aligned} \int_\Omega u_t^N \phi(t) \, dx &= \int_\Omega \left(\sum_{j=1}^N (a_j^N)'(t) \phi_j \right) \left(\sum_{i=1}^\infty \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \phi_i \right) dx \\ &= \int_\Omega \sum_{j=1}^N (a_j^N)'(t) \left(\sum_{i=1}^\infty \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \phi_i \phi_j \right) dx \\ &= \sum_{j=1}^N (a_j^N)'(t) \left(\sum_{i=1}^\infty \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \int_\Omega \phi_i \phi_j \, dx \right) \qquad (3.2) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} (a_j^N)'(t) \langle \phi(t), \phi_j \rangle_{L^2(\Omega)} = \sum_{j=1}^{N} (a_j^N)'(t) \left(\sum_{i=1}^{N} \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \delta_{ij} \right) \\
&= \int_\Omega \left(\sum_{j=1}^{N} (a_j^N)'(t) \phi_j \right) \left(\sum_{i=1}^{N} \langle \phi(t), \phi_i \rangle_{L^2(\Omega)} \phi_i \right) dx \\
&= \int_\Omega u_t^N (\Pi_N \phi).
\end{aligned}
$$

Following (3.1) and (3.2), for each $\phi \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
\int_\Omega u_t^N \cdot \phi &= \int_\Omega u_t^N (\Pi_N \phi) = \int_\Omega u_t^N \left(\sum_{j=1}^{N} \langle \phi, \phi_j \rangle_{L^2(\Omega)} \phi_j \right) dx \\
&= \sum_{j=1}^{N} \left(\int_\Omega \langle \phi, \phi_j \rangle_{L^2(\Omega)} u_t^N \phi_j \, dx \right) = \sum_{j=1}^{N} \langle \phi, \phi_j \rangle_{L^2(\Omega)} \left(\int_\Omega u_t^N \phi_j \right) \\
&= \sum_{j=1}^{N} \langle \phi, \phi_j \rangle_{L^2(\Omega)} \left(- \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla \phi_j - \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) \phi_j \right) \\
&= - \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla(\Pi_N \phi) \, dx - \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) (\Pi_N \phi) \, dx.
\end{aligned} \tag{3.3}
$$

The last equality is achieved by using

$$
\Pi_N \phi = \sum_{j=1}^{N} \langle \phi, \phi_j \rangle_{L^2(\Omega)} \phi_j \quad \text{and} \quad \nabla(\Pi_N \phi) = \sum_{j=1}^{N} \langle \phi, \phi_j \rangle_{L^2(\Omega)} \nabla \phi_j.
$$

Integrating (3.3) from 0 to T, we have

$$
\int_0^T \int_\Omega u_t^N \phi = - \int_0^T \int_\Omega E_\varepsilon^M(u^N) \cdot \varepsilon \nabla u^N \cdot \nabla(\Pi_N \phi) - \int_0^T \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) (\Pi_N \phi)
$$

for each $\phi \in L^2(0, T; H_0^1(\Omega))$.

$$
\begin{aligned}
&\int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla(\Pi_N \phi) \, dx \\
&= - \int_\Omega E_\varepsilon^M(u^N) \varepsilon (\Delta u^N) \cdot (\Pi_N \phi) \, dx + \int_{\partial\Omega} E_\varepsilon^M(u^N) \varepsilon \frac{\partial u^N}{\partial \nu} \cdot (\Pi_N \phi) \, ds \\
&= - \int_\Omega E_\varepsilon^M(u^N) \varepsilon (\Delta u^N) \cdot (\Pi_N \phi) \, dx.
\end{aligned} \tag{3.4}
$$

By (3.3) and (3.4), we have

$$
\int_\Omega u_t^N \phi \, dx = \int_\Omega E_\varepsilon^M(u^N) \left(\varepsilon (\Delta u^N) - \frac{1}{\varepsilon} f(u^N) \right) (\Pi_N \phi) \, dx \tag{3.5}
$$

for each $\phi \in L^2(\Omega)$. Suppose that $\phi \in L^2(0,T;H_0^1(\Omega))$. Then $\phi(t) \in H_0^1(\Omega)$ for each $t \in [0,T]$. By (3.5), we have

$$\begin{aligned}\left|\int_0^T\int_\Omega u_t^N \phi\, dxdt\right| &\le \underbrace{\int_0^T\int_\Omega \varepsilon E_\varepsilon^M(u^N)|\Delta u^N|\cdot|\Pi_N\phi|\, dxdt}_{\equiv(A)} \\ &\quad + \underbrace{\int_0^T\int_\Omega \frac{1}{\varepsilon}E_\varepsilon^M(u^N)|f(u^N)|\cdot|\Pi_N\phi|\, dxdt}_{\equiv(B)}.\end{aligned} \tag{3.6}$$

Let $h^N(t) \equiv \int_\Omega |\nabla u^N|^2(x,t)\, dx$, $\forall\, t \in [0,T]$. Equation (2.10) can be expressed as

$$-\frac{1}{2}(h^N)'(t) = \varepsilon b_\varepsilon^N(t)\int_\Omega (\Delta u^N)^2(x,t)\, dx + \frac{1}{\varepsilon}b_\varepsilon^N(t)\int_\Omega f'(u^N(x,t))|\nabla u^N|^2(x,t)\, dx \tag{3.7}$$

for all $t \in [0,T]$. Integrating (3.7) from 0 to τ, we have

$$\begin{aligned}-\frac{1}{2}\left[h^N(\tau) - h^N(0)\right] &= \underbrace{\int_0^\tau \varepsilon b_\varepsilon^N(t)\left(\int_\Omega (\Delta u^N)^2(x,t)\, dx\right) dt}_{\equiv\eta(\tau)\ge 0} \\ &\quad + \int_0^\tau \frac{1}{\varepsilon}b_\varepsilon^N(t)\left(\int_\Omega f'(u^N(x,t))|\nabla u^N|^2\, dx\right) dt.\end{aligned}$$

Hence we have

$$h^N(\tau) = (h^N(0) - 2\eta(\tau)) - \int_0^\tau \frac{2}{\varepsilon}b_\varepsilon^N(t)\left(\int_\Omega f'(u^N)|\nabla u^N|^2\, dx\right) dt \tag{3.8}$$

for all $\tau \in [0,T]$. By (3.8),

$$\begin{aligned}0 \le \eta(\tau) &= \frac{1}{2}\left[h^N(0)\right] - \frac{1}{2}h^N(\tau) - \int_0^\tau\int_\Omega \frac{1}{\varepsilon}b_\varepsilon^N(t)f'(u^N)|\nabla u^N|^2\, dxdt \\ &\le \frac{h^N(0)}{2} + \left|\int_0^\tau\int_\Omega \frac{1}{\varepsilon}b_\varepsilon^N(t)f'(u^N)|\nabla u^N|^2\, dxdt\right| \\ &\le \frac{h^N(0)}{2} + \int_0^\tau\int_\Omega \frac{1}{\varepsilon}M\|f\|_{C^1}|\nabla u^N|^2\, dxdt \\ &= \frac{h^N(0)}{2} + \frac{1}{\varepsilon}M\|f\|_{C^1}\int_0^\tau h^N(t)\, dt \\ &\le \frac{1}{2}\left[\sup_{\tau\in[0,T]}\int_\Omega |\nabla u^N|^2(x,t)\, dx\right] + \frac{1}{\varepsilon}M\|f\|_{C^1}T\left[\sup_{\tau\in[0,T]}\int_\Omega |\nabla u^N|^2(x,\tau)\, dx\right]\end{aligned} \tag{3.9}$$

for all $\tau \in [0,T]$. By Theorem 1.1(iii) and (3.9), we get

$$\begin{aligned}\eta(T) &= \int_0^T\int_\Omega \varepsilon b_\varepsilon^N(t)(\Delta u^N)^2(x,t)\, dxdt \\ &\le \left[\frac{1}{2} + \frac{M}{\varepsilon}\|f\|_{C^1}T\right]\cdot \sup_{\tau\in[0,T]}\int_\Omega |\nabla u^N|^2(x,\tau)\, dx \\ &\le \left(\frac{1}{2} + \frac{M}{\varepsilon}\|f\|_{C^1}T\right)e^{2\|f\|_{C^1}MT/\varepsilon}\cdot\left(\int_\Omega |\nabla u_0|^2\, dx\right) \equiv M_{\varepsilon,T}.\end{aligned} \tag{3.10}$$

By Hölder's inequality, (3.6) and (3.10), we have

$$
\begin{aligned}
(A) &\le \left(\int_0^T\int_\Omega \left(\varepsilon E_\varepsilon^M(u^N)|\Delta u^N|\right)^2\,dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |\Pi_N\phi|^2\,dxdt\right)^{1/2} \\
&\le (\varepsilon\cdot M)^{1/2}(\eta(T))^{1/2}\|\phi\|_{L^2(\Omega_T)} \\
&\le (\varepsilon\cdot M)^{1/2}(M_{\varepsilon,T})^{1/2}\|\phi\|_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}
\tag{3.11}
$$

By Hölder's inequality and $|\Pi_N\phi|\le|\phi|$, we have

$$
\begin{aligned}
(B) &\equiv \int_0^T\int_\Omega \frac{1}{\varepsilon}E_\varepsilon^M(u^N)|f(u^N)||\Pi_N\phi|\,dxdt \\
&\le \frac{1}{\varepsilon}M\|f\|_{C^0}\left(\int_0^T\int_\Omega |\Pi_N\phi|\,dxdt\right) \\
&\le \frac{1}{\varepsilon}M\|f\|_{C^0}\left(\int_0^T\int_\Omega |\phi|^2\,dxdt\right)^{1/2}\left(\int_0^T\int_\Omega 1^2\,dxdt\right)^{1/2} \\
&\le \frac{1}{\varepsilon}M\|f\|_{C^0}(|\Omega|T)^{1/2}\|\phi\|_{L^2(\Omega_T)} = \frac{1}{\varepsilon}M\|f\|_{C^0}(|\Omega|T)^{1/2}\|\phi\|_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}
\tag{3.12}
$$

Combining (3.6), (3.11) and (3.12) yields

$$
\left|\int_0^T\int_\Omega u_t^N\phi\,dxdt\right| \le C(\varepsilon,T)\|\phi\|_{L^2(0,T;H_0^1(\Omega))}
\tag{3.13}
$$

for all $\phi\in L^2(0,T;H_0^1(\Omega))$. Here $C(\varepsilon,T)$ is a constant which is independent of N. By (3.13), for each $N\in\mathbb{N}$, we have

$$
\|u_t^N\|_{L^2(0,T;H^{-1}(\Omega))} \equiv \sup\left\{\frac{\left|\int_0^T\int_\Omega u_t^N(t)\phi(t)\,dxdt\right|}{\|\phi\|_{L^2(0,T;H_0^1(\Omega))}}\;\middle|\;\phi\ne 0\right\} \le C(\varepsilon,T).
$$

This concludes the proof of the assertion (i). Next, using Cauchy-Schwarz inequality, Hölder's inequality, (3.3) and (2.20), we get, for each $\phi\in H_0^1(\Omega)$,

$$
\begin{aligned}
\left|\int_\Omega u_t^N(t)\cdot\phi\right| &\le \varepsilon M\int_\Omega |\nabla u^N\cdot\nabla(\Pi_N\phi)| + \frac{M}{\varepsilon}\int_\Omega |f(u^N)||\Pi_N\phi| \\
&\le \varepsilon M\int_\Omega |\nabla u^N|\cdot|\nabla(\Pi_N\phi)| + \frac{M}{\varepsilon}\|f\|_{C^0}\int_\Omega|\phi| \\
&\le \varepsilon M\|\nabla u^N\|_{L^2(\Omega)}\cdot\|\nabla\phi\|_{L^2(\Omega)} + \frac{M}{\varepsilon}\|f\|_{C^0}|\Omega|^{1/2}\|\phi\|_{L^2(\Omega)} \\
&\le \left(\varepsilon M\|u^N\|_{H_0^1(\Omega)} + \frac{M}{\varepsilon}\|f\|_{C^0}|\Omega|^{1/2}\right)\|\phi\|_{H_0^1(\Omega)} \\
&\le \left(\varepsilon M K_{\varepsilon,T}^* + \frac{M}{\varepsilon}\|f\|_{C^0}|\Omega|^{1/2}\right)\|\phi\|_{H_0^1(\Omega)} \\
&\equiv K(T)\cdot\|\phi\|_{H_0^1(\Omega)}.
\end{aligned}
$$

Thus we have for each $t \in [0, T]$,

$$\|u_t^N(t)\|_{H^{-1}(\Omega)} \leq K(T) \quad \text{for all } N \in \mathbb{N}.$$

Hence the conclusion of (ii) holds true.

Now we want to show $\|u^N\|_{L^2(0,T;H_0^1(\Omega))} \leq \widetilde{C}$ for all $N \in \mathbb{N}$, i.e., show that

$$\int_0^T \int_\Omega (u^N)^2 \, dxdt + \int_0^T \int_\Omega |\nabla u^N|^2 \, dxdt \leq C^* \quad \text{for all } N \in \mathbb{N}.$$

By (2.20),

$$\|u^N\|_{H_0^1(\Omega)}^2(t) \leq (K_{\varepsilon,T}^*)^2 \quad \text{for all } t \in [0,T], \text{ for all } N \in \mathbb{N},$$

$$\int_0^T \|u^N\|_{H_0^1(\Omega)}^2(t) \, dt \leq T \cdot (K_{\varepsilon,T}^*)^2 \equiv K_{\varepsilon,T}^{**} \quad \text{for each } N \in \mathbb{N},$$

$$\|u^N\|_{L^2(0,T;H_0^1(\Omega))} \leq (K_{\varepsilon,T}^{**})^{1/2} \quad \text{for each } N \in \mathbb{N}.$$

This yields the conclusion of (iii).

4. Proof of Theorem 1.3

First, we recall the following proposition:

Proposition 4.1 (Aubin-Lions compactness theorem)**.** [22] *Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Suppose*

$$X_0 \hookrightarrow X \hookrightarrow X_1,$$

with a compact embedding of X_0 into X. Let $\{u_n\}_{n=1}^\infty$ be a sequence that is bounded in $L^p(0,T;X_0)$ and for which $\{\partial_t u_n\}_{n=1}^\infty$ is bounded in $L^q(0,T;X_1)$, with $1 < p, q < +\infty$. Then $\{u_n\}_{n=1}^\infty$ is precompact in $L^p(0,T;X)$. This means that there exist a subsequence $\{u_{n_j}\}_{j=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $u \in L^p(0,T;X)$ such that

$$\lim_{j\to\infty} \|u_{n_j} - u\|_{L^p(0,T;X)} = 0.$$

By Theorem 1.2, the sequence $\{u^N\}_{N=1}^\infty$ is bounded in $L^2(0,T;H_0^1(\Omega))$, and $\{u_t^N\}_{N=1}^\infty$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. By the weak compactness theorem, there are a subsequence $\{u^{N_j}\}_{j=1}^\infty \subset \{u^N\}_{N=1}^\infty$ and a function $u \in L^2(0,T;H_0^1(\Omega))$, with $u_t \in L^2(0,T;H^{-1}(\Omega))$, such that

$$u^{N_j} \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$u_t^{N_j} \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)).$$

By Proposition 4.1, there exist a subsequence $\{u^{N_l}\}_{l=1}^{\infty} \subset \{u^N\}_{N=1}^{\infty}$ and a function $\widetilde{u} \in L^2(0,T;L^2(\Omega))$ such that

$$u^{N_l} \to \widetilde{u} \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Without loss of generality, there exist a subsequence of $\{u^N\}_{N=1}^{\infty}$, say still $\{u^N\}_{N=1}^{\infty}$, and a function $u \in L^2(0,T;H_0^1(\Omega))$, with $u_t \in L^2(0,T;H^{-1}(\Omega))$, such that

$$u^N \to u \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.1}$$

Note that if $u \in L^2(0,T;H_0^1(\Omega))$, then $u \in L^2(0,T;L^2(\Omega))$. Moreover,

$$L^2(0,T;L^2(\Omega)) = L^2(\Omega_T), \tag{4.2}$$

$$L^2(0,T;H_0^1(\Omega)) = H_0^1(\Omega_T). \tag{4.3}$$

Using the norm convergence (4.1) and (4.2), there exists a subsequence of $\{u^N\}_{N=1}^{\infty}$, say still $\{u^N\}_{N=1}^{\infty}$ such that

$$u^N \to u \quad \text{a.e. in } \Omega_T \equiv \Omega \times [0,T], \tag{4.4}$$

and

$$u^N \to u \quad \text{strongly in } L^2(\Omega_T).$$

Recalling that the boundedness of $\{u^N\}_{N=1}^{\infty}$ in $L^2(0,T;H_0^1(\Omega))$ and (4.3) imply that there exists a subsequence of $\{u^N\}_{N=1}^{\infty}$, say still $\{u^N\}_{N=1}^{\infty}$, such that

$$u^N \rightharpoonup u \quad \text{weakly in } L^2(\Omega_T),$$

and

$$\nabla u^N \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega_T),$$

by means of the weak compactness theorem.

Assertion (vi) follows from (4.4) and the continuity of function f. Let E_T be the subset of Ω_T consisting of the points for which the sequence $\{f(u^N)\}_{N=1}^{\infty}$ does not converge to $f(u)$ at that point. By assertion (vi), $|E_T| = 0$ and

$$f(u^N) \to f(u) \quad \text{at each point of } D_T \equiv \Omega_T - E_T.$$

For each $\varepsilon > 0$, there always exists a measurable set E_ε, say $E_\varepsilon \equiv D_T$, such that $|E_\varepsilon| = |D_T| = |\Omega_T - E_T| = |\Omega_T| < \infty$ and

$$\int_{\widetilde{E}_\varepsilon} |f(u^N)|^2\, dx = \int_{E_T} |f(u^N)|^2\, dx \le \|f\|_{C^0}^2 \cdot |E_T| = 0 < \varepsilon.$$

For each $\varepsilon > 0$, there exists $\delta_\varepsilon \equiv \varepsilon/(1+\|f\|_{C^0}^2) > 0$ such that if $E \subset \Omega_T$ with $|E| < \delta_\varepsilon$, then

$$\int_E |f(u^N)|^2\, dx \le \|f\|_{C^0}^2 \cdot |E| < \|f\|_{C^0}^2 \cdot \delta_\varepsilon = \left(\frac{\|f\|_{C^0}^2}{1+\|f\|_{C^0}^2}\right) \cdot \varepsilon < \varepsilon.$$

By assertion (vi) and the Vitali's Convergence Theorem [12], it follows that assertion (vii) holds. By Theorem 1.1(iv) and the weak compactness theorem, after taking subsequences, we have (viii). Applying the weak compactness theorem and Theorem 1.2(ii), after taking subsequences, we conclude that (ix) holds.

5. Proof of Theorem 1.4

By Theorem 1.1, for each $N \in \mathbb{N}$, we have

$$\int_\Omega u_t^N \phi^N = -\int_\Omega E_\varepsilon^M(u^N)\varepsilon\nabla u^N \cdot \nabla\phi^N - \int_\Omega E_\varepsilon^M(u^N)\frac{1}{\varepsilon}f(u^N)\phi^N, \tag{5.1}$$
$$\int_\Omega u^N(x,0)\phi^N = \int_\Omega u_0\phi^N$$

for each $\phi^N \in W^N \equiv \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \subset H_0^1(\Omega) \cap C^\infty(\Omega)$. It's well-known that if $H_0^1(\Omega)$ is provided with the inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} \equiv \int_\Omega \nabla u \cdot \nabla v,$$

then $\{\widehat{\phi}_k\}_{k=1}^\infty$ is an orthonormal basis for $H_0^1(\Omega)$, where $\widehat{\phi}_k = \lambda_{k+1}^{-1/2}\phi_{k+1}$ for each $k \in \mathbb{N}$. For each $\phi \in L^2(0,T; H_0^1(\Omega))$, and for each $N \in \mathbb{N}$, we define $\phi^N\colon [0,T] \mapsto H_0^1(\Omega)$ by

$$\phi^N(t) \equiv \sum_{k=1}^N \langle \phi(t), \widehat{\phi}_k\rangle_{H_0^1(\Omega)}\widehat{\phi}_k.$$

By the Fourier Series Theorem (see [18, p. 194]), for each $t \in [0,T]$, we have

$$\phi^N(t) \to \phi(t) \quad \text{in } H_0^1(\Omega) \text{ as } N \to +\infty. \tag{5.2}$$

By Theorem 1.3(ix), for each $t \in [0,T]$,

$$\langle u_t^N(t), \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \to \langle u_t(t), \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \quad \text{as } N \to +\infty. \tag{5.3}$$

Using Theorem 1.2(ii) and (5.2), we have

$$\langle u_t^N(t), \phi^N(t) - \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \to 0 \quad \text{as } N \to +\infty. \tag{5.4}$$

For each $t \in [0,T]$ and for each $N \in \mathbb{N}$, we have

$$\begin{aligned}\langle u_t^N(t), \phi^N(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \langle u_t^N(t), \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad + \langle u_t^N(t), \phi^N(t) - \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned} \tag{5.5}$$

Combining (5.3), (5.4) and (5.5) yields

$$\langle u_t^N(t), \phi^N(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \to \langle u_t(t), \phi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \quad \text{as } N \to +\infty \tag{5.6}$$

for each $t \in [0,T]$. We observe that for each $t \in [0,T]$,

$$\begin{aligned} \left| \langle u_t^N(t), \phi^N(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| &\le \|u_t^N(t)\|_{H^{-1}(\Omega)} \|\phi^N(t)\|_{H_0^1(\Omega)} \\ &\le K \|\phi(t)\|_{H_0^1(\Omega)} \end{aligned} \tag{5.7}$$

by using the uniform bound of $\{u_t^N\}_{N=1}^\infty$ on $[0,T]$ (see Theorem 1.2(ii)). Owing to $\phi \in L^2(0,T;H_0^1(\Omega))$, by Lebesgue's Dominated Convergence Theorem, (5.6) and (5.7), we have

$$\int_0^T \int_\Omega u_t^N(t) \phi^N(t)\, dxdt \to \int_0^T \int_\Omega u_t(t) \phi(t)\, dxdt \quad \text{as } N \to +\infty \tag{5.8}$$

for all $\phi \in L^2(0,T;H_0^1(\Omega))$. Now we have the following:

Claim 1. For each $\phi \in L^2(0,T;H_0^1(\Omega))$, we have

$$\int_0^T \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla \phi^N\, dxdt \to \int_0^T \int_\Omega E_\varepsilon^M(u) \varepsilon \nabla u \cdot \nabla \phi\, dxdt \quad \text{as } N \to +\infty. \tag{5.9}$$

By Theorem 1.3(viii),

$$\nabla u^N(t) \rightharpoonup \nabla u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^n).$$

It is readily seen that for each $\phi \in L^2(0,T;H_0^1(\Omega))$,

$$\int_\Omega \nabla u^N(t) \cdot \nabla \phi(t)\, dx \to \int_\Omega \nabla u(t) \cdot \nabla \phi(t)\, dx \quad \text{as } N \to +\infty \tag{5.10}$$

for each $t \in [0,T]$. Owing to $u^K \in L^2(0,T;H_0^1(\Omega))$ for each $K \in \mathbb{N}$, by (5.10), we have

$$\int_\Omega \nabla u^N(t) \cdot \nabla u^K(t)\, dx \to \int_\Omega \nabla u(t) \cdot \nabla u^K(t)\, dx \quad \text{as } N \to +\infty. \tag{5.11}$$

Utilizing (5.10) to (5.11), we obtain

$$\int_\Omega |\nabla u^N(t)|^2\, dx \to \int_\Omega |\nabla u(t)|^2\, dx \quad \text{as } N \to +\infty. \tag{5.12}$$

Based on Theorem 1.3(i), the continuity of function W, and $\|W\|_{C^0} < +\infty$, we can apply Lebesgue's Dominated Convergence Theorem to obtain

$$\int_\Omega W \circ u^N(t)\, dx \to \int_\Omega W \circ u(t)\, dx \quad \text{as } N \to +\infty \tag{5.13}$$

for each $t \in [0,T]$. Combining (5.12) and (5.13) and recalling that

$$E_\varepsilon^M(u^N(t)) = \frac{E_\varepsilon(u^N(t)) + M - |E_\varepsilon(u^N(t)) - M|}{2},$$

we have for each $t \in [0,T]$,

$$\lim_{N\to\infty} E_\varepsilon^M(u^N(t)) = E_\varepsilon^M(u(t)). \tag{5.14}$$

This, combined with (5.10), implies that

$$
\begin{aligned}
\int_\Omega E^M_\varepsilon(u^N(t))\varepsilon\nabla u^N(t)\cdot\nabla\phi(t)\,dx &= \varepsilon E^M_\varepsilon(u^N(t))\int_\Omega \nabla u^N(t)\cdot\nabla\phi(t)\,dx\\
&\to \varepsilon E^M_\varepsilon(u(t))\int_\Omega \nabla u(t)\cdot\nabla\phi(t)\,dx\\
&= \int_\Omega E^M_\varepsilon(u(t))\varepsilon\nabla u(t)\cdot\nabla\phi(t)\,dx \quad \text{as } N\to+\infty
\end{aligned}
\tag{5.15}
$$

for each $t\in[0,T]$. Since

$$
\begin{aligned}
\left|\int_\Omega \nabla u^N(t)\cdot\nabla(\phi^N(t)-\phi(t))\,dx\right| &\le \int_\Omega \left|\nabla u^N(t)\cdot\nabla(\phi^N(t)-\phi(t))\right|\,dx\\
&\le \|\nabla u^N(t)\|_{L^2(\Omega)}\cdot\|\nabla(\phi^N(t)-\phi(t))\|_{L^2(\Omega)}\\
&\le \|u^N(t)\|_{H^1_0(\Omega)}\cdot\|\phi^N(t)-\phi(t)\|_{H^1_0(\Omega)}
\end{aligned}
$$

by Theorem 1.1(iv) and the Fourier Series Theorem, we have

$$
\int_\Omega \nabla u^N(t)\cdot\nabla(\phi^N(t)-\phi(t))\,dx\to 0 \quad \text{as } N\to+\infty \tag{5.16}
$$

for each $t\in[0,T]$. Combining (5.14) and (5.16), we have that for each $t\in[0,T]$,

$$
\int_\Omega E^M_\varepsilon(u^N(t))\varepsilon\nabla u^N(t)\cdot\nabla(\phi^N(t)-\phi(t))\,dx\to 0 \quad \text{as } N\to+\infty. \tag{5.17}
$$

Putting (5.15) and (5.17) together, we obtain that

$$
\int_\Omega E^M_\varepsilon(u^N(t))\varepsilon\nabla u^N(t)\cdot\nabla\phi^N(t)\,dx\to\int_\Omega E^M_\varepsilon(u(t))\varepsilon\nabla u(t)\cdot\nabla\phi(t)\,dx \quad \text{as } N\to+\infty
$$

for each $t\in[0,T]$. By Theorem 1.1(iv), we have

$$
\begin{aligned}
\left|\int_\Omega E^M_\varepsilon(u^N(t))\varepsilon\nabla u^N(t)\cdot\nabla\phi^N(t)\,dx\right| &\le \varepsilon M\left(\sup_{t\in[0,T]}\|u^N(t)\|_{H^1_0(\Omega)}\right)\|\phi(t)\|_{H^1_0(\Omega)}\\
&\le \varepsilon M K^*_{\varepsilon,T}\cdot\|\phi\|_{H^1_0(\Omega)}
\end{aligned}
$$

for each $t\in[0,T]$ and each $N\in\mathbb{N}$. Since $\phi\in L^2(0,T;H^1_0(\Omega))$, $\|\phi\|_{H^1_0(\Omega)}$ is integrable on $[0,T]$. Now we can apply Lebesgue's Dominated Convergence Theorem to conclude that (5.9) holds.

Claim 2. For each $\phi\in L^2(0,T;H^1_0(\Omega))$, we have

$$
\int_0^T\int_\Omega E^M_\varepsilon(u^N)\frac{1}{\varepsilon}f(u^N)\phi^N\,dxdt\to\int_0^T\int_\Omega E^M_\varepsilon(u)\frac{1}{\varepsilon}f(u)\phi\,dxdt \quad \text{as } N\to+\infty. \tag{5.18}
$$

By Theorem 1.3(i), for a.e. $t\in[0,T]$, we have

$$
u^N(t)\to u(t) \quad \text{a.e. in } \Omega \text{ as } N\to+\infty. \tag{5.19}
$$

By the continuity of f and (5.19), we have

$$f \circ u^N(t) \to f \circ u(t) \quad \text{a.e. in } \Omega \text{ as } N \to +\infty.$$

Moreover,

$$f \circ u^N(t) \cdot \phi(t) \to f \circ u(t) \cdot \phi(t) \quad \text{a.e. in } \Omega \text{ as } N \to +\infty.$$

Since $|f \circ u^N(t) \cdot \phi(t)| \leq \|f\|_{C^0} \cdot |\phi(t)|$ on Ω, for each $N \in \mathbb{N}$, and $|\phi(t)| \in H_0^1(\Omega) \subset L^1(\Omega)$, we can apply Lebesgue's Dominated Convergence Theorem to obtain

$$\int_\Omega \frac{1}{\varepsilon} f \circ u^N(t) \cdot \phi(t)\, dx \to \int_\Omega \frac{1}{\varepsilon} f \circ u(t) \cdot \phi(t)\, dx \quad \text{as } N \to +\infty \tag{5.20}$$

for a.e. $t \in [0, T]$. On the other hand, for a.e. $t \in [0, T]$,

$$\left| \int_\Omega f \circ u^N(t) \cdot (\phi^N(t) - \phi(t))\, dx \right| \leq |\Omega|^{1/2} \cdot \|f\|_{C^0} \cdot \|\phi^N(t) - \phi(t)\|_{H_0^1(\Omega)}.$$

This, combined with the Fourier Series Theorem, implies that

$$\int_\Omega \frac{1}{\varepsilon} f \circ u^N(t) \cdot (\phi^N(t) - \phi(t))\, dx \to 0 \quad \text{as } N \to +\infty \tag{5.21}$$

for a.e. $t \in [0, T]$. Combining (5.14), (5.20) and (5.21) together yields

$$E_\varepsilon^M(u^N(t)) \int_\Omega \frac{1}{\varepsilon} f \circ u^N(t) \cdot \phi^N(t)\, dx \to E_\varepsilon^M(u(t)) \int_\Omega \frac{1}{\varepsilon} f \circ u(t) \cdot \phi(t)\, dx \quad \text{as } N \to +\infty$$

for a.e. $t \in [0, T]$. Since

$$\left| E_\varepsilon^M(u^N(t)) \int_\Omega \frac{1}{\varepsilon} f \circ u^N(t) \cdot \phi^N(t)\, dx \right| \leq \frac{M \cdot \|f\|_{C^0} \cdot |\Omega|^{1/2}}{\varepsilon} \cdot \|\phi(t)\|_{H_0^1(\Omega)} \quad \text{a.e. on } [0, T]$$

for each $N \in \mathbb{N}$, and $\|\phi\|_{H_0^1(\Omega)} \in L^1[0, T]$, we can apply Lebesgue's Dominated Convergence Theorem to conclude that (5.18) holds.

It follows from (5.1) that for each $N \in \mathbb{N}$,

$$\int_0^T \int_\Omega u_t^N \phi^N = -\int_0^T \int_\Omega E_\varepsilon^M(u^N) \varepsilon \nabla u^N \cdot \nabla \phi^N - \int_0^T \int_\Omega E_\varepsilon^M(u^N) \frac{1}{\varepsilon} f(u^N) \phi^N \tag{5.22}$$

for each $\phi \in L^2(0, T; H_0^1(\Omega))$. Finally, combining (5.22), (5.8), (5.9) and (5.18), we obtain that $u \in L^2(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H^{-1}(\Omega))$ and u satisfies (1.14) for all $\phi \in L^2(0, T; H_0^1(\Omega))$.

Next, claim that $u(0) = u_0$ in $L^2(\Omega)$. Choosing $v \in C^1([0, T]; H_0^1(\Omega))$ with $v(T) = 0$, we have

$$\frac{d}{dt} \langle u^N(t), v(t) \rangle_{L^2(\Omega)} = \langle u_t^N(t), v(t) \rangle_{L^2(\Omega)} + \langle u^N(t), v'(t) \rangle_{L^2(\Omega)},$$

and, as a consequence,

$$\begin{aligned}\int_0^T \langle u_t^N(t), v(t)\rangle_{L^2(\Omega)} dt &= \langle u^N(t), v(t)\rangle_{L^2(\Omega)}\Big|_{t=0}^{t=T} - \int_0^T \langle u^N(t), v'(t)\rangle_{L^2(\Omega)}\, dt \\ &= -\langle u^N(0), v(0)\rangle_{L^2(\Omega)} - \int_0^T \langle u^N(t), v'(t)\rangle_{L^2(\Omega)}\, dt.\end{aligned} \tag{5.23}$$

For each $N \in \mathbb{N}$, we have

$$u^N(0) = \sum_{k=1}^N \langle u^N(0), \phi_k\rangle_{L^2(\Omega)} \phi_k = \sum_{k=1}^N \langle u_0, \phi_k\rangle_{L^2(\Omega)} \phi_k$$

by using (1.13). Here $\{\phi_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ satisfying (1.7). By the Fourier Series Theorem, we find that

$$u^N(0) \to u_0 \quad \text{in } L^2(\Omega) \text{ as } N \to +\infty, \tag{5.24}$$

and we see that

$$\langle u^N(0), v(0)\rangle_{L^2(\Omega)} \to \langle u_0, v(0)\rangle_{L^2(\Omega)} \quad \text{as } N \to +\infty \tag{5.25}$$

by using (5.24) and Hölder's inequality. Using Theorem 1.3(viii), (ix), (5.23) and (5.25), we get

$$\int_0^T \langle u_t(t), v(t)\rangle_{L^2(\Omega)}\, dt = -\langle u_0, v(0)\rangle_{L^2(\Omega)} - \int_0^T \langle u(t), v'(t)\rangle_{L^2(\Omega)}\, dt.$$

By

$$\frac{d}{dt}\langle u(t), v(t)\rangle_{L^2(\Omega)} = \langle u'(t), v(t)\rangle_{L^2(\Omega)} + \langle u(t), v'(t)\rangle_{L^2(\Omega)},$$

we have

$$\langle u_0, v(0)\rangle_{L^2(\Omega)} = \langle u(0), v(0)\rangle_{L^2(\Omega)}$$

for each $v(0) \in H_0^1(\Omega)$. This yields the conclusion of

$$u(0) = u_0 \quad \text{in } L^2(\Omega).$$

This completes the proof of Theorem 1.4.

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