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Research Article

Almost Automorphic Random Functions in Probability

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We introduce the notion of almost automorphic random functions in probability. Some basic and fundamental properties of such functions are established.

1. Introduction

Since almost periodic random functions in probability are introduced in [1], several authors have made contributions on such almost periodic random functions (see, e.g., [2–5] and references therein). In fact, almost periodic random functions in probability are a natural generalization of the deterministic almost periodic functions.

On the other hand, just like that almost periodic functions are an important generalization of continuous periodic functions (see, e.g., [6], where it is shown that the space of continuous periodic functions is a set of first category in $AP(X)$), almost automorphic functions are an important generalization of almost periodic functions (cf. [7, 8] for some basic results and applications about almost automorphic functions). However, to the best of our knowledge, the notion of almost automorphic random functions in probability has not been introduced and studied. So, in this paper, we aim to study some basic and fundamental properties of almost automorphic random functions in probability.

It is needed to note that another kind of almost periodic random functions, which is called pth mean almost periodic random functions, has been introduced and studied by Bezandry and Diagana. We refer the reader to the monograph of Bezandry and Diagana [9] for a detailed knowledge on pth mean almost periodic random functions. In addition, the notion of square-mean almost automorphic random functions is introduced recently by Fu and Liu [10]; the notion of square-mean pseudo almost automorphic random functions is introduced by Chen and Lin [11], and the concept of distributional almost automorphy for stochastic processes is introduced by Fu and Chen in [12]. For other results concerning such functions and their applications, we refer the reader to [13–15] and references therein for some recent works. But, as one will see in the next section, our notion of almost automorphic random functions in probability is different from the notion of square-mean almost automorphic random functions.

2. Almost Automorphic Random Functions in Probability

Throughout the rest of this paper, let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of positive integers.

Definition 1. A random function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is called almost automorphic in probability provided that, for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\} \subset \{s'_n\}$ and a random function $g : \mathbb{R} \times \Omega \to \mathbb{R}$, such that, for every $t \in \mathbb{R}$, $\varepsilon > 0$, and $\eta > 0$, there corresponds a positive integer $N(t, \varepsilon, \eta)$ with the property that, for all $n > N$,

$$\begin{aligned} P\{\omega; |f(t+s_n,\omega) - g(t,\omega)| \ge \varepsilon\} < \eta, \\ P\{\omega; |g(t-s_n,\omega) - f(t,\omega)| \ge \varepsilon\} < \eta. \end{aligned} \tag{1}$$

We denote all such functions by $AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$.

Remark 2. In the above definition, we do not assume that f is continuous in probability on $\mathbb{R}$ since our main interest here is the recurrent property of such functions.

Before we make further study on the properties of almost automorphic random functions in probability, we would like first to compare our notion with the notion of square-mean almost automorphism.

Definition 3 (see [10]). A random function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is called square-mean almost automorphic provided that $\tilde{f} \in AA(\mathbb{R}, L^2(\Omega, \mathbb{R}))$, where

$$\left[\tilde{f}(t)\right](\omega) = f(t, \omega), \quad t \in \mathbb{R}, \ \omega \in \Omega, \tag{2}$$

and $AA(\mathbb{R}, L^2(\Omega, \mathbb{R}))$ means the space of all classical almost automorphic functions from $\mathbb{R}$ to $L^2(\Omega, \mathbb{R})$ (cf. [7]). For convenience, we denote the set of all such functions by $AA(\mathbb{R}, L^2(\Omega, \mathbb{R}))$.

Remark 4. It is not difficult to show that if $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$ and if f is continuous in probability on $\mathbb{R}$, then $f \in AA(\mathbb{R}, L^2(\Omega, \mathbb{R}))$ by using the definitions. However, the contrary is not true in general. In fact, for a random function from $\mathbb{R} \times \Omega$ to $\mathbb{R}$, convergence in probability does not necessarily mean square-mean convergence.

Definition 5. A random function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be bounded in probability, if for every $\eta > 0$ there exists a number $M > 0$ such that

$$P\{\omega; |f(t, \omega)| \geq M\} < \eta \quad \forall t \in \mathbb{R}. \tag{3}$$

Theorem 6. *Let $f, \overline{f} \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$. Then, the following assertions hold true:*

(i) $f + \overline{f} \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$;

(ii) *for every* $c \in \mathbb{R}$, $cf \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$;

(iii) *for every* $a \in \mathbb{R}$, $f(\cdot + a, \cdot) \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$;

(iv) f *is bounded in probability, and* g *is also bounded in probability, where* g *is the function in Definition 1;*

(v) $f \cdot \overline{f} \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$;

(vi) $f/\overline{f} \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$ *provided that* $1/\overline{f}$ *is bounded in probability;*

(vii) *if there exists a constant* $\alpha \in \mathbb{R}$ *such that for every* $t > \alpha$

$$P\{\omega \in \Omega; f(t, \omega) = 0\} = 1, \tag{4}$$

then for every $t \in \mathbb{R}$, $P\{\omega \in \Omega; f(t, \omega) = 0\} = 1$.

Proof. One can show (i)–(iii) by the definition of $AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$. We omit the details here. Next, let us prove (iv). We prove it by contradiction. Suppose that f is not bounded in probability. Then, there exists a number $\eta_0 > 0$ such that for every $n \in \mathbb{N}$, there corresponds a number $t_n \in \mathbb{R}$ with the property that

$$P\{\omega; |f(t_n, \omega)| \geq n\} \geq \eta_0. \tag{5}$$

Since $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$, there exist a subsequence of $\{t_n\}$ (for convenience, we still denote it by $\{t_n\}$) and a random function $g : \mathbb{R} \times \Omega \to \mathbb{R}$, such that for the above η_0, there corresponds a positive integer N_1 with the property that

$$P\{\omega; |f(t_n, \omega) - g(0, \omega)| \geq 1\} < \frac{\eta_0}{3}, \quad n > N_1. \tag{6}$$

On the other hand, it is easy to see that there exists a positive integer $N_2 > 0$ such that

$$P\{\omega; |g(0, \omega)| \geq n - 1\} < \frac{\eta_0}{3}, \quad n > N_2. \tag{7}$$

Combining the above two inequalities, we get for all $n > \max\{N_1, N_2\}$

$$\begin{aligned} P\{\omega; |f(t_n, \omega)| \geq n\} &\leq P\{\omega; |g(0, \omega)| \geq n - 1\} \\ &\quad + P\{\omega; |f(t_n, \omega) - g(0, \omega)| \geq 1\} \\ &< \frac{2\eta_0}{3}, \end{aligned} \tag{8}$$

which contradicts with (5). In addition, the boundedness of g follows from the boundedness of f and the fact that, for every $t \in \mathbb{R}$ and $\eta > 0$, there exists $s_{t,\eta} \in \mathbb{R}$ such that

$$P\left\{\omega; \left|f\left(t + s_{t,\eta}, \omega\right) - g(t, \omega)\right| \geq 1\right\} < \eta. \tag{9}$$

Now, let us prove (v). By the definition, we know that for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\} \subset \{s'_n\}$ and two random functions $g, \overline{g} : \mathbb{R} \times \Omega \to \mathbb{R}$, such that, for every $t \in \mathbb{R}$, $\varepsilon > 0$, and $\eta > 0$, there corresponds a positive integer $N(t, \varepsilon, \eta)$ with the property that for all $n > N$

$$\begin{aligned} &P\{\omega; |f(t + s_n, \omega) - g(t, \omega)| \geq \varepsilon\} < \eta, \\ &P\left\{\omega; \left|\overline{f}(t + s_n, \omega) - \overline{g}(t, \omega)\right| \geq \varepsilon\right\} < \eta, \\ &P\{\omega; |g(t - s_n, \omega) - f(t, \omega)| \geq \varepsilon\} < \eta, \\ &P\left\{\omega; \left|\overline{g}(t - s_n, \omega) - \overline{f}(t, \omega)\right| \geq \varepsilon\right\} < \eta. \end{aligned} \tag{10}$$

By (iv), we know that for every $\eta > 0$ there exists a number $M_\eta > 0$ such that

$$\begin{aligned} &P\left\{\omega; \left|\overline{f}(t, \omega)\right| \geq M_\eta\right\} < \frac{\eta}{4}, \\ &P\left\{\omega; |g(t, \omega)| \geq M_\eta\right\} < \frac{\eta}{4}, \end{aligned} \tag{11}$$

for all $t \in \mathbb{R}$. Then, for every $t \in \mathbb{R}$, $\varepsilon > 0$, $\eta > 0$, and $n > N(t, \varepsilon/2M_\eta, \eta/4)$, we have

$$\begin{aligned} &P\left\{\omega; \left|f(t+s_n,\omega)\cdot\overline{f}(t+s_n,\omega)-g(t,\omega)\cdot\overline{g}(t,\omega)\right|\geq\varepsilon\right\}\\ &\leq P\left\{\omega;\ \left|f(t+s_n,\omega)\cdot\overline{f}(t+s_n,\omega)\right.\right.\\ &\qquad\left.\left.-g(t,\omega)\cdot\overline{f}(t+s_n,\omega)\right|\geq\frac{\varepsilon}{2}\right\}\\ &\quad+P\left\{\omega;\left|g(t,\omega)\cdot\overline{f}(t+s_n,\omega)-g(t,\omega)\cdot\overline{g}(t,\omega)\right|\geq\frac{\varepsilon}{2}\right\}\\ &\leq P\left\{\omega;\left|f(t+s_n,\omega)-g(t,\omega)\right|\geq\frac{\varepsilon}{2M_\eta}\right\}\\ &\quad+P\left\{\omega;\left|\overline{f}(t+s_n,\omega)\right|\geq M_\eta\right\}\\ &\quad+P\left\{\omega;\left|\overline{f}(t+s_n,\omega)-\overline{g}(t,\omega)\right|\geq\frac{\varepsilon}{2M_\eta}\right\}\\ &\quad+P\left\{\omega;\left|g(t,\omega)\right|\geq M_\eta\right\}<\eta, \end{aligned} \tag{12}$$

and by a similar argument, we can also get

$$P\left\{\omega;\left|g(t-s_n,\omega)\cdot\overline{g}(t-s_n,\omega)-f(t,\omega)\cdot\overline{f}(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{13}$$

Thus, $f\cdot\overline{f}\in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$.

The proof of (vi) is similar to that of (v). So we omit the details.

It remains to show (vii). By Definition 1, we can choose a subsequence $\{s_n\}$ with $\lim_{n\to\infty}s_n = +\infty$ and a random function $g:\mathbb{R}\times\Omega\to\mathbb{R}$, such that for every $t\in\mathbb{R}$, $\varepsilon>0$, and $\eta>0$ there corresponds a positive integer $N(t,\varepsilon,\eta)$ with the property that, for all $n>N$,

$$P\left\{\omega;\left|f(t+s_n,\omega)-g(t,\omega)\right|\geq\varepsilon\right\}<\eta, \tag{14}$$

$$P\left\{\omega;\left|g(t-s_n,\omega)-f(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{15}$$

Noting that $t+s_n>\alpha$ for sufficiently large n and

$$P\left\{\omega\in\Omega;f(t,\omega)=0\right\}=1,\quad t>\alpha, \tag{16}$$

we conclude by (14) that, for every $t\in\mathbb{R}$, $\varepsilon>0$, and $\eta>0$, there holds

$$P\left\{\omega;\left|g(t,\omega)\right|\geq\varepsilon\right\}<\eta, \tag{17}$$

which means that

$$P\left\{\omega;\left|g(t,\omega)\right|=0\right\}=1, \tag{18}$$

for every $t\in\mathbb{R}$. Combining this with (15), by a similar argument to the above proof, we can get

$$P\left\{\omega;\left|f(t,\omega)\right|=0\right\}=1, \tag{19}$$

for all $t\in\mathbb{R}$. □

Theorem 7. *Let $\{f_k\}\subset AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$ be uniformly convergent in probability on $\mathbb{R}$ to a random function $f:\mathbb{R}\times\Omega\to\mathbb{R}$. Then, $f\in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$.*

Proof. For every sequence of real numbers $\{s_n'\}$, by the diagonal method, one can choose a subsequence $\{s_n\}\subset\{s_n'\}$ such that, for every $k\in\mathbb{N}$, there is a random function $g_k:\mathbb{R}\times\Omega\to\mathbb{R}$, that satisfying for every $t\in\mathbb{R}$, $\varepsilon>0$, and $\eta>0$, there corresponds a positive integer $N(k,t,\varepsilon,\eta)$ with the property that, for all $n>N$,

$$P\left\{\omega;\left|f_k(t+s_n,\omega)-g_k(t,\omega)\right|\geq\varepsilon\right\}<\eta, \tag{20}$$

$$P\left\{\omega;\left|g_k(t-s_n,\omega)-f_k(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{21}$$

By using the fact that $\{f_k\}$ is uniformly convergent in probability on $\mathbb{R}$ to f, (20), and

$$\begin{aligned} &\left|g_k(t,\omega)-g_l(t,\omega)\right|\\ &\leq\left|g_k(t,\omega)-f_k(t+s_n,\omega)\right|+\left|f_k(t+s_n,\omega)-f_l(t+s_n,\omega)\right|\\ &\quad+\left|f_l(t+s_n,\omega)-g_l(t,\omega)\right|, \end{aligned} \tag{22}$$

we can prove that, for every $\varepsilon>0$ and $\eta>0$, there exists a positive integer $N'>0$ such that, for all $k,l>N'$ and $t\in\mathbb{R}$, there holds

$$P\left\{\omega;\left|g_k(t,\omega)-g_l(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{23}$$

Then, by [9, Proposition 3.7], there exists a subsequence (we still denote it by g_k for convenience) such that for every $t\in\mathbb{R}$, $\lim_{k\to\infty}g_k(t,\omega)$ exists for a.e. $\omega\in\Omega$.

Let $g(t,\omega)=\lim_{k\to\infty}g_k(t,\omega)$. Then, again by the fact that $\{f_k\}$ is uniformly convergent in probability on $\mathbb{R}$ to f, (20), and

$$\begin{aligned} &\left|f(t+s_n,\omega)-g(t,\omega)\right|\\ &\leq\left|f(t+s_n,\omega)-f_k(t+s_n,\omega)\right|+\left|f_k(t+s_n,\omega)-g_k(t,\omega)\right|\\ &\quad+\left|g_k(t,\omega)-g(t,\omega)\right|, \end{aligned} \tag{24}$$

we conclude that, for every $t\in\mathbb{R}$, $\varepsilon>0$, and $\eta>0$, there corresponds a positive integer $N(t,\varepsilon,\eta)$ with the property that, for all $n>N$,

$$P\left\{\omega;\left|f(t+s_n,\omega)-g(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{25}$$

Similarly, we can obtain

$$P\left\{\omega;\left|g(t-s_n,\omega)-f(t,\omega)\right|\geq\varepsilon\right\}<\eta. \tag{26}$$

Thus, $f\in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$. This completes the proof. □

Lemma 8. *Let $f\in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$. Then, for every $t\in\mathbb{R}$ and $\varepsilon,\eta>0$, there exist finite real numbers $s_1,\ldots,s_n$ such that*

$$\bigcup_{i=1}^{n}(s_i+C)=\mathbb{R}, \tag{27}$$

where

$$C = C(t, \varepsilon, \eta) = \{\tau \in \mathbb{R} : P\{\omega; |f(t+\tau, \omega) - f(t, \omega)| \geq \varepsilon\} < \eta\}. \tag{28}$$

Proof. We prove it by contradiction. Assume that there do not exist finite real numbers $s_1, \ldots, s_n$ such that

$$\bigcup_{i=1}^{n} (s_i + C) = \mathbb{R}. \tag{29}$$

Taking an arbitrary real number s_1, there exists a real number $s_2 \notin (s_1 + C)$; that is, $s_2 - s_1 \notin C$. Then, we can choose a real number $s_3 \notin (s_1 + C) \bigcup (s_2 + C)$; that is,

$$s_3 - s_1, s_3 - s_2 \notin C. \tag{30}$$

Continuing by this way, we can get a real number sequence $\{s_n\}$ satisfying that for every $n, k \in \mathbb{N}$ with $n > k$, there holds

$$s_n - s_k \notin C. \tag{31}$$

Now, since $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$, there exist $N_1 \in \mathbb{N}$ and a random function $g : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$P\left\{\omega; \left|g\left(t - s_{N_1}, \omega\right) - f(t, \omega)\right| \geq \frac{\varepsilon}{2}\right\} < \frac{\eta}{2}. \tag{32}$$

Again by $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$, one can choose a positive integer $N_2 > N_1$ such that

$$P\left\{\omega; \left|f\left(t - s_{N_1} + s_{N_2}, \omega\right) - g\left(t - s_{N_1}, \omega\right)\right| \geq \frac{\varepsilon}{2}\right\} < \frac{\eta}{2}. \tag{33}$$

Combining the above two inequalities, we get

$$P\left\{\omega; \left|f\left(t - s_{N_1} + s_{N_2}, \omega\right) - f(t, \omega)\right| \geq \varepsilon\right\} < \eta. \tag{34}$$

Then, we have $s_{N_2} - s_{N_1} \in C$, which contradicts with (31). This completes the proof. □

Theorem 9. *Let $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$ and let C be the set as in Lemma 8. Then, for every $L > 0$, there exists $L' > 0$ such that for every $a \in \mathbb{R}$ there holds*

$$\operatorname{mes}\left(C \cap \left[a, a + L'\right]\right) \geq L. \tag{35}$$

Proof. By Lemma 8, there exist finite real numbers $s_1, \ldots, s_n$ such that

$$\bigcup_{i=1}^{n} (s_i + C) = \mathbb{R}. \tag{36}$$

Fix $L > 0$. Let

$$L' = 2 \max_{1 \leq i \leq n} \left|s_i - t\right| + nL, \tag{37}$$

and for $a \in \mathbb{R}$ and $i = 1, 2, \ldots, n$,

$$C_i^a := \left[a + \max_{1 \leq i \leq n} \left|s_i - t\right| - s_i + t, \; a + \max_{1 \leq i \leq n} \left|s_i - t\right| + nL - s_i + t\right] \bigcap C. \tag{38}$$

It is easy to see that for every $a \in \mathbb{R}$ and $i = 1, 2, \ldots, n$, $C_i^a \subset [a, a + L'] \cap C$. So, for every $a \in \mathbb{R}$, we have

$$\begin{aligned}
&\operatorname{mes}\left(C \cap \left[a, a + L'\right]\right) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} \operatorname{mes} C_i^a \\
&= \frac{1}{n} \sum_{i=1}^{n} \operatorname{mes}\left(C_i^a + s_i - t\right) \\
&\geq \frac{1}{n} \operatorname{mes}\left[\bigcup_{i=1}^{n} \left(C_i^a + s_i - t\right)\right] \\
&= \frac{1}{n} \operatorname{mes}\left\{\left[a + \max_{1 \leq i \leq n} \left|s_i - t\right|, a + \max_{1 \leq i \leq n} \left|s_i - t\right| + nL\right] \right. \\
&\qquad \left. \bigcap \left(\bigcup_{i=1}^{n} \left(C + s_i - t\right)\right)\right\} = L.
\end{aligned} \tag{39}$$

□

Definition 10. Let $E \subset \mathbb{R}$. A random function $f : \mathbb{R} \times E \times \Omega \to \mathbb{R}$ is called almost automorphic in probability uniformly for $x \in E$ provided that for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\} \subset \{s_n'\}$ and a random function $g : \mathbb{R} \times E \times \Omega \to \mathbb{R}$, such that for every $t \in \mathbb{R}$, $x \in E$, $\varepsilon > 0$, and $\eta > 0$ there corresponds a positive integer $N(t, x, \varepsilon, \eta)$ with the property that, for all $n > N$,

$$\begin{aligned}
&P\{\omega; |f(t + s_n, x, \omega) - g(t, x, \omega)| \geq \varepsilon\} < \eta, \\
&P\{\omega; |g(t - s_n, x, \omega) - f(t, x, \omega)| \geq \varepsilon\} < \eta.
\end{aligned} \tag{40}$$

We denote all such functions by $AA\mathscr{R}(\mathbb{R} \times E \times \Omega, \mathbb{R})$. In addition, we denote by $\mathscr{H}(f)$ all the random functions g in the definition.

Next, for convenience, we denote $\operatorname{Lip}(\mathbb{R} \times E \times \Omega, \mathbb{R})$ by the set of all random functions $f : \mathbb{R} \times E \times \Omega \to \mathbb{R}$ satisfying that there exist a constant $L > 0$ and $\Omega' \subset \Omega$ with $P(\Omega') = 0$ such that

$$\left|f(t, x, \omega) - f(t, y, \omega)\right| \leq L\left|x - y\right|, \tag{41}$$

for all $t \in \mathbb{R}$, $x, y \in E$, and $\omega \in \Omega \setminus \Omega'$.

Lemma 11. *Let $f \in AA\mathscr{R}(\mathbb{R} \times \Omega, \mathbb{R})$. Then for every $\varepsilon, \eta > 0$ and $t \in \mathbb{R}$ there exists finite numbers $x_1, \ldots, x_n \in \mathbb{R}$ such that*

$$P\{\omega; |f(t, \omega) - x_i| \geq \varepsilon, i = 1, 2 \ldots, n\} < \eta. \tag{42}$$

Proof. Since

$$\lim_{M \to +\infty} P\{\omega; |f(t, \omega)| > M\} = 0, \tag{43}$$

there exists $M_0 > 0$ such that $P\{\omega; |f(t, \omega)| > M_0\} < \eta$. On the other hand, there exist finite numbers $x_1, \ldots, x_n \in \mathbb{R}$ such that $[-M_0, M_0] \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. Noting that

$$\{\omega; |f(t, \omega) - x_i| \geq \varepsilon, i = 1, 2 \ldots, n\} \subset \{\omega; |f(t, \omega)| > M_0\}, \tag{44}$$

we conclude that

$$P\{\omega; |f(t,\omega) - x_i| \geq \varepsilon, i = 1, 2\ldots, n\} < \eta. \tag{45}$$

□

Theorem 12. *Let $f \in AA\mathscr{R}(\mathbb{R}\times\mathbb{R}\times\Omega,\mathbb{R})$ with $\mathscr{H}(f) \subset \mathrm{Lip}(\mathbb{R}\times\mathbb{R}\times\Omega,\mathbb{R})$, and let $\varphi \in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$. Then, $F \in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$, where $F(t,\omega) = f(t,\varphi(t,\omega),\omega)$ for $t \in \mathbb{R}$ and $\omega \in \Omega$.*

Proof. Let $\{s_n'\}$ be a sequence of real numbers. Then, there exist a subsequence $\{s_n\} \subset \{s_n'\}$ and two random functions $g : \mathbb{R}\times E\times\Omega \to \mathbb{R}$ and $\psi : \mathbb{R}\times\Omega \to \mathbb{R}$. Here, g is the function in Definition 10 and ψ corresponds g in Definition 1 (φ corresponds f in Definition 1).

Fix $t \in \mathbb{R}$ and $\varepsilon,\eta > 0$. By Lemma 11, there exist finite numbers $x_1,\ldots,x_k \in \mathbb{R}$ such that

$$P\{\omega; |\psi(t,\omega) - x_i| \geq \varepsilon,\ i = 1, 2\ldots, k\} < \eta. \tag{46}$$

Since $\mathscr{H}(f) \subset \mathrm{Lip}(\mathbb{R}\times\mathbb{R}\times\Omega,\mathbb{R})$, there exist a constant $L > 0$ and $\Omega' \subset \Omega$ with $P(\Omega') = 0$ such that

$$\begin{gathered} |f(t,x,\omega) - f(t,y,\omega)| \leq L|x-y|, \\ |g(t,x,\omega) - g(t,y,\omega)| \leq L|x-y|, \end{gathered} \tag{47}$$

for all $t \in \mathbb{R}$, $x, y \in E$, and $\omega \in \Omega\setminus\Omega'$. In addition, there exists $N > 0$ such that, for all $n > N$, there hold

$$\begin{gathered} P\{\omega; |\varphi(t+s_n,\omega) - \psi(t,\omega)| \geq 3\varepsilon\} < \eta, \\ P\{\omega; |f(t+s_n,x_i,\omega) - g(t,x_i,\omega)| \geq L\varepsilon\} < \frac{\eta}{k}, \quad i = 1, 2\ldots, k. \end{gathered} \tag{48}$$

Let

$$\begin{gathered} \Omega_0 = \{\omega; |\psi(t,\omega) - x_i| \geq \varepsilon,\ i = 1, 2\ldots, k\}, \\ \Omega_i = \{\omega; |\psi(t,\omega) - x_i| < \varepsilon\}, \quad i = 1, 2\ldots, k. \end{gathered} \tag{49}$$

Then we have $P(\Omega_0) < \eta$ and $\bigcup_{i=0}^{k}\Omega_i = \Omega$. Combining all the above assertions, for all $n > N$, we have

$$\begin{aligned} &P\{\omega; |f(t+s_n,\varphi(t+s_n,\omega),\omega) - g(t,\psi(t,\omega),\omega)| \geq 6L\varepsilon\} \\ &\leq P\{\omega; |f(t+s_n,\varphi(t+s_n,\omega),\omega) \\ &\qquad - f(t+s_n,\psi(t,\omega),\omega)| \geq 3L\varepsilon\} \\ &\quad + P\{\omega; |f(t+s_n,\psi(t,\omega),\omega) - g(t,\psi(t,\omega),\omega)| \geq 3L\varepsilon\} \\ &\leq P\{\omega; |\varphi(t+s_n,\omega) - \psi(t,\omega)| \geq 3\varepsilon\} \\ &\quad + P\{\omega; |f(t+s_n,\psi(t,\omega),\omega) - g(t,\psi(t,\omega),\omega)| \geq 3L\varepsilon\} \\ &\leq \eta + P\left(\bigcup_{i=0}^{k}\Omega_i \cap \{\omega; |f(t+s_n,\psi(t,\omega),\omega) \right. \\ &\qquad \left. -g(t,\psi(t,\omega),\omega)| \geq 3L\varepsilon\}\right) \\ &\leq \eta + \sum_{i=0}^{k} P(\Omega_i \cap \{\omega; |f(t+s_n,\psi(t,\omega),\omega) \\ &\qquad - g(t,\psi(t,\omega),\omega)| \geq 3L\varepsilon\}) \\ &\leq 2\eta + \sum_{i=1}^{k} P(\Omega_i \cap \{\omega; |f(t+s_n,\psi(t,\omega),\omega) \\ &\qquad -g(t,\psi(t,\omega),\omega)| \geq 3L\varepsilon\}) \\ &\leq 2\eta + \sum_{i=1}^{k} P(\Omega_i \cap \{\omega; |f(t+s_n,x_i,\omega) - g(t,x_i,\omega)| \geq L\varepsilon\}) \\ &\leq 2\eta + \sum_{i=1}^{k} P\{\omega; |f(t+s_n,x_i,\omega) - g(t,x_i,\omega)| \geq L\varepsilon\} \\ &\leq 2\eta + \sum_{i=1}^{k}\frac{\eta}{k} = 3\eta. \end{aligned} \tag{50}$$

Similarly, one can show that

$$\begin{aligned} &P\{\omega; |g(t-s_n,\psi(t-s_n,\omega),\omega) - f(t,\varphi(t,\omega),\omega)| \geq 6L\varepsilon\} \\ &\quad \leq 3\eta, \end{aligned} \tag{51}$$

for sufficiently large n. Thus, we conclude that $F \in AA\mathscr{R}(\mathbb{R}\times\Omega,\mathbb{R})$. □

Remark 13. Let $f \in \mathrm{Lip}(\mathbb{R}\times\mathbb{R}\times\Omega,\mathbb{R})$ and $g \in \mathscr{H}(f)$. Then, we can obtain that there exists a constant $L > 0$ such that, for every $t \in \mathbb{R}$ and $x, y \in E$, there holds

$$P\{\omega \in \Omega; |g(t,x,\omega) - g(t,y,\omega)| \leq L|x-y|\} = 1. \tag{52}$$

But, we are not sure if $g \in \mathrm{Lip}(\mathbb{R}\times\mathbb{R}\times\Omega,\mathbb{R})$. We leave it as a problem to the reader.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] O. Onicescu and V. I. Istrătescu, "Approximation theorems for random functions," *Rendiconti di Matematica*, vol. 8, pp. 65–81, 1975.

[2] Gh. Cenuşă and I. Săcuiu, "Some properties of random functions almost periodic in probability," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 25, no. 9, pp. 1317–1325, 1980.

[3] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, NY, USA, 2nd edition, 1989.

[4] C. Deng and H. S. Ding, "Vector valued almost periodic random functionsinprobability," *Journal of Nonlinear Evolution Equations and Applications*. In press.

[5] Y. Han and J. Hong, "Almost periodic random sequences in probability," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 962–974, 2007.

[6] Z.-M. Zheng, H.-S. Ding, and G. M. N'Guérékata, "The space of continuous periodic functions is a set of first category in $AP(X)$," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 275702, 3 pages, 2013.

[7] G. M. N'Guerekata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic, New York, NY, USA, 2001.

[8] G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, NY, USA, 2005.

[9] P. H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*, Springer, New York, NY, USA, 2011.

[10] M. Fu and Z. Liu, "Square-mean almost automorphic solutions for some stochastic differential equations," *Proceedings of the American Mathematical Society*, vol. 138, no. 10, pp. 3689–3701, 2010.

[11] Z. Chen and W. Lin, "Square-mean pseudo almost automorphic process and its application to stochastic evolution equations," *Journal of Functional Analysis*, vol. 261, no. 1, pp. 69–89, 2011.

[12] M. Fu and F. Chen, "Almost automorphic solutions for some stochastic differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 80, pp. 66–75, 2013.

[13] P. Bezandry and T. Diagana, "Square-mean almost periodic solutions to some classes of nonautonomous stochastic evolution equations with finite delay," *Journal of Applied Functional Analysis*, vol. 7, no. 4, pp. 345–366, 2012.

[14] Y.-K. Chang, R. Ma, and Z.-H. Zhao, "Almost periodic solutions to a stochastic differential equation in Hilbert spaces," *Results in Mathematics*, vol. 63, no. 1-2, pp. 435–449, 2013.

[15] H. Zhou, Z. Zhou, and Z. Qiao, "Mean-square almost periodic solution for impulsive stochastic Nicholson's blowflies model with delays," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5943–5948, 2013.