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Blow up Solutions to a System of Higher-order Kirchhoff-type Equations with Positive Initial Energy

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Abstract. In this paper we investigate blow up property of solutions for a system of nonlinear higher order Kirchhoff equations with nonlinear dissipations and positive initial energy. Some estimates for lower bound of the blow up time are also given. This improves and extends the blow up results in [16] by Liu and Wang (2006) and Gao et al. [7] (2011).

1. Introduction

In this paper we are concerned with the following system of higher order Kirchhoff type equations with damping

$$\begin{cases} u_{tt} + M(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)(-\Delta)^{m_1}u + a_1 |u_t|^{q-2} u_t = f_1(u,v) & \text{in } \Omega_T, \\ v_{tt} + M(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)(-\Delta)^{m_2}v + a_2 |v_t|^{r-2} v_t = f_2(u,v) & \text{in } \Omega_T, \end{cases} \tag{1.1}$$

and initial-boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) & \text{in } \Omega, \\ \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,\ldots,m_1-1 & \text{on } \Gamma_T, \\ \frac{\partial^i v}{\partial \nu^i} = 0, \quad i = 0,1,\ldots,m_2-1 & \text{on } \Gamma_T, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$, T is a positive constant, ν represents the unit outward normal on the boundary, $\Omega_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$, $m_i \geq 1$ ($i = 1,2$) are positive integers, $q, r \geq 2$, $a_i > 0$ ($i = 1,2$) are positive constants, M is a locally Lipschitz function which satisfies in some conditions (to be specified later). The functions $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\begin{aligned} f_1(u,v) &= a\,|u+v|^{2(p-1)}\,(u+v) + b\,|u|^{p-2}\,u\,|v|^p, \\ f_2(u,v) &= a\,|u+v|^{2(p-1)}\,(u+v) + b\,|v|^{p-2}\,v\,|u|^p, \end{aligned} \tag{1.3}$$

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which satisfy

$$uf_1(u,v) + vf_2(u,v) = 2pF(u,v), \quad \forall\,(u,v) \in \mathbb{R}^2,$$

where $a, b > 0$, $p > 1$ and

$$F(u,v) = \frac{a}{2p}\,|u+v|^{2p} + \frac{b}{p}\,|uv|^p\,.$$

One can easily verify that $\partial_u F = f_1$ and $\partial_v F = f_2$.

Consider a problem of a single wave equation of the form

$$\begin{aligned} & u_{tt} + M(\|D^m\|_2^2)(-\Delta)^m u + \delta\,|u_t|^{q-2}\,u_t = \mu\,|u|^{p-2}\,u, && t \ge 0,\ x \in \Omega, \\ & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x \in \Omega, \\ & \frac{\partial^i u}{\partial \nu^i}(x,t) = 0, \quad i = 0,1,\ldots,m-1, && t \ge 0,\ x \in \partial\Omega, \end{aligned} \tag{1.4}$$

where $\delta, \mu > 0$, $p > 2$, $q \ge 2$ and $m \ge 1$. When $M = 1$ and $m = 1$, (1.4) has been investigated by many authors. In [12] Levine showed the nonexistence of solutions in presence the linear damping case ($q = 2$). Gorgiev and Todorva [8] extended this result to nonlinear damping case ($p > q > 2$) where initial energy is negative. Ikehata in [10] considered (1.4) when $q = 2$ and obtained blow up result with small positive initial energy in some sense. Later, Levin and Todorva [14] proved that the solutions can not exist globally if $p > q \ge 2$ and the initial energy is positive. In connecting with global nonexistence and blow up of solutions we refer to the studies [3, 13, 20, 26, 30] and the references cited in this works. In the case $M = 1$ and $m = 2$ the problem (1.4) deals whit Petrovsky wave equations. In this regard we may also recall the works by Komornik [11], Guesmia [9], Wu and Tsai [29], Messaoudi [18], Chen and Zhou [6] and the references therein.

For the case $M \neq 1$ the equation in (1.4) converts to a Kirchhoff type. Matsuyama and Ikehata in [17] considered (1.4) with $m = 1$ when M is a C^1-class function for $s \ge 0$ and $M(s) \ge m_0 > 0$. They obtained a global solvability in the class $H^2 \times H_0^1$ and energy decay. In the same time, Ono [21] obtained the global existence and decay properties of solutions when $q = 2$ with

$$M(s) = a + bs^\gamma, \quad a \ge 0,\ b \ge 0,\ a + b > 0,\ \gamma \ge 1, \tag{1.5}$$

for degenerate ($a = 0$) and non-degenerate ($a > 0$) equations. In the immediate work by Ono [22] we can see global existence, decay and blow up of solutions for the nonlinear damping case $q > 2$ and $a > 0$. In this regard, we may also mention to some other works by Benaissa and Messaoudi [4, 5] and Ono [23].

When $m > 1$, Li [15] considered (1.4) with $M(s) = s^\gamma$, $\gamma > 0$ and proved that the solution exists globally if $p \le q$ while if $p > \max\{q, 2\gamma\}$, then for any initial data with negative initial energy, the solution blows up at finite time. Later Messaoudi and Houari in [19] improved this result and showed that under some considerations on initial data solutions also blow up in finite time with positive initial energy. Recently, Gao et. al [7] improved some results in the literature [6, 15, 18] and obtained local existence and blow up of solutions where M is a locally Lipschitz function satisfying some conditions. More recently, for $M(s) = s^{2\gamma}$, $\gamma > 0$, Ye in [31] by constructing stable set in H_0^m showed that the solutions exists globally in time if $p \leq q$ and proved the global nonexistence under some consideration on initial data when $p > 2(\gamma + 1)$.

In connecting with the systems of wave equations of Kirchhoff type Park and Bae [24, 25] investigated the existence of solutions for (1.1)–(1.2) with $m_1 = m_2 = 1$ in degenerate case $M(s) = s^\gamma$, $\gamma > 1$ and non-degenerate case (1.5). Later, with $\gamma = 1$ in (1.5), Liu [16] obtained global existence for nonlinear damping terms and proved blow up results in linear damping case ($q = r = 2$) for some class of sources. In an other work, when $m_1 = m_2 = 1$ and nonlinear damping terms in (1.1) are replaced with strong damping terms, Wu in [27] proved that the local solution blows up in finite time by applying concave method. Very recently, Ye [32] considered the problem (1.1)–(1.2) and proved decay and global existence of solutions in $H_0^{m_1} \times H_0^{m_2}$ where M is a locally Lipschitz function such as $M(s) = a + bs^\gamma$ with the source terms defined in (1.3). However, blow up properties has been not considered. Our main aim in this paper is to investigate blow up properties for the solutions of (1.1)–(1.2). More precisely, for a locally Lipschitz function M, we prove that the L^2 norm of solutions $(\|u\|_2^2 + \|v\|_2^2)$ blows up at a finite time $T^\star > 0$. This extends and improves some results in the literature such as the one in [7] in which the blow up result obtained only for a single higher order Kirchhoff type wave equation and the nonexistence results in [16] for $q, r \geq 2$, $m_1, m_2 \geq 1$ and more general M. Some estimates for lower bounds of the blow up time are also given.

This paper is organized as follows: In Section 2 we give some preliminary materials needed throughout our proofs. In Section 3 we prove a local existence result (Theorem 2.3). In Section 4 we state and prove our main result on the blow up of solutions. In Section 5 we obtain lower bounds for the blow up time.

2. Preliminaries

In this section we present some notations, assumptions and lemmas needed for our work. First of all we state the following Sobolev-Poincaré inequality which will be used frequently throughout our proofs.

Lemma 2.1. (Sobolev-Poincaré inequality [1]) *Let $2 \leq s \leq 2N/(N-2k)$ if $N > 2k$ and $2 \leq s < +\infty$ if $N \leq 2k$. Then there exists a constant B depending only on Ω, N, k and s*

such that

$$\|u\|_s \leq B \left\|(-\Delta)^{k/2} u\right\|_2$$

holds for all $u \in H_0^k(\Omega)$.

In order to obtain our results we consider the following assumptions on the problem (1.1)–(1.2):

(H$_1$) $M \in C^1([0,+\infty),\mathbb{R})$ is a locally Lipschitz function satisfying

$$M(\tau) \geq m_0, \quad \mathcal{M}(\tau) \geq \tau M(\tau), \quad \forall\, \tau \in \mathbb{R}_+, \tag{2.1}$$

where m_0 is a positive constant and $\mathcal{M}(\tau) = \int_0^\tau M(s)\,ds$.

(H$_2$) $q, r \geq 2$, $m_i \geq 1$ $(i = 1,2)$ and

$$\begin{aligned} &1 < p < +\infty, && N \leq 2\min\{m_1, m_2\}, \\ &1 < p \leq \min\left\{\frac{N}{2(N-2m_1)}, \frac{N}{2(N-2m_2)}\right\}, && N > 2\max\{m_1, m_2\}. \end{aligned}$$

(H$_3$) $u_0 \in H_0^{m_1}(\Omega) \cap H^{2m_1}(\Omega)$, $v_0 \in H_0^{m_2}(\Omega) \cap H^{2m_2}(\Omega)$, $u_1, v_1 \in L^2(\Omega)$.

(H$_4$) There exist two positive constants c_0 and c_1 such that

$$c_0(|u|^{2p} + |v|^{2p}) \leq 2pF(u,v) \leq c_1(|u|^{2p} + |v|^{2p}). \tag{2.2}$$

Next, same as in [32], we define the following functionals on $H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$:

$$E(t) = E(u,v) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) + J(u,v), \tag{2.3}$$

$$J(t) = J(u,v) = \frac{1}{2}\mathcal{M}(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2) - \int_\Omega F(u,v)\,dx,$$

$$K(t) = K(u,v) = \mathcal{M}(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2) - 2p\int_\Omega F(u,v)\,dx. \tag{2.4}$$

Lemma 2.2. *Let* (u,v) *be a solution of* (1.1)–(1.2) *and* (H$_3$) *holds. Then* $E(t)$ *is a non-increasing function for* $t > 0$ *and*

$$E(t) - E(0) = -a_1 \int_0^t\int_\Omega |u_t(s)|^q\,dxds - a_2\int_0^t\int_\Omega |v_t(s)|^r\,dxds. \tag{2.5}$$

Proof. Multiplying the first equation in (1.1) by u_t and the second one by v_t, integrating over Ω and using the initial-boundary conditions (1.2) we obtain (2.5). $\square$

Local existence result associated to (1.1)–(1.2) can be established by combining the arguments in [2,7,8,18,21,22]. However, we give a proof of the following result in Section 3.

Theorem 2.3. *Suppose that the assumptions* (H_1)–(H_4) *hold. Then there exists a unique local solution* (u,v) *of* (1.1)–(1.2) *in the class*

$$u \in C([0,T), H_0^{m_1}(\Omega)), \quad v \in C([0,T), H_0^{m_2}(\Omega)),$$
$$u_t \in C([0,T), L^2(\Omega)) \cap L^q(\Omega \times [0,T)), \quad v_t \in C([0,T), L^2(\Omega)) \cap L^r(\Omega \times [0,T))$$

for some $T > 0$.

Consider the space

$$\begin{aligned}\mathcal{W}_T = \big\{(u,v) : \, & u \in C([0,T), H_0^{m_1}(\Omega) \cap H^{2m_1}(\Omega)),\\ & v \in C([0,T), H_0^{m_2}(\Omega) \cap H^{2m_2}(\Omega)),\\ & u_t \in C([0,T), L^2(\Omega)) \cap L^q(\Omega \times [0,T)),\\ & v_t \in C([0,T), L^2(\Omega)) \cap L^r(\Omega \times [0,T))\big\},\end{aligned}$$

with the norm

$$\begin{aligned}\|(u,v)\|_{\mathcal{W}_T}^2 = \max_{0\le t\le T} & \left(\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2\right)\\ & + \|u_t\|_{L^q(\Omega\times[0,T))}^2 + \|v_t\|_{L^r(\Omega\times[0,T))}^2 .\end{aligned}$$

Definition 2.4. Let the assumptions (H_1)–(H_4) hold, (u,v) be a solution of (1.1)–(1.2) and

$$T^\star = \sup\left\{T > 0 : (u,v) \in \mathcal{W}_T \text{ exists on } [0,T)\right\}.$$

If $T^\star = +\infty$ then we say that the solution of (1.1)–(1.2) exists globally and if $T^\star < +\infty$ we say that the solutions blow up at the finite time $T^\star$ in the sense

$$\|u_t\|_2^2 + \|v_t\|_2^2 + \|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2 \to +\infty \quad \text{as } t \to {T^\star}^- .$$

Remark 2.5. In the case $T^\star = +\infty$ and under the hypotheses (H_1)–(H_4) the problem (1.1)–(1.2) has been investigated in [32].

3. Local existence

First, note that in what follows C_i are various positive constants which may be different at different occurrences. To prove the Theorem 2.3 we first state the following lemma which can be obtained by exploiting the Faedo-Galerkin method and using the similar arguments as in [1, 28]:

Lemma 3.1. *Suppose that $(u_0,u_1)\in H^{2m}(\Omega)\cap H_0^m(\Omega)\times L^2(\Omega)$, then there exists a unique solution u of*

$$\begin{cases} u_{tt}+M(t)(-\Delta)^m u + aQ_r(u_t)=f(x,t), & (x,t)\in\Omega\times[0,T],\\ u(0)=u_0, \quad u_t(0)=u_1, & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0, \end{cases}$$

satisfying

$$u\in C([0,T],H^{2m}(\Omega)\cap H_0^m(\Omega)) \quad \textit{and} \quad u_t\in C([0,T],L^2(\Omega))\cap L^r(\Omega\times[0,T]),$$

where $a>0$, $m\geq 1$, M is a positive locally Lipschitz function, $Q_r(z)=|z|^{r-2}z$ $(r>2)$ and $f\in H^1([0,T],L^2(\Omega))$.

Similar as in [22, 27], for $R>0$ and $T>0$ we define

$$X_{T,R}=\left\{(u,v)\in\mathcal{W}_T : e(u,v)\leq R^2,\ u,\ v \text{ satisfy the initial conditions in (1.2)}\right\},$$

where

$$\begin{aligned} e(u,v) &= \|u_t\|_2^2+\|v_t\|_2^2+\|D^{m_1}u_t\|_2^2+\|D^{m_2}v_t\|_2^2\\ &\quad+\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2+\|(-\Delta)^{m_1}u\|_2^2+\|(-\Delta)^{m_2}v\|_2^2. \end{aligned}$$

Then, $X_{T,R}$ is a complete metric space with the distance

$$\begin{aligned} &d(w_1,w_2)\\ &=\sup_{0\leq t\leq T}\left(\|(u_1-v_1)_t\|_2^2+\|D^{m_1}(u_1-v_1)\|_2^2+\|(u_2-v_2)_t\|_2^2+\|D^{m_2}(u_2-v_2)\|_2^2\right)^{1/2}, \end{aligned}$$

where $w_1=(u_1,u_2), w_2=(v_1,v_2)\in X_{T,R}$. Next, for $(\widehat{u},\widehat{v})\in X_{T,R}$ we consider the following system

$$\begin{cases} u_{tt}+M(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)(-\Delta)^{m_1}u+a_1Q_q(u_t)=f_1(\widehat{u},\widehat{v}),\\ v_{tt}+M(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)(-\Delta)^{m_2}v+a_2Q_r(v_t)=f_2(\widehat{u},\widehat{v}), \end{cases} \tag{3.1}$$

with initial and boundary conditions (1.2). By Lemma 3.1 this problem has a unique solution (u,v). We define a nonlinear mapping Ψ in the following way: For $(\widehat{u},\widehat{v})\in X_{T,R}$, $(u,v)=\Psi(\widehat{u},\widehat{v})$ is the unique solution of the problem (1.1)–(1.2). We show that there exists $T>0$ and $R>0$ such that Ψ maps $X_{T,R}$ into itself and Ψ is a contraction mapping in $X_{T,R}$ with respect to the metric $d(\cdot,\cdot)$.

For simplicity in computations we let $a_1=a_2=1$. Multiplying the first equation in (3.1) by u_t, the second by v_t, integrating over Ω and summing up the results with together we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_t\|_2^2+\|v_t\|_2^2+M(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)\right)\\
&\quad+2\left\langle u_t,Q_q(u_t)\right\rangle+2\left\langle v_t,Q_r(v_t)\right\rangle\\
&=2\left\langle u_t,f_1(\widehat{u},\widehat{v})\right\rangle+2\left\langle v_t,f_2(\widehat{u},\widehat{v})\right\rangle\\
&\quad+2\left(\left\langle D^{m_1}\widehat{u},D^{m_1}\widehat{u_t}\right\rangle+\left\langle D^{m_2}\widehat{v},D^{m_2}\widehat{v_t}\right\rangle\right)M'(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)\\
&\qquad\times(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2).
\end{aligned}
\tag{3.2}
$$

For the first term on the right-hand side of (3.2), by using Hölder's inequality, (H_2), Lemma 2.1 and using the same way followed in [2] we have

$$
\begin{aligned}
\int_\Omega u_t f_1(\widehat{u},\widehat{v})\,dx &\le C_1\left(\|\widehat{u}\|_{4p-2}^{4p-2}+\|\widehat{v}\|_{4p-2}^{4p-2}+\|\widehat{u}\|_{4p-4}^{2p-2}\|\widehat{v}\|_{4p}^{2p}\right)^{1/2}\|u_t\|_2\\
&\le C_2\left(\|D^{m_1}\widehat{u}\|_2^{2p-1}+\|D^{m_2}\widehat{v}\|_2^{2p-1}+\|D^{m_1}\widehat{u}\|_2^{p-1}\|D^{m_2}\widehat{v}\|_2^{p}\right)\|u_t\|_2\\
&\le 3C_2R^{2p-1}\|u_t\|_2\,.
\end{aligned}
\tag{3.3}
$$

Similarly,

$$
\int_\Omega v_t f_2(\widehat{u},\widehat{v})\,dx\le 3C_3R^{2p-1}\|v_t\|_2\,.
\tag{3.4}
$$

Also, by using Young's inequality we have

$$
\begin{aligned}
\left\langle D^{m_1}\widehat{u},D^{m_1}\widehat{u_t}\right\rangle+\left\langle D^{m_2}\widehat{v},D^{m_2}\widehat{v_t}\right\rangle &\le \|D^{m_1}\widehat{u}\|_2\|D^{m_1}\widehat{u_t}\|_2+\|D^{m_2}\widehat{v}\|_2\|D^{m_2}\widehat{v_t}\|_2\\
&\le 2R^2.
\end{aligned}
\tag{3.5}
$$

Letting $M_0'=\sup_{0\le s\le R^2}|M'(s)|$, using (H_1) and (3.2)–(3.5), by integrating over $(0,t)$ we get

$$
\begin{aligned}
&\|u_t\|_2^2+\|v_t\|_2^2+\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2\\
&\quad+2\widehat{m}_0\int_0^t\left(\left\langle u_t(s),Q_q(u_t(s))\right\rangle+\left\langle v_t(s),Q_r(v_t(s))\right\rangle\right)ds\\
&\le L_1+12C_4\widehat{m}_0R^{2p-1}\int_0^t(\|u_t(s)\|_2+\|v_t(s)\|_2)\,ds\\
&\quad+4R^2\widehat{m}_0M_0'\int_0^t(\|D^{m_1}u(s)\|_2^2+\|D^{m_2}v(s)\|_2^2)\,ds,
\end{aligned}
\tag{3.6}
$$

where $\widehat{m}_0=(\min\{1,m_0\})^{-1}$, $C_4=\max\{C_2,C_3\}$ and

$$
L_1=\widehat{m}_0\left(\|u_1\|_2^2+\|v_1\|_2^2+M(\|D^{m_1}\widehat{u_0}\|_2^2+\|D^{m_2}\widehat{v_0}\|_2^2)\,\|D^{m_1}u_0\|_2^2+\|D^{m_2}v_0\|_2^2\right).
$$

Multiplying first equation in (3.1) by $(-\Delta)^{m_1}u_t$, the second by $(-\Delta)^{m_2}v_t$, integrating over

Ω and summing up the results we gain

$$\begin{aligned}
&\frac{d}{dt}\left(\|D^{m_1}u_t\|_2^2+\|D^{m_2}v_t\|_2^2+M(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)(\|(-\Delta)^{m_1}u\|_2^2+\|(-\Delta)^{m_2}v\|_2^2)\right)\\
&\quad+2\left\langle Q_q(u_t),(-\Delta)^{m_1}u_t\right\rangle+2\left\langle Q_r(v_t),(-\Delta)^{m_2}v_t\right\rangle\\
&=2\left\langle f_1(\widehat{u},\widehat{v}),(-\Delta)^{m_1}u_t\right\rangle+2\left\langle f_2(\widehat{u},\widehat{v}),(-\Delta)^{m_2}v_t\right\rangle\\
&\quad+2\left(\langle D^{m_1}\widehat{u},D^{m_1}\widehat{u_t}\rangle+\langle D^{m_2}\widehat{v},D^{m_2}\widehat{v_t}\rangle\right)M'(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)\\
&\qquad\times(\|(-\Delta)^{m_1}u\|_2^2+\|(-\Delta)^{m_2}v\|_2^2).
\end{aligned}$$

Integrating over $(0,t)$, using (3.5) and (H_1) we obtain

$$\begin{aligned}
&\|D^{m_1}u_t\|_2^2+\|D^{m_2}v_t\|_2^2+M(\|D^{m_1}\widehat{u}\|_2^2+\|D^{m_2}\widehat{v}\|_2^2)(\|(-\Delta)^{m_1}u\|_2^2+\|(-\Delta)^{m_2}v\|_2^2)\\
&\quad+2\int_0^t\left\langle Q_q(u_t(s)),(-\Delta)^{m_1}u_t(s)\right\rangle ds+2\int_0^t\left\langle Q_r(v_t(s)),(-\Delta)^{m_2}v_t(s)\right\rangle ds\\
&\le L_2+2\int_0^t\left\langle f_1(\widehat{u}(s),\widehat{v}(s)),(-\Delta)^{m_1}u_t(s)\right\rangle ds+2\int_0^t\left\langle f_2(\widehat{u}(s),\widehat{v}(s)),(-\Delta)^{m_2}v_t(s)\right\rangle ds\\
&\quad+4R^2M_0'\int_0^t(\|(-\Delta)^{m_1}u(s)\|_2^2+\|(-\Delta)^{m_2}v(s)\|_2^2)\,ds,
\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}
L_2&=\|D^{m_1}u_1\|_2^2+\|D^{m_2}v_1\|_2^2\\
&\quad+M(\|D^{m_1}\widehat{u}_0\|_2^2+\|D^{m_2}\widehat{v}_0\|_2^2)(\|(-\Delta)^{m_1}u_0\|_2^2+\|(-\Delta)^{m_2}v_0\|_2^2).
\end{aligned}$$

For the second term on the right-hand side of (3.7), using integration by parts, we have

$$\begin{aligned}
&\int_0^t\int_\Omega f_1(\widehat{u}(s),\widehat{v}(s))(-\Delta)^{m_1}u_t(s)\,dxds\\
&=\int_\Omega f_1(\widehat{u},\widehat{v})(-\Delta)^{m_1}u\,dx-\int_\Omega f_1(\widehat{u}_0,\widehat{v}_0)(-\Delta)^{m_1}u_0\,dx\\
&\quad-\int_0^t\int_\Omega\left(\frac{\partial f_1}{\partial u}(\widehat{u}(s),\widehat{v}(s))\widehat{u}_t(s)+\frac{\partial f_1}{\partial v}(\widehat{u}(s),\widehat{v}(s))\widehat{v}_t(s)\right)(-\Delta)^{m_1}u(s)\,dxds\\
&=I_1+I_2+I_3.
\end{aligned}\tag{3.8}$$

By Young's inequality and Hölder's inequality, we then get

$$I_1\le\varepsilon\|(-\Delta)^{m_1}u\|_2^2+\frac{1}{4\varepsilon}\|f_1(\widehat{u},\widehat{v})\|_2^2,\tag{3.9}$$

$$I_2\le\|f_1(\widehat{u}_0,\widehat{v}_0)\|_2\|(-\Delta)^{m_1}u_0\|_2,\tag{3.10}$$

$$I_3\le\int_0^t\left(\left\|\frac{\partial f_1}{\partial u}(\widehat{u}(s),\widehat{v}(s))\widehat{u}_t(s)\right\|_2+\left\|\frac{\partial f_1}{\partial v}(\widehat{u}(s),\widehat{v}(s))\widehat{v}_t(s)\right\|_2\right)\|(-\Delta)^{m_1}u(s)\|_2\,ds.\tag{3.11}$$

To estimate the terms in (3.11), without lose of generality we suppose that $m_1 \geq m_2$. Then, by (H_2) and Lemma 2.1 we have

$$
\begin{aligned}
&\left\| \frac{\partial f_1}{\partial u}(\widehat{u},\widehat{v})\widehat{u}_t \right\|_2 \\
&\leq C_5 \left[\int_\Omega \left(|\widehat{u}+\widehat{v}|^{4(p-1)} + |\widehat{u}|^{2(p-2)} |\widehat{v}|^{2p} \right) (\widehat{u}_t)^2 \, dx \right]^{1/2} \\
&\leq C_6 \left[\int_\Omega \left(|\widehat{u}|^{4(p-1)} + |\widehat{v}|^{4(p-1)} + |\widehat{u}|^{4(p-2)} + |\widehat{v}|^{4p} \right) (\widehat{u}_t)^2 \, dx \right]^{1/2} \\
&\leq C_6 \Big[\left(\|\widehat{u}\|^{4(p-1)}_{4(p-1)N/m_1} + \|\widehat{u}\|^{4(p-2)}_{4(p-2)N/m_1} \right) \|\widehat{u}_t\|^2_{2N/(N-m_1)} \\
&\qquad + \left(\|\widehat{v}\|^{4(p-1)}_{4(p-1)N/m_2} + \|\widehat{v}\|^{4p}_{4pN/m_2} \right) \|\widehat{u}_t\|^2_{2N/(N-m_2)} \Big]^{1/2} \\
&\leq C_7 \left(\|D^{m_1}\widehat{u}\|_2^{2(p-1)} + \|D^{m_1}\widehat{u}\|_2^{2(p-2)} + \|D^{m_2}\widehat{v}\|_2^{2(p-1)} + \|D^{m_1}\widehat{u}\|_2^{2p} \right) \|D^{m_1}\widehat{u}_t\|_2 \\
&\leq C_7 \left(R^{2(p-2)} + 2R^{2(p-1)} + R^{2p} \right) R,
\end{aligned} \tag{3.12}
$$

where we have used $2N/(N-m_2) \leq 2N/(N-m_1)$. We also have

$$
\begin{aligned}
\left\| \frac{\partial f_1}{\partial v}(\widehat{u},\widehat{v})\widehat{v}_t \right\|_2 &\leq C_8 \left[\int_\Omega \left(|\widehat{u}+\widehat{v}|^{4(p-1)} + |\widehat{u}|^{2(p-1)} |\widehat{v}|^{2(p-1)} \right) (\widehat{v}_t)^2 \, dx \right]^{1/2} \\
&\leq C_9 \left[\int_\Omega \left(|\widehat{u}|^{4(p-1)} + |\widehat{v}|^{4(p-1)} \right) (\widehat{v}_t)^2 \, dx \right]^{1/2} \\
&\leq C_9 \left(\|\widehat{u}\|^{4(p-1)}_{4(p-1)N/m_2} + \|\widehat{v}\|^{4(p-1)}_{4(p-1)N/m_2} \right)^{1/2} \|\widehat{v}_t\|_{2N/(N-m_2)} \\
&\leq C_{10} \left(\|D^{m_2}\widehat{u}\|_2^{2(p-1)} + \|D^{m_2}\widehat{v}\|_2^{2(p-1)} \right) \|D^{m_2}\widehat{v}_t\|_2 .
\end{aligned} \tag{3.13}
$$

For the first term on the right-hand side of the last inequality in (3.13) we have

$$
\begin{aligned}
\|D^{m_2}\widehat{u}\|_2^2 = \int_\Omega \widehat{u}(-\Delta)^{m_2}\widehat{u} \, dx &\leq \|\widehat{u}\|_2 \|(-\Delta)^{m_2}\widehat{u}\|_2 \\
&\leq B \|D^{m_1}\widehat{u}\|_2 \|(-\Delta)^{m_2}\widehat{u}\|_2 \leq \widehat{B}R^2,
\end{aligned} \tag{3.14}
$$

where $\widehat{B}$ depends on B and Ω. Therefore, by (3.13) and (3.14) we get

$$
\left\| \frac{\partial f_1}{\partial v}(\widehat{u},\widehat{v})\widehat{v}_t \right\|_2 \leq C_{11} R^{2(p-1)} R. \tag{3.15}
$$

Thus, by (3.12) and (3.15) we get

$$
I_3 \leq C_{12} C(R) \int_0^t \|(-\Delta)^{m_1} u(s)\|_2 \, ds, \tag{3.16}
$$

where $C(R) = (R^{2(p-2)} + R^{2(p-1)} + R^{2p})R$. By similar way followed in (3.8)–(3.13), using again (H_2) and considering (3.14), we can see

$$\begin{aligned}
&\int_0^t\int_\Omega f_2(\widehat{u}(s),\widehat{v}(s))(-\Delta)^{m_2}v_t(s)\,dxds \\
&\le \varepsilon\left\|(-\Delta)^{m_2}v\right\|_2^2+\frac{1}{4\varepsilon}\left\|f_2(\widehat{u},\widehat{v})\right\|_2^2+\left\|f_2(\widehat{u}_0,\widehat{v}_0)\right\|_2\left\|(-\Delta)^{m_2}v_0\right\|_2 \\
&\quad+C_{13}C(R)\int_0^t\left\|(-\Delta)^{m_2}v(s)\right\|_2 ds.
\end{aligned}\tag{3.17}$$

Therefore, by (3.7)–(3.10), (3.16) and (3.17), using similar argument as in [7] for nonlinear damping terms and taking (3.3) into account, for $\varepsilon=m_0/2$, we get

$$\begin{aligned}
e(u,v)&\le L_1+\check{m}_0L_2+L(R) \\
&\quad+\left(12C_4\widehat{m}_0R^{2p-1}+2(C_{12}+C_{13})\check{m}_0C(R)\right)\int_0^t e^{1/2}(u(s),v(s))\,ds \\
&\quad+4R^2\widehat{m}_0M_0'\int_0^t e(u(s),v(s))\,ds,
\end{aligned}\tag{3.18}$$

where $\check{m}_0=(\min\{1,m_0/2\})^{-1}$ and

$$\begin{aligned}
L(R)=\frac{9\check{m}_0(C_2^2+C_3^2)R^{2(2p-1)}}{\varepsilon}&+2\left\|f_1(\widehat{u}_0,\widehat{v}_0)\right\|_2\left\|(-\Delta)^{m_1}u_0\right\|_2 \\
&+2\left\|f_2(\widehat{u}_0,\widehat{v}_0)\right\|_2\left\|(-\Delta)^{m_2}v_0\right\|_2.
\end{aligned}$$

Then, by (3.18) we get

$$e(u,v)\le\xi(u_0,v_0,\widehat{u}_0,\widehat{v}_0,u_1,v_1,R)^2e^{4R^2\widehat{m}_0M_0'T},\quad\forall\,t\in(0,T],\tag{3.19}$$

where

$$\begin{aligned}
\xi(u_0,v_0,\widehat{u}_0,\widehat{v}_0,u_1,v_1,R)&=\sqrt{L_1+\check{m}_0L_2+L(R)} \\
&\qquad+\frac{12C_4\widehat{m}_0R^{2p-1}+2(C_{12}+C_{13})\check{m}_0C(R)}{4R^2\widehat{m}_0M_0'}.
\end{aligned}$$

If T and R satisfy $\xi(u_0,v_0,\widehat{u}_0,\widehat{v}_0,u_1,v_1,R)^2e^{4R^2\widehat{m}_0M_0'T}\le R^2$, then we have $e(u,v)\le R^2$. Thus, the solution (u,v) satisfies the regularities described in $\mathcal{W}_T$. Specifically, by Lemma 3.1, (3.6) and (3.19) it follows that $u_t\in C([0,T],L^2(\Omega))\cap L^q(\Omega\times[0,T])$ and $v_t\in C([0,T],L^2(\Omega))\cap L^r(\Omega\times[0,T])$. Hence, Ψ maps $X_{T,R}$ into itself. Next, we show that Ψ is a contraction mapping with respect to $d(\cdot\,,\cdot)$.

Assume that $(\widehat{u}_1,\widehat{v}_1),(\widehat{u}_2,\widehat{v}_2)\in X_{T,R}$. Let (u_1,v_1) and (u_2,v_2) be two solutions of (3.1)–(1.2) in $X_{T,R}$. Suppose that $w=(w_1,w_2)$, where $w_1=u_1-u_2$, $w_2=v_1-v_2$. We then have

$$\begin{aligned}
&(w_1)_{tt}+M(\|D^{m_1}\widehat{u}_1\|_2^2+\|D^{m_2}\widehat{v}_1\|_2^2)(-\Delta)^{m_1}w_1 \\
&\quad+\left[M(\|D^{m_1}\widehat{u}_1\|_2^2+\|D^{m_2}\widehat{v}_1\|_2^2)-M(\|D^{m_1}\widehat{u}_2\|_2^2+\|D^{m_2}\widehat{v}_2\|_2^2)\right](-\Delta)^{m_1}u_2 \\
&\quad+Q_q((u_1)_t)-Q_q((u_2)_t) \\
&=f_1(\widehat{u}_1,\widehat{v}_1)-f_1(\widehat{u}_2,\widehat{v}_2)
\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}(w_2)_{tt} &+ M(\|D^{m_1}\widehat{u}_1\|_2^2 + \|D^{m_2}\widehat{v}_1\|_2^2)(-\Delta)^{m_2}w_2 \\ &+ \left[M(\|D^{m_1}\widehat{u}_1\|_2^2 + \|D^{m_2}\widehat{v}_1\|_2^2) - M(\|D^{m_1}\widehat{u}_2\|_2^2 + \|D^{m_2}\widehat{v}_2\|_2^2)\right](-\Delta)^{m_2}v_2 \\ &+ Q_r((v_1)_t) - Q_r((v_2)_t) \\ =\ & f_2(\widehat{u}_1,\widehat{v}_1) - f_2(\widehat{u}_2,\widehat{v}_2),\end{aligned} \tag{3.21}$$

with the initial conditions

$$w_1(0) = (w_1)_t(0) = 0, \quad w_2(0) = (w_2)_t(0) = 0. \tag{3.22}$$

Multiplying (3.20) by $(w_1)_t$ and then integrating over Ω we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|(w_1)_t\|_2^2 + M(\|D^{m_1}\widehat{u}_1\|_2^2 + \|D^{m_2}\widehat{v}_1\|_2^2)\,\|D^{m_1}w_1\|_2^2\right) \\ &+ \langle Q_q((u_1)_t) - Q_q((u_2)_t), (w_1)_t\rangle \\ =\ &\left[M(\|D^{m_1}\widehat{u}_2\|_2^2 + \|D^{m_2}\widehat{v}_2\|_2^2) - M(\|D^{m_1}\widehat{u}_1\|_2^2 + \|D^{m_2}\widehat{v}_1\|_2^2)\right]\langle(-\Delta)^{m_1}u_2, (w_1)_t\rangle \\ &+ \frac{1}{2}\frac{d}{dt}M(\|D^{m_1}\widehat{u}_1\|_2^2 + \|D^{m_2}\widehat{v}_1\|_2^2)\,\|D^{m_1}w_1\|_2^2 + \langle f_1(\widehat{u}_1,\widehat{v}_1) - f_1(\widehat{u}_2,\widehat{v}_2), (w_1)_t\rangle \\ =\ &J_1 + J_2 + J_3.\end{aligned} \tag{3.23}$$

We have

$$\begin{aligned}J_1 &\le L\,[(\|D^{m_1}\widehat{u}_2\|_2 - \|D^{m_1}\widehat{u}_1\|_2)(\|D^{m_1}\widehat{u}_2\|_2 + \|D^{m_1}\widehat{u}_1\|_2) \\ &\qquad + (\|D^{m_2}\widehat{v}_2\|_2 - \|D^{m_2}\widehat{v}_1\|_2)(\|D^{m_2}\widehat{v}_2\|_2 + \|D^{m_2}\widehat{v}_1\|_2)]\,\|(-\Delta)^{m_1}u_2\|_2\,\|(w_1)_t\|_2 \\ &\le 4RL(\|D^{m_1}\widehat{u}_1 - D^{m_1}\widehat{u}_2\|_2 + \|D^{m_2}\widehat{v}_1 - D^{m_2}\widehat{v}_2\|_2)\,\|(-\Delta)^{m_1}u_2\|_2\,\|(w_1)_t\|_2 \\ &\le 4R^2L\widetilde{e}^{1/2}(\widehat{u}_1 - \widehat{u}_2, \widehat{v}_1 - \widehat{v}_2)\widetilde{e}^{1/2}(w_1, w_2),\end{aligned} \tag{3.24}$$

where L is the Lipschits constant of M in $[0, R]$ and

$$\widetilde{e}(z_1, z_2) = \|(z_1)_t\|_2^2 + \|(z_2)_t\|_2^2 + \|D^{m_1}z_1\|_2^2 + \|D^{m_2}z_2\|_2^2\,.$$

Using (3.5) we have

$$J_2 \le 2R^2M_0'\widetilde{e}(w_1, w_2). \tag{3.25}$$

To estimate J_3 first, from the relations (1.9) and (1.10) in [2], we have

$$\begin{aligned}&|f_1(\widehat{u}_1,\widehat{v}_1) - f_1(\widehat{u}_2,\widehat{v}_2)| \\ &\le C_{14}(|\widehat{u}_1 - \widehat{u}_2| + |\widehat{v}_1 - \widehat{v}_2|)\left(|\widehat{u}_1|^{2(p-1)} + |\widehat{v}_1|^{2(p-1)} + |\widehat{u}_2|^{2(p-1)} + |\widehat{v}_2|^{2(p-1)}\right) \\ &\quad + C_{15}\left[|\widehat{u}_1 - \widehat{u}_2|\,|\widehat{v}_1|^p\,(|\widehat{u}_1|^{p-1} + |\widehat{u}_2|^{p-1}) + |\widehat{v}_1 - \widehat{v}_2|\,|\widehat{u}_2|^p\,(|\widehat{v}_1|^{p-1} + |\widehat{v}_2|^{p-1})\right].\end{aligned} \tag{3.26}$$

Then, as a typical estimate, we have

$$
\begin{aligned}
&\int_\Omega |\widehat{u}_1-\widehat{u}_2|\,|\widehat{u}_1|^{2(p-1)}\,|(w_1)_t|\,dx \\
&\le \|\widehat{u}_1-\widehat{u}_2\|_{2N/(N-m_1)}\,\|\widehat{u}_1\|_{4(p-1)N/m_1}^{2(p-1)}\,\|(w_1)_t\|_2 \\
&\le B^{2p-1}\,\|D^{m_1}(\widehat{u}_1-\widehat{u}_2)\|_2\,\|D^{m_1}\widehat{u}_1\|_2^{2(p-1)}\,\|(w_1)_t\|_2 \\
&\le C_{16}R^{2(p-1)}\widetilde{e}^{1/2}(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)\widetilde{e}^{1/2}(w_1,w_2).
\end{aligned}
\tag{3.27}
$$

Recalling $m_1 \ge m_2$ and taking (3.14) into account we can obtain the same estimates as in (3.27) for other similar terms in (3.26). From (H$_2$), (3.14), for the following typical term, we get

$$
\begin{aligned}
&\int_\Omega |\widehat{u}_1-\widehat{u}_2|\,|\widehat{v}_1|^{p}\,|\widehat{u}_1|^{p-1}\,|(w_1)_t|\,dx \\
&\le \|\widehat{u}_1-\widehat{u}_2\|_{2N/(N-m_2)}\,\|\widehat{v}_1\|_{4pN/m_2}^{p}\,\|\widehat{u}_1\|_{4(p-1)N/m_2}^{p-1}\,\|(w_1)_t\|_2 \\
&\le B\,\|D^{m_2}(\widehat{u}_1-\widehat{u}_2)\|_2\,B^p\,\|D^{m_2}\widehat{v}_1\|_2^{p}\,B^{p-1}\,\|D^{m_2}\widehat{u}_1\|_2^{p-1}\,\|(w_1)_t\|_2 \\
&\le 2B\widehat{B}R\,\|D^{m_1}(\widehat{u}_1-\widehat{u}_2)\|_2\,(B^pR^p)(B^{3(p-1)/2}R^{p-1})\,\|(w_1)_t\|_2 \\
&\le C_{17}R^{2p}\widetilde{e}^{1/2}(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)\widetilde{e}^{1/2}(w_1,w_2).
\end{aligned}
\tag{3.28}
$$

Following the same steps in (3.28), it is easy to see

$$
\int_\Omega |\widehat{v}_1-\widehat{v}_2|\,|\widehat{v}_1|^{p}\,|\widehat{u}_1|^{p-1}\,|(w_1)_t|\,dx \le C_{18}R^{2p-1}\widetilde{e}^{1/2}(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)\widetilde{e}^{1/2}(w_1,w_2).
$$

Therefore,

$$
J_3 \le C_{19}\widetilde{C}(R)\widetilde{e}^{1/2}(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)\widetilde{e}^{1/2}(w_1,w_2),
\tag{3.29}
$$

where $\widetilde{C}(R) = R^{2(p-1)} + R^{2p-1} + R^{2p}$. Thus, by (3.24), (3.25), (3.29) and using the fact that

$$
(Q_q((u_1)_t) - Q_q((u_2)_t))((u_1)_t - (u_2)_t) \ge 0,
$$

from (3.23), (H$_1$) and (3.22), we get

$$
\begin{aligned}
&\|(w_1)_t\|_2^2 + \|D^{m_1}w_1\|_2^2 \\
&\le C_{20}R^2M_0'\int_0^t \widetilde{e}(w_1(s),w_2(s))\,ds \\
&\quad + C_{21}\left(4R^2L+\widetilde{C}(R)\right)\int_0^t \widetilde{e}^{1/2}(\widehat{u}_1(s)-\widehat{u}_2(s),\widehat{v}_1(s)-\widehat{v}_2(s))\widetilde{e}^{1/2}(w_1(s),w_2(s))\,ds.
\end{aligned}
\tag{3.30}
$$

Analogously, by the same way followed in (3.23)–(3.30), from (3.21) we obtain

$$
\begin{aligned}
&\|(w_2)_t\|_2^2 + \|D^{m_2}w_2\|_2^2 \\
&\le C_{22}R^2M_0'\int_0^t \widetilde{e}(w_1(s),w_2(s))\,ds \\
&\quad + C_{23}\left(4R^2L+\widetilde{C}(R)\right)\int_0^t \widetilde{e}^{1/2}(\widehat{u}_1(s)-\widehat{u}_2(s),\widehat{v}_1(s)-\widehat{v}_2(s))\widetilde{e}^{1/2}(w_1(s),w_2(s))\,ds.
\end{aligned}
\tag{3.31}
$$

Finally, by (3.30), (3.31) and applying Gronwall's inequality, we find

$$\widetilde{e}(w_1, w_2) \le \frac{C_{24}}{(M_0')^2}\left(L + \frac{\widetilde{C}(R)}{R^2}\right)^2 e^{C_{25}M_0'R^2T} \sup_{0\le t\le T} \widetilde{e}^{1/2}(\widehat{u}_1 - \widehat{u}_2, \widehat{v}_1 - \widehat{v}_2),$$

which gives us

$$d((u_1, v_1), (u_2, v_2)) \le K(T, R)d((\widehat{u}_1, \widehat{v}_1), (\widehat{u}_2, \widehat{v}_2)),$$

where $K(T, R) = (\sqrt{C_{24}}/M_0')(L + \widetilde{C}(R)/R^2)e^{C_{25}M_0'R^2T/2}$. Now, we choose R sufficient large and T sufficient small so that

$$K(T, R) < 1 \quad \text{and} \quad \xi(u_0, v_0, \widehat{u}_0, \widehat{v}_0, u_1, v_1, R)^2 e^{4R^2\widehat{m}_0M_0'T} \le R^2.$$

Thus, the map Ψ is contraction. Therefore, applying the Banach fixed point theorem completes the proof of Theorem 2.3.

4. Blow up

In this section, we study the blow up of the solutions to the system (1.1)–(1.2). First we introduce the following:

$$B_1 = \frac{m_0}{2c_1}B^{-2p}, \quad \alpha_1 = B_1^{1/(2p-2)}, \quad E_1 = \frac{m_0}{2}\left(1 - \frac{1}{p}\right)\alpha_1^2. \tag{4.1}$$

Our main result reads in the following theorem.

Theorem 4.1. *Suppose that the assumptions* (H$_1$)–(H$_4$) *hold and* $p > \frac{1}{2}\max\{q, r\}$. *Assume further that*

$$(\|D^{m_1}u_0\|_2^2 + \|D^{m_2}v_0\|_2^2)^{1/2} > \alpha_1, \quad E(0) < E_1. \tag{4.2}$$

Then any solution of (1.1)–(1.2) *can not exist for all time.*

To prove above theorem we need the following lemma.

Lemma 4.2. *Suppose that assumptions* (H$_1$)–(H$_4$) *hold. Let* (u, v) *be a solution of* (1.1)–(1.2). *Moreover, assume that* $E(0) < E_1$ *and* $(\|D^{m_1}u_0\|_2^2 + \|D^{m_2}v_0\|_2^2)^{1/2} > \alpha_1$. *Then there exists a constant* $\alpha_2 > \alpha_1$ *such that*

$$(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)^{1/2} > \alpha_2 \tag{4.3}$$

and

$$\frac{1}{B}\sqrt[2p]{\frac{p}{c_1}}\left(\int_\Omega F(u(t), v(t))\,dx\right)^{1/(2p)} \ge \alpha_2, \quad \forall\, t \ge 0. \tag{4.4}$$

Proof. By the assumptions (H$_1$), (H$_2$), (H$_4$), Lemma 2.1 and (2.3) we have

$$
\begin{aligned}
E(t) &\geq \frac{1}{2}\mathcal{M}(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2) - \int_\Omega F(u,v)\,dx \\
&\geq \frac{m_0}{2}(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2) - \frac{c_1}{2p}(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) \\
&\geq \frac{m_0}{2}(\alpha(t))^2 - \frac{c_1}{p}B^{2p}(\alpha(t))^{2p} =: G(\alpha(t)),
\end{aligned}
\tag{4.5}
$$

where $\alpha(t) = (\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)^{1/2}$ and $G(\alpha) = \frac{m_0}{2}\alpha^2 - \frac{c_1}{p}B^{2p}\alpha^{2p}$. It is not difficult to see that G is strictly increasing in $(0,\alpha_1)$, strictly decreasing in $(\alpha_1,+\infty)$ and $G(\alpha)\to-\infty$ as $\alpha\to+\infty$. By a simple computation we can also see

$$G(\alpha_1) = E_1.$$

There exists $\alpha_2 > \alpha_1$ such that $G(\alpha_2) = E(0)$. This is possible since $E(0) < E_1$. Therefore, by (4.5) we have

$$G(\alpha(0)) \leq E(0) = G(\alpha_2).$$

Thus $\alpha(0) \geq \alpha_2$. To show (4.3) we suppose that there exists $t_0 > 0$ such that $\alpha(t_0) \leq \alpha_2$ and by continuity of $\alpha(\cdot)$ we can choose t_0 such that $\alpha_1 < \alpha(t_0)$. Since G is decreasing on $(\alpha_1,+\infty)$ we have $G(\alpha(t_0)) \geq G(\alpha_2) = E(0)$ and by (4.5) we know that $G(\alpha(t_0)) \leq E(t_0)$ which yields $E(t_0) \geq E(0)$ and this contradicts (2.5). Hence (4.3) holds.

To establish (4.4), we use (H$_1$), (2.3) and (2.5) to obtain

$$E(0) + \frac{1}{2p}(a\,\|u(t)+v(t)\|_{2p}^{2p} + 2b\,\|u(t)v(t)\|_p^p) \geq \frac{m_0^2}{2}(\alpha(t))^2.$$

Then, from (4.3) we yield

$$\int_\Omega F(u(t),v(t))\,dx \geq \frac{m_0^2}{2}\alpha_2^2 - G(\alpha_2) = \frac{c_1}{p}B^{2p}\alpha_2^{2p}.$$

Therefore, (4.4) follows. This completes the proof of Lemma 4.2. □

Proof of Theorem 4.1. We set

$$L(t) = \int_\Omega (u^2+v^2)\,dx,$$

then

$$L'(t) = 2\int_\Omega (uu_t + vv_t)\,dx$$

and

$$
\begin{aligned}
L''(t) &= 2(\|u_t\|_2^2 + \|v_t\|_2^2) + 4p\int_\Omega F(u,v)\,dx \\
&\quad - 2M(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2) \\
&\quad - 2a_1\int_\Omega uu_t\,|u_t|^{q-2}\,dx - 2a_2\int_\Omega vv_t\,|v_t|^{r-2}\,dx.
\end{aligned}
\tag{4.6}
$$

Using Hölder's inequality and the left inequality in (2.2) we get

$$\begin{aligned}\left|\int_\Omega u u_t\,|u_t|^{q-2}\,dx\right| &\le \|u\|_q\,\|u_t\|_q^{q-1} \le |\Omega|^{(2p-q)/(2pq)}\,\|u\|_{2p}\,\|u_t\|_q^{q-1}\\ &\le |\Omega|^{(2p-q)/(2pq)}\left(\frac{2p}{c_0}\right)^{1/(2p)}\left(\int_\Omega F(u,v)\,dx\right)^{1/(2p)}\|u_t\|_q^{q-1}.\end{aligned} \tag{4.7}$$

Then, by (4.4), the inequality (4.7) turns into

$$\left|\int_\Omega u u_t\,|u_t|^{q-2}\,dx\right| \le k_1\left(\int_\Omega F(u,v)\,dx\right)^{1/q}\|u_t\|_q^{q-1}. \tag{4.8}$$

Similarly,

$$\left|\int_\Omega v v_t\,|v_t|^{r-2}\,dx\right| \le k_2\left(\int_\Omega F(u,v)\,dx\right)^{1/r}\|u_t\|_r^{r-1}, \tag{4.9}$$

where

$$k_i = |\Omega|^{(2p-\kappa_i)/(2p\kappa_i)}\left(\frac{2p}{c_0}\right)^{1/(2p)}\left(\frac{c_1}{p}\alpha_2^{2p}B^{2p}\right)^{1/(2p)-1/\kappa_i}, \quad \kappa_1 = q, \quad \kappa_2 = r, \quad i = 1,2.$$

By applying Young's inequality to (4.8) and (4.9) we have

$$\left|\int_\Omega u u_t\,|u_t|^{q-2}\,dx\right| \le k_1\left\{\frac{\varepsilon_1^q}{q}\int_\Omega F(u,v)\,dx + \varepsilon_1^{-q/(q-1)}\left(\frac{q-1}{q}\right)\int_\Omega |u_t|^q\,dx\right\} \tag{4.10}$$

and

$$\left|\int_\Omega v v_t\,|v_t|^{r-2}\,dx\right| \le k_2\left\{\frac{\varepsilon_2^r}{r}\int_\Omega F(u,v)\,dx + \varepsilon_2^{-r/(r-1)}\left(\frac{r-1}{r}\right)\int_\Omega |v_t|^r\,dx\right\}, \tag{4.11}$$

where $\varepsilon_1, \varepsilon_2 > 0$ will be chosen later. Then, by (H_1), (2.4), (4.10) and (4.11), the equality (4.6) turns into following inequality

$$\begin{aligned}L''(t) \ge{}& 2(\|u_t\|_2^2 + \|v_t\|_2^2) - 2K(t) - 2\left(a_1k_1\frac{\varepsilon_1^q}{q} + a_2k_2\frac{\varepsilon_2^r}{r}\right)\int_\Omega F(u,v)\,dx\\ &- 2a_1k_1\left(\frac{q-1}{q}\right)\varepsilon_1^{-q/(q-1)}\|u_t\|_q^q - 2a_2k_2\left(\frac{r-1}{r}\right)\varepsilon_2^{-r/(r-1)}\|v_t\|_r^r.\end{aligned} \tag{4.12}$$

By the definition of $E(t)$ we have

$$\begin{aligned}-2K(t) &\ge -2K(t) + 2\sigma(E(t) - E(0))\\ &= \sigma(\|u_t\|_2^2 + \|v_t\|_2^2) + (\sigma - 2)\mathcal{M}(\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2)\\ &\quad + 2(2p - \sigma)\int_\Omega F(u,v)\,dx - 2\sigma E(0),\end{aligned} \tag{4.13}$$

where σ is a positive constant to be specified later. Therefore, by (4.12) and (4.13) we arrive at

$$\begin{aligned}L''(t) &\geq (\sigma+2)(\|u_t\|_2^2+\|v_t\|_2^2)+(\sigma-2)\mathcal{M}(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)\\&\quad+2\left[(2p-\sigma)-\left(a_1k_1\frac{\varepsilon_1^q}{q}+a_2k_2\frac{\varepsilon_2^r}{r}\right)\right]\int_\Omega F(u,v)\,dx-2\sigma E(0)\\&\quad-2a_1k_1\left(\frac{q-1}{q}\right)\varepsilon_1^{-q/(q-1)}\|u_t\|_q^q-2a_2k_2\left(\frac{r-1}{r}\right)\varepsilon_2^{-r/(r-1)}\|v_t\|_r^r.\end{aligned} \tag{4.14}$$

Since $E(0)<E_1$ we can choose σ such that

$$\frac{2pE_1}{p(E_1-E(0))+E(0)}<\sigma<2p.$$

Then, by Lemma 4.2, (2.1), (4.1) and (4.3) we have

$$\begin{aligned}(\sigma-2)\mathcal{M}(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)-2\sigma E(0) &\geq (\sigma-2)m_0\alpha_1^2-2\sigma E(0)\\&=2\left(\frac{pE_1}{p-1}-E(0)\right)\sigma-\frac{4pE_1}{p-1}>0.\end{aligned}$$

We now fix ε_1 and ε_2 such that

$$\mu:=2p-\sigma-\left(a_1k_1\frac{\varepsilon_1^q}{q}+a_2k_2\frac{\varepsilon_2^r}{r}\right)>0.$$

Integrating (4.14) over $(0,t)$ we get

$$\begin{aligned}L'(t) &> 2\mu\int_0^t\int_\Omega F(u(s),v(s))\,dxds\\&\quad-C(\varepsilon_1,q)\int_0^t\|u_t(s)\|_q^q\,ds-C(\varepsilon_2,r)\int_0^t\|v_t(s)\|_r^r\,ds+L'(0),\end{aligned} \tag{4.15}$$

where $C(\varepsilon_i,s)=2a_ik_i(\frac{s-1}{s})\varepsilon_i^{-s/(s-1)}$, $i=1,2$. Taking (4.4) and (2.5) into account and using the fact that $E(0)-E(t)<E_1$, the inequality (4.15) takes the form

$$L'(t)>2\mu\left(\frac{c_1}{p}B^{2p}\alpha_2^2\right)t-E_1\left(\frac{C(\varepsilon_1,q)}{a_1}+\frac{C(\varepsilon_2,r)}{a_2}\right)+L'(0). \tag{4.16}$$

Finally, by integrating (4.16) from 0 to t we find

$$L(t)>\mu\left(\frac{c_1}{p}B^{2p}\alpha_2^2\right)t^2+\left\{L'(0)-E_1\left(\frac{C(\varepsilon_1,q)}{a_1}+\frac{C(\varepsilon_2,r)}{a_2}\right)\right\}t+L(0), \tag{4.17}$$

which shows that $\|u(t)\|_2^2+\|v(t)\|_2^2$ has quadratic growth for $t\geq 0$. On the other hand by using Hölder's inequality we have

$$\begin{aligned}\|u(t)\|_2 &\leq \|u_0\|_2+\int_0^t\|u_t(s)\|_2\,ds\\&\leq \|u_0\|_2+C\int_0^t\|u_t(s)\|_q\,ds\leq\|u_0\|_2+C\left(\frac{E_1}{a_1}\right)^{1/q}t^{(q-1)/q},\end{aligned} \tag{4.18}$$

where C is some positive constant. Similarly,

$$\|v(t)\|_2 \le \|v_0\|_2 + C\left(\frac{E_1}{a_2}\right)^{1/r} t^{(r-1)/r}. \tag{4.19}$$

By (4.18) and (4.19) we obtain

$$L(t) \le 2(\|u_0\|_2^2 + \|v_0\|_2^2) + 2C^2\left[\left(\frac{E_1}{a_1}\right)^{2/q} t^{2(q-1)/q} + \left(\frac{E_1}{a_2}\right)^{2/r} t^{2(r-1)/r}\right],$$

which contradicts (4.17). Hence, the solution $(u(t), v(t))$ of (1.1)–(1.2) can not be extended to the whole interval $[0, +\infty)$. This completes the proof of Theorem 4.1. $\square$

Remark 4.3. By Theorem 4.1 we showed that the L^2 norm of solution $\|(u,v)\|_2^2 := \|u\|_2^2 + \|v\|_2^2$ blows up in a finite time $T^\star > 0$. Therefore, by Lemma 2.1

$$\|D^{m_1}u\|_2^2 + \|D^{m_2}v\|_2^2 \to +\infty \quad \text{as } t \to {T^\star}^-. \tag{4.20}$$

5. Lower bounds for the blow up time

In this section we obtain lower bounds for the blow up time. To prove main results we need the following assumption instead of (H_2):

$(\mathrm{H}_2)'$ $q, r \ge 2$, $m_i \ge 1$ $(i = 1, 2)$ and

$$\begin{aligned} &1 < p < +\infty, && N \le 2\min\{m_1, m_2\}, \\ &1 < p \le \min\left\{\frac{N-m_1}{2(N-2m_1)}, \frac{N-m_2}{2(N-2m_2)}\right\}, && N > 2\max\{m_1, m_2\}. \end{aligned}$$

Remark 5.1. Under the hypotheses (H_1) and $(\mathrm{H}_2)'$–(H_4) the results of Theorem 2.3 still hold because $N - m_i < N$, $i = 1, 2$.

Our main results are given in two following theorems:

Theorem 5.2. *Suppose* (H_1), $(\mathrm{H}_2)'$–(H_4) *and* (4.2) *hold. Assume further that* $p > \frac{1}{2}\max\{q, r\}$. *Then the finite blow-up time* $T^\star$ *satisfies the following estimate:*

$$T^\star > \int_{\Theta(0)}^{+\infty} \frac{m_0^{2p-1}\, d\zeta}{m_0^{2p-1}(E(0)+\zeta) + 2^{4(p-1)}(\gamma_1+\gamma_2)\left((E(0))^{2p-1} + \zeta^{2p-1}\right)}, \tag{5.1}$$

where $\Theta(0) = \int_\Omega F(u(0), v(0))\, dx$ *and the positive constants* γ_i $(i = 1, 2)$ *are specified in* (5.3).

Theorem 5.3. *Suppose that the assumptions of Theorem* 5.2 *hold. Then the finite blow-up time $T^\star$ satisfies the following estimate:*

$$T^\star > \frac{1}{2p}\log\left[1+\left(\frac{(\Phi(0))^{-2p}}{\gamma_1+\gamma_2}\right)m_0^{2p-1}\right], \tag{5.2}$$

where the positive constants γ_i $(i=1,2)$ are specified in (5.3) *and*

$$\Phi(0)=\|u_1\|_2^2+\|v_1\|_2^2+\mathcal{M}(\|D^{m_1}u_0\|_2^2+\|D^{m_2}v_0\|_2^2).$$

To prove the above theorems, we first prove the following lemma (in the proof C_i, $i=1,\dots,5$ are some positive constants):

Lemma 5.4. *Assume that* $(\mathrm{H}_2)'$ *hold. Then, there exist tow positive constants γ_1 and γ_2 such that*

$$\int_\Omega |f_i(u,v)|^2\,dx \le \gamma_i\left(\int_\Omega(|D^{m_1}u|^2+|D^{m_2}v|^2)\,dx\right)^{2p-1}, \quad i=1,2. \tag{5.3}$$

Proof. Obviously, we have

$$\begin{aligned}|f_1(u,v)| &\le C_1(|u+v|^{2p-1}+|u|^{p-1}|v|^p)\\ &\le C_2(|u|^{2p-1}+|v|^{2p-1}+|u|^{p-1}|v|^p).\end{aligned}$$

By Young's inequality we obtain

$$|u|^{p-1}|v|^p \le C_3|u|^{2p-1}+C_4|v|^{2p-1}.$$

Therefore,

$$\int_\Omega |f_1(u,v)|^2\,dx \le C_4\int_\Omega(|u|^{4p-2}+|v|^{4p-2})\,dx. \tag{5.4}$$

Using $(\mathrm{H}_2)'$ and the embedding $H_0^{m_i}(\Omega)\hookrightarrow L^{4p-2}(\Omega)$ $(i=1,2)$ from (5.4) we get

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2\,dx &\le C_4B^{4p-2}(\|D^{m_1}u\|_2^{4p-2}+\|D^{m_2}v\|_2^{4p-2})\\ &\le C_5(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)^{2p-1}.\end{aligned}$$

Therefore (5.3) follows. The same way can be followed to obtain similar inequality for f_2. □

Proof of Theorem 5.2. Theorem 4.1 guarantees the existence of $T^\star$. We define

$$\Theta(t)=\int_\Omega F(u(t),v(t))\,dx.$$

Then, by using Young's inequality and Lemma 5.4 we have

$$\begin{aligned}\Theta'(t) &= \int_\Omega (u_t f_1 + v_t f_2)\,dx \\ &\le \frac{1}{2}\int_\Omega (u_t^2+v_t^2)\,dx + \frac{1}{2}\int_\Omega (f_1^2+f_2^2)\,dx \\ &\le \frac{1}{2}\int_\Omega (u_t^2+v_t^2)\,dx + \frac{1}{2}(\gamma_1+\gamma_2)(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)^{2p-1}.\end{aligned} \tag{5.5}$$

By (2.1), (2.3) and Lemma 2.2 we obtain

$$\begin{aligned}\int_\Omega (u_t^2+v_t^2)\,dx + m_0(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2) &\le 2E(t) + 2\int_\Omega F(u,v)\,dx \\ &\le 2E(0) + 2\int_\Omega F(u,v)\,dx.\end{aligned} \tag{5.6}$$

Consequently, by (5.5) and (5.6) we get

$$\begin{aligned}\Theta'(t) &\le E(0)+\Theta(t)+2^{2p-2}m_0^{1-2p}(\gamma_1+\gamma_2)[E(0)+\Theta(t)]^{2p-1} \\ &\le E(0)+\Theta(t)+2^{4(p-1)}m_0^{1-2p}(\gamma_1+\gamma_2)[(E(0))^{2p-1}+(\Theta(t))^{2p-1}].\end{aligned} \tag{5.7}$$

Integrating (5.7) over $(0,t)$ we get

$$t > \int_{\Theta(0)}^{\Theta(t)} \frac{m_0^{2p-1}\,d\zeta}{m_0^{2p-1}(E(0)+\zeta)+2^{4(p-1)}(\gamma_1+\gamma_2)((E(0))^{2p-1}+\zeta^{2p-1})}. \tag{5.8}$$

From (4.20) and (5.6) we see that $\Theta(t)\to+\infty$ as $t\to T^{\star -}$. Hence, (5.1) follows by letting $t\to T^{\star -}$ in (5.8). Thus, the proof of Theorem 5.2 is complete. □

Proof of Theorem 5.3. We set

$$\Phi(t) = \int_\Omega (u_t^2+v_t^2)\,dx + \mathcal{M}(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2).$$

We have

$$\Phi'(t) = -2a_1\,\|u_t\|_q^q - 2a_2\,\|v_t\|_r^r + 2\int_\Omega (u_t f_1 + v_t f_2)\,dx.$$

Using Young's inequality, Lemma 5.4 and (H_1) we obtain

$$\begin{aligned}\Phi'(t) &\le \int_\Omega (u_t^2+v_t^2)\,dx + \int_\Omega (f_1^2+f_2^2)\,dx \\ &\le \int_\Omega (u_t^2+v_t^2)\,dx + (\gamma_1+\gamma_2)(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)^{2p-1} \\ &\le \int_\Omega (u_t^2+v_t^2)\,dx + m_0^{1-2p}(\gamma_1+\gamma_2)[\mathcal{M}(\|D^{m_1}u\|_2^2+\|D^{m_2}v\|_2^2)]^{2p-1} \\ &\le \Phi(t) + m_0^{1-2p}(\gamma_1+\gamma_2)(\Phi(t))^{2p-1}.\end{aligned} \tag{5.9}$$

Integrating (5.9) over $(0, t)$ we get

$$\begin{aligned}(\Phi(t))^{2(1-p)} \geq & -m_0^{1-2p}(\gamma_1 + \gamma_2) \\ & + [(\Phi(0))^{2(1-p)} + m_0^{1-2p}(\gamma_1 + \gamma_2)] \exp(2(1-p)t).\end{aligned} \tag{5.10}$$

By (4.20) and (2.1) we can easily see that if $t \to T^{\star^-}$ then $\Phi(t) \to +\infty$. Hence, (5.2) holds by letting $t \to T^{\star^-}$ in (5.10). □

Remark 5.5. Theorem 4.1 guarantees the existence of $T^\star$ in Theorems 5.2 and 5.3.

Remark 5.6. By (2.3) we have

$$\Phi(t) = 2E(t) + 2\Theta(t) \leq 2E(0) + 2\Theta(t).$$

Hence, the estimate (5.2) is also valid for Θ.

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