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Research Article

On an Inverse Boundary Value Problem for a Fourth Order Elliptic Equation with Integral Condition

Yashar T. Mehraliyev

Baku State University, 1148 Baku, Azerbaijan

Correspondence should be addressed to Yashar T. Mehraliyev; yashar_aze@mail.ru

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An inverse boundary value problem for a fourth order elliptic equation is investigated. At first the initial problem is reduced to the equivalent problem for which the existence and uniqueness theorem of the solution is proved. Further, using these facts, the existence and uniqueness of the classic solution of the initial problem are proved.

1. Introduction

The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science.

Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of Tikhonov [1], Lavrent'ev [2, 3], Denisov [4], Ivanchov [5], and their followers.

The goal of our paper is to prove the uniqueness and existence of the solution of a boundary value problem for a fourth order elliptic equation with integral condition.

The inverse problems with an integral predetermination condition for parabolic equations were investigated in [6–10].

In the papers [11–15] the inverse boundary value problems were investigated for a second order elliptic equation in a rectangular domain.

2. Problem Statement and Its Reduction to Equivalent Problem

Consider the following equation:

$$u_{tttt}(x,t) + u_{xxxx}(x,t) = a(t)\,u(x,t) + f(x,t) \qquad (1)$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ an inverse boundary problem with the boundary conditions

$$u(x,0) = \varphi_0(x), \qquad u_t(x,T) = \varphi_1(x),$$
$$u_{tt}(x,0) = \varphi_2(x), \qquad u_{ttt}(x,T) = \varphi_3(x) \quad (0 \le x \le 1), \qquad (2)$$

$$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = 0, \quad (0 \le t \le T), \qquad (3)$$

the integral condition

$$\int_0^1 u(x,t)\,dx = 0 \quad (0 \le t \le T), \qquad (4)$$

and with the additional condition

$$u(0,t) = h(t) \quad (0 \le t \le T), \qquad (5)$$

where $f(x,t)$, $\varphi_i(x)$ $(i = \overline{0,3})$, $h(t)$ are the given functions and $u(x,t)$, $a(t)$ are the required functions.

Definition 1. The classic solution of problems (1)–(5) is the pair $\{u(x,t), a(t)\}$ of the functions $u(x,t)$ and $a(t)$ possessing the following properties:

(1) the function $u(x,t)$ is continuous in D_T together with all its derivatives contained in (1);

(2) the function $a(t)$ is continuous on $[0,T]$;

(3) all the conditions of (1)–(5) are satisfied in the ordinary sense.

For investigating problems (1)–(5), at first consider the following problem:

$$y^{(4)}(t) = a(t)\,y(t) \quad (0 \le t \le T), \tag{6}$$

$$y(0) = 0, \qquad y'(T) = 0, \qquad y''(0) = 0, \qquad y'''(T) = 0, \tag{7}$$

where $a(t) \in C[0,T]$ is a given function, $y = y(t)$ is a desired function, and under the solution of problems (6) and (7) we will understand a function $y(t) \in C^4[0,T]$ satisfied in $[0,T]$ equation (6) and conditions (7).

The following lemma is valid.

Lemma 2 (see [16, 17]). *Let the function $a(t) \in C[0,T]$ be such that*

$$\|a(t)\|_{C[0,T]} \le R = const. \tag{8}$$

Furthermore,

$$\frac{5}{12}T^4 R < 1. \tag{9}$$

Then problems (6), (7) *have only a trivial solution.*

Alongside with inverse boundary value problem, consider the following auxiliary inverse boundary value problem. It is required to determine the pair $\{u(x,t), a(t)\}$ of the functions $u(x,t)$ and $a(t)$ possessing the properties (1) and (2) of definition of the classic solution of problems (1)–(5) from relations (1)–(3) and

$$u_{xxx}(1,t) = 0 \quad (0 \le t \le T), \tag{10}$$

$$h^{(4)}(t) + u_{xxxx}(0,t) = a(t)\,h(t) + f(0,t) \quad (0 \le t \le T). \tag{11}$$

The following lemma is valid.

Lemma 3. *Let $\varphi_i(x) \in C[0,1]$ $(i = \overline{0,3})$ $h(t) \in C^4[0,T]$, $h(t) \ne 0$ $(0 \le t \le T)$, $f(x,t) \in C(D_T)$, let $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, and let the following consistency conditions be fulfilled:*

$$\int_0^1 \varphi_i(x)\,dx = 0 \quad \left(i = \overline{0,3}\right), \tag{12}$$

$$\varphi_0(0) = h(0), \qquad \varphi_1(0) = h'(T),$$
$$\varphi_2(0) = h''(0), \qquad \varphi_3(0) = h'''(T). \tag{13}$$

Then the following statements are true.

(1) *Each classic solution $u(x,t)$, $a(t)$ of problems* (1)–(5) *is the solution of problems* (1)–(3), (10), *and* (11) *as well.*

(2) *Each solution $u(x,t), a(t)$ of problems* (1)–(3), (10), *and* (11) *such that*

$$\frac{5}{12}T^4\|a(t)\|_{C[0,T]} < 1, \tag{14}$$

is the classic solution of problems (1)–(5).

Proof. Let $u(x,t), a(t)$ be a solution of problems (1)–(5). Integrating equation (1) with respect to x from 0 to 1, we have

$$\frac{d^4}{dt^4}\int_0^1 u(x,t)\,dx + u_{xxx}(1,t) - u_{xxx}(0,t) = a(t)\int_0^1 u(x,t)\,dx + \int_0^1 f(x,t)\,dx \quad (0 \le t \le T). \tag{15}$$

Hence, by means of $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$ and (3) we obtain (10).

Substituting $x = 0$ in (1), we find

$$u_{tttt}(0,t) + u_{xxxx}(0,t) = a(t)\,u(0,t) + f(0,t) \quad (0 \le t \le T). \tag{16}$$

Further assuming $h(t) \in C^4[0,T]$ and differentiating (5) four times, we have

$$u_{tttt}(0,t) = h^{(4)}(t) \quad (0 \le t \le T). \tag{17}$$

Taking into account last relation and condition (5) in (16) we obtain (11).

Now suppose that $u(x,t), a(t)$ is a solution of problems (1)–(3), (10), and (11); moreover, (14) is fulfilled. Then, taking into account (3) and (10) in (15) we find

$$\frac{d^4}{dt^4}\int_0^1 u(x,t)\,dx - a(t)\int_0^1 u(x,t)\,dx = 0 \quad (0 \le t \le T). \tag{18}$$

By (2) and (12), it is obvious that

$$\int_0^1 u(x,0)\,dx = \int_0^1 \varphi_0(x)\,dx = 0,$$
$$\int_0^1 u_t(x,T)\,dx = \int_0^1 \varphi_1(x)\,dx = 0,$$
$$\int_0^1 u_{tt}(x,0)\,dx = \int_0^1 \varphi_2(x)\,dx = 0,$$
$$\int_0^1 u_{ttt}(x,T)\,dx = \int_0^1 \varphi_3(x)\,dx = 0. \tag{19}$$

Since by Lemma 2, problems (18), (19) have only a trivial solution, then $\int_0^1 u(x,t)dx = 0$ $(0 \le t \le T)$; that is, condition (4) is fulfilled.

Further, from (11) and (16) we get

$$\frac{d^4}{dt^4}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)) \quad (0 \le t \le T). \tag{20}$$

By (2) and consistency conditions (13), we have

$$\begin{aligned} u(0,0) - h(0) &= \varphi_0(0) - h(0) = 0, \\ u_t(0,T) - h'(T) &= \varphi_1(0) - h'(T) = 0, \\ u_{tt}(0,0) - h''(T) &= \varphi_2(0) - h''(0) = 0, \\ u_t(0,T) - h'''(T) &= \varphi_1(0) - h'''(T) = 0. \end{aligned} \tag{21}$$

From (20) and (21), by Lemma 3 we conclude that condition (5) is fulfilled. The lemma is proved. □

3. Investigation of the Existence and Uniqueness of the Classic Solution of the Inverse Boundary Value Problem

We will look for the first component $u(x,t)$ of the solution $u(x,t), a(t)$ of problems (1)–(3), (10), and (11) in the following form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi), \tag{22}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x \, dx \quad (k = 0,1,2,\ldots), \tag{23}$$

and moreover,

$$m_k = \begin{cases} 1, & k = 0 \\ 2, & k = 1,2,\ldots. \end{cases} \tag{24}$$

Then applying the formal scheme of the Fourier method, from (1), (2) we get

$$\begin{aligned} u_k^{(4)}(t) + \lambda_k^4 u_k(t) &= F_k(t;u,a) \\ (k = 0,1,2,\ldots; 0 \le t \le T), \end{aligned} \tag{25}$$

$$\begin{aligned} u_k(0) = \varphi_{0k}, \quad u_k'(T) = \varphi_{1k}, \\ u_k''(0) = \varphi_{2k}, \quad u_k'''(T) = \varphi_{3k} \\ (k = 0,1,2,\ldots), \end{aligned} \tag{26}$$

where

$$\begin{aligned} F_k(t;u,a) &= f_k(t) + a(t) u_k(t), \\ f_k(t) &= m_k \int_0^1 f(x,t) \cos \lambda_k x \, dx, \\ \varphi_{ik} &= m_k \int_0^1 \varphi_i(x) \cos \lambda_k x \, dx \quad \left(i = \overline{0,3}; k = 0,1,2,\ldots\right). \end{aligned} \tag{27}$$

From (25), (26) we obtain

$$\begin{aligned} u_0(t) &= \varphi_{00} + t\varphi_{10} + \left(\frac{1}{2}t^2 - tT\right)\varphi_{20} \\ &\quad + \left(\frac{1}{6}t^3 - \frac{1}{2}T^2 t\right)\varphi_{30} + \int_0^T G_0(t,\tau) F_0(\tau;u) \, d\tau, \end{aligned} \tag{28}$$

$$\begin{aligned} u_k(t) &= \frac{1}{\alpha_k(T)} \\ &\times \left\{ \frac{2}{\rho_k} \left[-\left(4\rho_k^4 + 1\right)\left(ch2\rho_k T + \cos 2\rho_k T\right) \right.\right. \\ &\quad + ch\rho_k t \cos \rho_k (2T - t) \\ &\quad \left. + ch\rho_k (2T - t) \cos \rho_k t \right] \varphi_{0k} \\ &\quad + \left[-\frac{2t}{\rho_k}\left(4\rho_K^4 + 1\right)\left(ch2\rho_k T + \cos 2\rho_k T\right) \right. \\ &\quad + \frac{2}{\rho_k^2}\left(ch\rho_k (T - t) \sin \rho_k (T + t) \right. \\ &\quad + ch\rho_k (T + t) \sin \rho_k (T - t) \\ &\quad - sh\rho_k (T - t) \cos \rho_k (T + t) \\ &\quad \left.\left. + sh\rho_k (T + t) \cos \rho_k (T - t)\right) \right] \varphi_{1k} \\ &\quad + \left[-\frac{1}{\rho_k}\left(t^2 - 2tT\right)\left(4\rho_K^4 + 1\right) \right. \\ &\quad \times \left(ch2\rho_k T + \cos 2\rho_k T\right) - \frac{2}{\rho_k^3} \\ &\quad \times \left(sh\rho_k (2T - t) \sin \rho_k t + sh\rho_k t (2T - t) \right. \\ &\quad \left. \left. \times \sin \rho_k (2T - t)\right) \right] \varphi_{2k} \\ &\quad + \left[-\frac{1}{\rho_k}\left(\frac{t^3}{3} - tT^2\right)\left(4\rho_K^4 + 1\right) \right. \\ &\quad \times \left(ch2\rho_k T + \cos 2\rho_k T\right) + \frac{1}{2\rho_k^4} \\ &\quad \times \left(sh\rho_k (T - t) \sin \rho_k (T + t) \right. \\ &\quad - sh\rho_k (T - t) \cos \rho_k (T + t) \\ &\quad \left. \left. - sh\rho_k (T + t) \cos \rho_k (T - t)\right) \right] \varphi_{3k} \\ &\quad \left. + \int_0^T G_k(t,\tau) F_k(\tau;u,a) \, d\tau \right\}, \end{aligned} \tag{29}$$

where

$$G_0(t,\tau)=\begin{cases}-\dfrac{t^3}{6}+\left(T\tau-\dfrac{\tau^2}{2}\right)t, & t\in[0,\tau],\\ -\dfrac{\tau^3}{6}+\left(Tt-\dfrac{t^2}{2}\right)\tau, & t\in[\tau,T],\end{cases}$$

$$G_k(t,\tau)=\begin{cases}\dfrac{1}{\alpha_k(T)}\left[ch\rho_k(2T+t-\tau)\sin\rho_k(t-\tau)\right.\\ \quad+ch\rho_k(2T-t-\tau)\sin\rho_k(t+\tau)\\ \quad-sh\rho_k(2T+t-\tau)\cos\rho_k(t-\tau)\\ \quad+sh\rho_k(2T-t-\tau)\cos\rho_k(t+\tau)\\ \quad-ch\rho_k(t+\tau)\sin\rho_k(2T-t-\tau)\\ \quad+ch\rho_k(t-\tau)\sin\rho_k(2T+t-\tau)\\ \quad-sh\rho_k(t+\tau)\cos\rho_k(2T-t-\tau)\\ \quad\left.-sh\rho_k(t-\tau)\cos\rho_k(2T+t-\tau)\right], \quad t\in[0,\tau],\\ \dfrac{1}{\alpha_k(T)}\left[-ch\rho_k(2T-t+\tau)\sin\rho_k(t-\tau)\right.\\ \quad+ch\rho_k(2T-t-\tau)\sin\rho_k(t+\tau)\\ \quad-sh\rho_k(2T-t+\tau)\cos\rho_k(t-\tau)\\ \quad+sh\rho_k(2T-t-\tau)\cos\rho_k(t+\tau)\\ \quad-ch\rho_k(t+\tau)\sin\rho_k(2T-t-\tau)\\ \quad+ch\rho_k(t-\tau)\sin\rho_k(2T-t+\tau)\\ \quad-sh\rho_k(t+\tau)\cos\rho_k(2T-t-\tau)\\ \quad\left.+sh\rho_k(t-\tau)\cos\rho_k(2T-t+\tau)\right], \quad t\in[\tau,T],\end{cases}\tag{30}$$

and moreover,

$$\alpha_k(T)=-8\rho_k^3\left(ch2\rho_kT+\cos2\rho_kT\right),\quad \rho_k=\frac{\sqrt{2}}{2}\lambda_k.\tag{31}$$

After substituting the expressions from (28), (29) into (22), for determining the component of the solution of problems (1)–(3), (10), and (11) we get

$$\begin{aligned}u(x,t)&=\varphi_{00}+t\varphi_{10}+\left(\frac{1}{2}t^2-tT\right)\varphi_{20}\\&+\left(\frac{1}{6}t^3-\frac{1}{2}T^2t\right)\varphi_{30}+\int_0^T G_0(t,\tau)F_0(\tau;u)\,d\tau\\&+\sum_{k=1}^{\infty}\left\{\frac{1}{\alpha_k(T)}\left\{\frac{2}{\rho_k}\left[-\left(4\rho_K^4+1\right)\left(ch2\rho_kT+\cos2\rho_kT\right)\right.\right.\right.\\&\quad+ch\rho_kt\cos\rho_k(2T-t)\\&\quad\left.+ch\rho_k(2T-t)\cos\rho_kt\right]\varphi_{0k}\\&\quad+\left[-\frac{2t}{\rho_k}\left(4\rho_K^4+1\right)\left(ch2\rho_kT+\cos2\rho_kT\right)\right.\\&\quad+\frac{2}{\rho_k^2}\left(ch\rho_k(T-t)\sin\rho_k(T+t)\right.\\&\quad+ch\rho_k(T+t)\sin\rho_k(T-t)\\&\quad-sh\rho_k(T-t)\cos\rho_k(T+t)\\&\quad\left.\left.+sh\rho_k(T+t)\cos\rho_k(T-t)\right)\right]\varphi_{1k}\\&\quad+\left[-\frac{1}{\rho_k}\left(t^2-2tT\right)\left(4\rho_K^4+1\right)\right.\\&\quad\times\left(ch2\rho_kT+\cos2\rho_kT\right)-\frac{2}{\rho_k^3}\\&\quad\times\left(sh\rho_k(2T-t)\sin\rho_kt+sh\rho_kt(2T-t)\right.\\&\quad\left.\left.\times\sin\rho_k(2T-t)\right)\right]\varphi_{2k}\\&\quad+\left[-\frac{1}{\rho_k}\left(\frac{t^3}{3}-tT^2\right)\left(4\rho_K^4+1\right)\right.\\&\quad\times\left(ch2\rho_kT+\cos2\rho_kT\right)+\frac{1}{2\rho_k^4}\\&\quad\times\left(sh\rho_k(T-t)\sin\rho_k(T+t)\right.\\&\quad-sh\rho_k(T-t)\cos\rho_k(T+t)\\&\quad\left.\left.-sh\rho_k(T+t)\cos\rho_k(T-t)\right)\right]\varphi_{3k}\\&\quad\left.+\int_0^T G_k(t,\tau)F_k(\tau;u,a)\,d\tau\right\}\Bigg\}\cos\lambda_kx,\end{aligned}\tag{32}$$

where

$$\begin{aligned}F_k(t;u,a)&=f_k(t)+a(t)u_k(t)\\&=m_k\int_0^1\left(f(x,t)+a(t)u(x,t)\right)\cos\lambda_kx\,dx\\&\qquad(k=0,1,2,\ldots).\end{aligned}\tag{33}$$

Now, from (11) and (22) we have

$$a(t)=h^{-1}(t)\left\{h^{(4)}(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^4u_k(t)\right\}.\tag{34}$$

For obtaining an equation for the second component $a(t)$ of the solution $\{u(x,t), a(t)\}$ of problems (1)–(3), (10), and (11), substitute expression (29) into (34):

$$\begin{aligned}
a(t) &= h^{-1}(t)\left\{h^{(4)}(t) - f(0,t)\right\} \\
&+ \sum_{k=1}^{\infty}\lambda_k^4\left\{\frac{1}{\alpha_k(T)}\right. \\
&\times\left\{\frac{2}{\rho_k}\left[-\left(4\rho_K^4+1\right)\left(ch2\rho_kT+\cos 2\rho_kT\right)\right.\right. \\
&\quad +ch\rho_kt\cos\rho_k(2T-t) \\
&\quad \left.+ch\rho_k(2T-t)\cos\rho_kt\right]\varphi_{0k} \\
&+\left[-\frac{2t}{\rho_k}\left(4\rho_K^4+1\right)\right. \\
&\quad \times\left(ch2\rho_kT+\cos 2\rho_kT\right) \\
&\quad +\frac{2}{\rho_k^2}\left(ch\rho_k(T-t)\sin\rho_k(T+t)\right. \\
&\quad\quad +ch\rho_k(T+t)\sin\rho_k(T-t) \\
&\quad\quad -sh\rho_k(T-t)\cos\rho_k(T+t) \\
&\quad\quad +sh\rho_k(T+t) \\
&\quad\quad \left.\left.\times\cos\rho_k(T-t)\right)\right]\varphi_{1k} \\
&+\left[-\frac{1}{\rho_k}\left(t^2-2tT\right)\left(4\rho_K^4+1\right)\right. \\
&\quad \times\left(ch2\rho_kT+\cos 2\rho_kT\right)-\frac{2}{\rho_k^3} \\
&\quad \times\left(sh\rho_k(2T-t)\sin\rho_kt\right. \\
&\quad\quad +sh\rho_kt(2T-t) \\
&\quad\quad \left.\left.\times\sin\rho_k(2T-t)\right)\right]\varphi_{2k} \\
&+\left[-\frac{1}{\rho_k}\left(\frac{t^3}{3}-tT^2\right)\left(4\rho_K^4+1\right)\right. \\
&\quad \times\left(ch2\rho_kT+\cos 2\rho_kT\right)+\frac{1}{2\rho_k^4} \\
&\quad \times\left(sh\rho_k(T-t)\sin\rho_k(T+t)\right. \\
&\quad\quad -sh\rho_k(T-t)\cos\rho_k(T+t) \\
&\quad\quad -sh\rho_k(T+t) \\
&\quad\quad \left.\left.\left.\times\cos\rho_k(T-t)\right)\right]\varphi_{3k} \right. \\
&\left.\left.+\int_0^T G_k(t,\tau)F_k(\tau;u,a)\,d\tau\right\}\right\},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
F_k(t;u,a) &= f_k(t)+a(t)u_k(t) \\
&= 2\int_0^1\left(f(x,t)+a(t)u(x,t)\right)\cos\lambda_kx\,dx \\
&\qquad (k=1,2,\ldots).
\end{aligned} \tag{36}$$

Thus, problems (1)–(3), (10), and (11) were reduced to systems (32), (35) with respect to the unknown functions $u(x,t)$ and $a(t)$.

The following lemma is important for studying the uniqueness of the solution of problems (1)–(3), (10), and (11).

Lemma 4. *If $\{u(x,t), a(t)\}$ is any solution of problems* (1)–(3), (10), *and* (11), *then the function*

$$u_k(t) = m_k\int_0^1 u(x,t)\cos\lambda_kx\,dx \quad (k=0,1,2,\ldots) \tag{37}$$

satisfies systems (28), (29) *in* $[0,T]$.

Proof. Let $\{u(x,t), a(t)\}$ be any solution of problems (1)–(3), (10), and (11). Then, having multiplied the both sides of (1) by the function $m_k\cos\lambda_kx$ $(k=0,1,2,\ldots)$, integrating the obtained equality with respect to x from 0 to 1, and using the relations

$$\begin{aligned}
&m_k\int_0^1 u_{tt}(x,t)\cos\lambda_kx\,dx \\
&\quad = \frac{d^4}{dt^4}\left(m_k\int_0^1 u(x,t)\cos\lambda_kx\,dx\right) = u_k^{(4)}(t) \\
&\qquad\qquad (k=0,1,2,\ldots), \\
&m_k\int_0^1 u_{xxxx}(x,t)\cos\lambda_kx\,dx \\
&\quad = \lambda_k^4\left(m_k\int_0^1 u(x,t)\cos\lambda_kx\,dx\right) = \lambda_k^4u_k(t) \\
&\qquad\qquad (k=0,1,2,\ldots),
\end{aligned} \tag{38}$$

we get that (25) is satisfied.

Similarly, from (2) we get that condition (26) is fulfilled.

Thus, $u_k(t)$ $(k=0,1,2,\ldots)$ is a solution of problems (25), (26). Hence, it directly follows that the function $u_k(t)$ $(k=0,1,2,\ldots)$ satisfies $[0,T]$ in systems (28), (29). The lemma is proved. □

Remark 5. From Lemma 4 it follows that for proving the uniqueness of the solution of problems (1)–(3), (10), and (11), it suffices to prove the uniqueness of the solution of systems (32), (35).

In order to investigate problems (1)–(3), (10), and (11), consider the following spaces.

Denote $B^5_{2,T}$ the set of all the functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = \pi k) \tag{39}$$

considered in D_T, where each of the functions $u_k(t)$ ($k = 0, 1, 2, \ldots$) is continuous on $[0, T]$ and

$$J_T(u) \equiv \|u_0(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{1/2} < \infty. \tag{40}$$

In this set, we determine the operation of addition and multiplication by the number (real) in the usual way: under the zero element of this set we will understand the function $u(x,t) \equiv 0$ on D_T, and determine the norm in this set by the following formula:

$$\|u(x,t)\|_{B^5_{2,T}} = J_T(u). \tag{41}$$

Prove that all these spaces are Banach spaces. Indeed, the validity of the first two axioms of the norms is obvious, and validity of the third axiom of the norm is easily established by means of the summation inequality of Minkovsky; consequently, $B^5_{2,T}$ is a linear normalized space. Now prove its completeness. Let

$$u_n(x,t) = \sum_{k=0}^{\infty} u_{k,n}(t) \cos \lambda_k x \quad (n = 1, 2, \ldots) \tag{42}$$

be any sequence fundamental in $B^5_{2,T}$. Then for any $\varepsilon > 0$ there exists a number n_ε such that

$$\begin{aligned} &\|u_n(x,t) - u_m(x,t)\|_{B^5_{2,T}} \\ &\quad = \|u_{0,n}(t) - u_{0,m}(t)\|_{C[0,T]} \\ &\qquad + \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} < \varepsilon \\ &\qquad\qquad \forall n, m \geq n_\varepsilon. \end{aligned} \tag{43}$$

Consequently, for any fixed k ($k = 1, 2, \ldots$),

$$\begin{aligned} &\|u_{0,n}(t) - u_{0,m}(t)\|_{C[0,T]} < \varepsilon, \\ &\|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} < \varepsilon \quad \forall n, m \geq n_\varepsilon. \end{aligned} \tag{44}$$

This means that for the sequence $\{u_{0,n}(t)\}_{n-1}^{\infty}$ and for any fixed k ($k = 1, 2, \ldots$), the sequences $\{u_{0,n}(t)\}_{n-1}^{\infty}$ are fundamental in $C[0,T]$ and consequently by the completeness of $C[0,T]$ they converge in the space $C[0,T]$:

$$\begin{aligned} &u_{0,n}(t) \overset{C[0,T]}{\longrightarrow} u_{0,0}(t) \in C[0,T] \text{ as } n \longrightarrow \infty, \\ &u_{k,n}(t) \overset{C[0,T]}{\longrightarrow} u_{k,0}(t) \in C[0,T] \text{ as } n \longrightarrow \infty. \end{aligned} \tag{45}$$

Further, by (43), for any fixed number N,

$$\begin{aligned} &\|u_{0,n}(t) - u_{0,m}(t)\|_{C[0,T]} \\ &\quad + \left(\sum_{k=1}^{N} \left(\lambda_k^5 \|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} < \varepsilon \\ &\qquad\qquad \forall n, m \geq n_\varepsilon. \end{aligned} \tag{46}$$

Using relations (45) and passing to limit as $m \to \infty$ in (46), we get

$$\begin{aligned} &\|u_{0,n}(t) - u_{0,0}(t)\|_{C[0,T]} \\ &\quad + \left(\sum_{k=1}^{N} \left(\lambda_k^5 \|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} < \varepsilon \\ &\qquad\qquad \forall n \geq n_\varepsilon. \end{aligned} \tag{47}$$

Hence, by arbitrariness of N (or the same, passing to limit as $N \to \infty$), we get

$$\begin{aligned} &\|u_{0,n}(t) - u_{0,0}(t)\|_{C[0,T]} \\ &\quad + \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_{k,n}(t) - u_{k,0}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} < \varepsilon \\ &\qquad\qquad \forall n \geq n_\varepsilon. \end{aligned} \tag{48}$$

Accept the denotation

$$u_0(x,t) = \sum_{k=0}^{\infty} u_{k,0}(t) \cos \lambda_k x. \tag{49}$$

Since $u_0(x,t) = [u_0(x,t) - u_{n_\varepsilon}(x,t)] + u_{n_\varepsilon}(x,t)$ and by (48) $u_0(x,t) - u_{n_\varepsilon}(x,t) \in B^5_{2,T}$ and also $u_{n_\varepsilon}(x,t) \in B^5_{2,T}$, we get that

$$u_0(x,t) \in B^5_{2,T}. \tag{50}$$

Then, by (48) for any $\varepsilon > 0$ there exists a number n_ε such that

$$\|u_n(x,t) - u_0(x,t)\|_{B^5_{2,T}} \leq \varepsilon \quad \forall n \geq n_\varepsilon. \tag{51}$$

And this means that the sequence $u_n(x,t)$ converges in $B^5_{2,T}$ to the element $u_0(x,t) \in B^5_{2,T}$. This proves the completeness and consequently the Banach property of the space $B^5_{2,T}$.

Denote by E^5_T the space $B^5_{2,T} \times C[0,T]$ of the vector functions $z(x,t) = \{u(x,t), a(t)\}$ with the norm

$$\|z(x,t)\|_{E^5_T} = \|u(x,t)\|_{B^5_{2,T}} + \|a(t)\|_{C[0,T]}. \tag{52}$$

It is known that $B^5_{2,T}$ and E^5_T are Banach spaces.

Now, in the space E^5_T consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}, \tag{53}$$

where

$$\Phi_1(u,a) = \widetilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \widetilde{u}_k(t)\cos\lambda_k x, \qquad \Phi_2(u,a) = \widetilde{a}(t), \tag{54}$$

$\widetilde{u}_0(t), \widetilde{u}_k(t)$ $(k = 1,2,\ldots)$, and $\widetilde{a}(t)$ are equal to the right hand sides of (28), (29), and (35), respectively.

It is easy to see that

$$ch2\rho_k T + \cos 2\rho_k T \geq \frac{1}{2} ch2\rho_k T,$$

$$\frac{ch\rho_k t}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(2T-t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq T),$$

$$\frac{ch\rho_k(T-t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(T+t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq T),$$

$$\frac{sh\rho_k t}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(2T-t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq T),$$

$$\frac{sh\rho_k(T-t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(T+t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq T),$$

$$\frac{ch\rho_k(2T+t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(2T-t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq \tau \leq T),$$

$$\frac{sh\rho_k(2T+t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(2T-t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq \tau \leq T),$$

$$\frac{ch\rho_k(t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq \tau \leq T),$$

$$\frac{sh\rho_k(t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq t \leq \tau \leq T),$$

$$\frac{ch\rho_k(2T-t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(2T-t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq \tau \leq t \leq T),$$

$$\frac{sh\rho_k(2T-t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(2T-t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq \tau \leq t \leq T),$$

$$\frac{ch\rho_k(t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{ch\rho_k(t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq \tau \leq t \leq T),$$

$$\frac{sh\rho_k(t+\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2, \quad \frac{sh\rho_k(t-\tau)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2 \quad (0 \leq \tau \leq t \leq T). \tag{55}$$

Taking into account these relations, by means of simple transformations we find

$$\begin{aligned}
&\|\widetilde{u}_0(t)\|_{C[0,T]} \\
&\leq |\varphi_{00}| + T|\varphi_{01}| + T^2|\varphi_{02}| + T^3|\varphi_{03}| \\
&\quad + \frac{7}{3}T^3\sqrt{T}\left(\int_0^t |f_0(\tau)|^2 d\tau\right)^{1/2} \\
&\quad + \frac{7}{3}T^4\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}, \\
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|\widetilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{1/2} \\
&\leq \frac{9\sqrt{6}}{4}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{0k}|\right)^2\right)^{1/2} \\
&\quad + \frac{\sqrt{6}}{4}(5T+8)\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{1k}|\right)^2\right)^{1/2} \\
&\quad + \frac{\sqrt{6}}{4}\left(5T^2+2\sqrt{2}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{2k}|\right)^2\right)^{1/2} \\
&\quad + \frac{\sqrt{6}}{4}\left(5T^3+8\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{3k}|\right)^2\right)^{1/2} \\
&\quad + 16\sqrt{3T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2|f_k(\tau)|\right)^2 d\tau\right)^{1/2} \\
&\quad + 16\sqrt{3}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\|\widetilde{a}(t)\|_{C[0,T]} &\le \left\|h^{-1}(t)\right\|_{C[0,T]} \\
&\times \left\{ \left\|h^{(4)}(t) - f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{1/2} \right. \\
&\quad \times \left[\frac{9}{4} \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_{0k}|\right)^2 \right)^{1/2} + \frac{1}{4}(5T+8) \right. \\
&\qquad \times \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_{1k}|\right)^2 \right)^{1/2} + \frac{1}{4}\left(5T^2 + 2\sqrt{2}\right) \\
&\qquad \times \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_{2k}|\right)^2 \right)^{1/2} + \frac{1}{4}\left(5T^3 + 8\right) \\
&\qquad \times \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_{3k}|\right)^2 \right)^{1/2} + 8\sqrt{2T} \\
&\qquad \times \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^2 |f_k(\tau)|\right)^2 d\tau \right)^{1/2} + 8\sqrt{2}T \|a(t)\|_{C[0,T]} \\
&\qquad \left. \left. \times \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_k(t)\|_{C[0,T]}\right)^2 \right)^{1/2} \right] \right\}.
\end{aligned} \tag{56}$$

Suppose that the data of problems (1)–(3), (10), and (11) satisfy the following conditions:

(1) $\varphi_i(x) \in C^4[0,1]$, $\varphi_i^{(5)}(x) \in L_2(0,1)$, and $\varphi_i'(0) = \varphi_i'(1) = \varphi_i'''(0) = \varphi_i''(1) = \varphi_i'''(0) = \varphi_i'''(1) = \varphi_i^{(4)}(0) = \varphi_i^{(4)}(1) = 0$ $(i = \overline{0,3})$.

(2) $f(x,t)$, $f_x(x,t) \in C(D_T)$, $f_{xx}(x,t) \in L_2(D_T)$, and $f_x(0,t) = f_x(1,t) = 0$ $(0 \le t \le T)$.

(3) $h(t) \in C^4[0,T]$, $h(t) \ne 0$ $(0 \le t \le T)$.

Then from (56), we get

$$\begin{aligned}
\|\widetilde{u}(x,t)\|_{B_{2,T}^5} &\le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^5}, \\
\|\widetilde{a}(t)\|_{C[0,T]} &\le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^5},
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
A_1(T) &= \|\varphi_0(x)\|_{L_2(0,1)} + T\|\varphi_1(x)\|_{L_2(0,1)} + T^2\|\varphi_2(x)\|_{L_2(0,1)} \\
&\quad + T^3\|\varphi_3(x)\|_{L_2(0,1)} + \frac{7}{3}T^3\sqrt{T}\|f(x,t)\|_{L_2(D_T)} \\
&\quad + \frac{9\sqrt{6}}{4}\left\|\varphi_0^{(6)}(x)\right\|_{L_2(0,1)} + \frac{\sqrt{6}}{4}(5T+8) \\
&\quad \times \left\|\varphi_1^{(5)}(x)\right\|_{L_2(0,1)} + \frac{\sqrt{6}}{4}\left(5T^2 + 2\sqrt{2}\right) \\
&\quad \times \left\|\varphi_2^{(5)}(x)\right\|_{L_2(0,1)} + \frac{\sqrt{6}}{4}\left(5T^3 + 8\right)\left\|\varphi_3^{(5)}(x)\right\|_{L_2(0,1)} \\
&\quad + 16\sqrt{3T}\|f_{xx}(x,t)\|_{L_2(D_T)}, \\
B_1(T) &= 16\sqrt{3}T + \frac{7}{3}T^4, \\
A_2(T) &= \left\|h^{-1}(t)\right\|_{C[0,T]} \\
&\quad \times \left\{ \left\|h^{(4)}(t) - f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{1/2} \right. \\
&\qquad \times \left[\frac{1}{4}\left\|\varphi_0^{(5)}(x)\right\|_{L_2(0,1)} + \frac{1}{4}(5T+8) \right. \\
&\qquad \times \left\|\varphi_1^{(5)}(x)\right\|_{L_2(0,1)} + \frac{1}{4}\left(5T^2 + 2\sqrt{2}\right) \\
&\qquad \times \left\|\varphi_2^{(5)}(x)\right\|_{L_2(0,1)} + \frac{1}{4}\left(5T^3 + 8\right) \\
&\qquad \times \left\|\varphi_3^{(5)}(x)\right\|_{L_2(0,1)} + 8\sqrt{2T} \\
&\qquad \left. \left. \times \|f_{xx}(x,t)\|_{L_2(D_T)} \right] \right\}, \\
B_2(T) &= \left\|h^{-1}(t)\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{1/2} 8\sqrt{2}T.
\end{aligned} \tag{58}$$

From inequalities (57) we get

$$\begin{aligned}
&\|\widetilde{u}(x,t)\|_{B_{2,T}^5} + \|\widetilde{a}(t)\|_{C[0,T]} \\
&\quad \le A(T) + B(T)\|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^5},
\end{aligned} \tag{59}$$

where

$$A(T) = A_1(T) + A_2(T), \qquad B(T) = B_1(T) + B_2(T). \tag{60}$$

So we can prove the following theorem.

Theorem 6. *Let conditions (1)–(3) be fulfilled and*

$$(A(T) + 2)^2 B(T) < 1. \tag{61}$$

Then problems (1)–(3), (10), and (11) have a unique solution in the ball $K = K_R(\|z\|_{E_T^5} \le R = A(T) + 2)$ of the space E_T^5.

Proof. In the space E_T^5 consider

$$z = \Phi z, \tag{62}$$

where the components $\Phi_i(u,a)$ $(i = 1,2)$ of the operator $\Phi(u,a)$ are defined from the right sides of (32) and (35).

Consider the operator $\Phi(u,a)$ in the ball $K = K_R$ from E_T^3. Similarly to (59), we get that for any $z, z_1, z_2 \in K_R$ the following estimates are valid:

$$\begin{aligned}
\|\Phi z\|_{E_T^5} &\le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^5} \\
&\le A(T) + B(T)\,R^2 \\
&\le A(T) + B(T)\,(A(T)+2)^2, \\
&\left\|\Phi z_1 - \Phi z_2\right\|_{E_T^5} \\
&\le B(T)\,R\left(\left\|a_1(t) - a_2(t)\right\|_{C[0,T]}\right. \\
&\qquad \left. + \left\|u_1(x,t) - u_2(x,t)\right\|_{B_{2,T}^5}\right) \\
&\le B(T)\,(A(T)+2)\left\|z_1 - z_2\right\|_{E_T^3}.
\end{aligned} \tag{63}$$

Then taking into account (61) in (63) it follows that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u,a\}$ that is a unique solution of (62) in the ball $K = K_R$; that is, it is a unique solution of systems (32), (35) in the ball $K = K_R$.

The function $u(x,t)$ as an element of the space $B_{2,T}^5$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t) u_{xx}(x,t), u_{xxxx}(x,t)$ in D_T.

From (25) it in easy to see that

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} \\
&\le \sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2} \\
&\quad + \sqrt{2}\left\|\left\|a(t)\,u_x(x,t) + f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}.
\end{aligned} \tag{64}$$

Hence it follows that $u_{tttt}(x,t)$ is continuous in D_T.

It is easy to verify that (1) and conditions (2), (3), (10), and (11) are satisfied in the ordinary sense.

Consequently, $u(x,t), a(t)$ is a solution of problems (1)–(3), (10), and (11), and by Lemma 4 it is unique in the ball $K = K_R$. The theorem is proved. □

By means of Lemma 3, a unique solvability of initial problems (1)–(5) follows from the last theorem.

Theorem 7. *Let all the conditions of Theorem 6 be fulfilled:*

$$\begin{aligned}
&\int_0^1 f(x,t)\,dx = 0 \quad (0 \le t \le T), \\
&\int_0^1 \varphi_i(x)\,dx = 0 \quad \left(i = \overline{0,3}\right), \\
&\varphi_0(0) = h(0), \quad \varphi_1(0) = h'(T), \quad \varphi_2(0) = h''(0), \\
&\varphi_3(0) = h'''(T), \quad \frac{5}{12}(A(T)+2)\,T^4 < 1.
\end{aligned} \tag{65}$$

Then problems (1)–(5) have a unique classic solution in the ball $K = K_R(\|z\|_{E_T^5} \le R = A(T) + 2)$ of the space E_T^5.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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