Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 205875, 6 pages
http://dx.doi.org/10.1155/2014/205875

Research Article

Strong Convergence on Iterative Methods of Cesàro Means for Nonexpansive Mapping in Banach Space

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Received 23 March 2014; Accepted 4 May 2014; Published 12 May 2014

Academic Editor: Jen-Chih Yao

Two new iterations with Cesàro's means for nonexpansive mappings are proposed and the strong convergence is obtained as $n \to \infty$. Our main results extend and improve the corresponding results of Xu (2004), Song and Chen (2007), and Yao et al. (2009).

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E and let T be nonexpansive mapping from C into itself (recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in C$). We denote fixed points of T as $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$.

Recall that a mapping $f : C \to C$ is contractive if there exists a constant $\beta \in (0,1)$ such that $\|f(x) - f(y)\| \le \beta\|x - y\|$, $\forall x, y \in C$.

In 1975, Baillon [1] proved the first nonlinear ergodic theorem.

Theorem 1. *Suppose that C is a nonempty closed convex subset of Hilbert space E and $T : C \to C$ mapping such that $F(T) \ne \phi$; then $\forall x \in C$, and the Cesàro means*

$$T_n x = \frac{1}{n+1} \sum_{i=0}^{n} T^i x \tag{1}$$

weakly converges to a fixed point of T.

In 1979, Bruck [2] showed the nonlinear ergodic theorem for nonexpansive mapping in uniformly convex Banach space with *Fréchet* differentiable norms.

In 2004, Xu [3] introduced the following viscosity iterative scheme $\{x_n\}$ given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \tag{2}$$

where parameter $\{\alpha_n\} \subset [0,1]$, satisfying

(X1) $\lim_{n\to\infty} \alpha_n = 0$;
(X2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(X3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

He proved that the explicit iterative scheme $\{x_n\}$ converges strongly to a fixed point p of T in uniformly smooth Banach space.

In 2007, Song and Chen [4] defined the following viscosity iteration $\{x_n\}$ of Cesàro means for nonexpansive mapping T:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{i=0}^{n} T^i x_n, \tag{3}$$

and they proved that the sequence $\{x_n\}$ converges strongly to some point in $F(T)$ in a uniformly convex Banach space with weakly sequentially continuous duality mapping.

In 2009, Yao et al. [5] introduced the following process $\{x_n\}$:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \ge 0. \tag{4}$$

They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T under the following control conditions of parameters:

(YLZ1) $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \ge 0$;
(YLZ2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(YLZ3) $\lim_{n\to\infty} \gamma_n = 0$.

Motivated by the above results, we propose the following new iterations with Cesàro's means for nonexpansive mappings:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \frac{1}{n+1} \sum_{i=0}^{n} T^i x_n, \quad n \geq 0, \tag{5}$$

and viscosity iteration:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{1}{n+1} \sum_{i=0}^{n} T^i x_n, \quad n \geq 0. \tag{6}$$

Some examples are given to show the generation of our new iterations with Cesàro's means as follows.

Example 2. If $n = 1$, iteration (5) with Cesàro's means for nonexpansive mappings is $x_2 = \alpha_1 u + \beta_1 x_1 + \gamma_1 \cdot (1/2)(x_1 + Tx_1)$, which is reduced as the same iteration of Yao et al. [5]. If $n \geq 2$, iteration (5) can be written as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n \\ &\quad + \gamma_n \cdot \frac{1}{n+1} \left(x_n + Tx_n + T^2 x_n + \cdots + T^n x_n \right), \end{aligned} \tag{7}$$

which is a generation of Yao et al. [5].

Example 3. Let $E = R$ with the usual metric, nonexpansive mapping defined by $Tx = \sin x$, the fixed contractive mapping $f(x) = (1/2)x$, and the parameters are defined as $\alpha_n = 1/2n$, $\gamma_n = 1/4n$, and $\beta_n = (4n-3)/4n$. The new iterations with Cesàro's means which is related to iterative step n can be written as follows:

$$\begin{aligned} x_{n+1} &= \frac{1}{2n} u + \frac{4n-3}{4n} x_n \\ &\quad + \frac{1}{4n(n+1)} \left(x_n + Tx_n + T^2 x_n + \cdots + T^n x_n \right), \\ x_{n+1} &= \frac{1}{2n} \cdot \frac{1}{2} x_n + \frac{4n-3}{4n} x_n \\ &\quad + \frac{1}{4n(n+1)} \left(x_n + Tx_n + T^2 x_n + \cdots + T^n x_n \right). \end{aligned} \tag{8}$$

2. Preliminaries

Throughout the paper, let E be a real Banach space with norm $\|\cdot\|$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^*, \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\| \right\}, \quad \forall x \in E, \tag{9}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued normalized duality mapping by j.

Let $S := \{x \in E : \|x\| = 1\}$ be the unit sphere of a Banach space. The space is said to have a Gâteaux differentiable norm (or E is said to be smooth), if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{10}$$

exists for every $x, y \in S$, and E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit (10) is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit (10) exists uniformly for $(x, y) \in S \times S$.

The following two results can be found in [6].

If E is smooth the duality mapping J is single-valued and strong-weak* continuous.

If E is Banach space with uniformly Gâteaux differentiable norm, then duality mapping $J : E \to E^*$ is single-valued and norm to weak star uniformly continuous on bounded sets of E.

In order to prove our main results, the following lemmas will be used.

Lemma 4 (see [7]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $z_t \in C$ of the contraction $C \ni x \vdash tu + (1-t)Tx$ as $t \to 0$ converges strongly to $x^* \in F(T)$ which is the nearest to u.*

Lemma 5 (see [1]). *Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \phi$, and $f : C \to C$ a fixed contraction. Then x_t defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a unique fixed point in $F(T)$ as $t \to 0$.*

Lemma 6 (see [3]). *Let E be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in E$, one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x+y) \rangle, \quad \forall j(x+y) \in J(x+y). \tag{11}$$

Lemma 7 (see [8]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \geq 0, \tag{12}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n \to \infty} \delta_n \leq 0$.

Then $\lim_{n \to \infty} a_n = 0$.

3. Main Results

Let C be a nonempty closed convex subset of a uniformly smooth Banach space. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \phi$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in (0,1) satisfying

(i) $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 0$;

(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n \to \infty} \gamma_n = 0$.

In the following, we will present the first main result. For each $n \geq 0$ and $t \in (0,1)$, let $x_{t,n}$ be the unique fixed point of the contractive mapping $T_{t,n}$ given by

$$T_{t,n}x = \frac{(1-\alpha_n)t}{\gamma_n + t\beta_n}u + \frac{(1-t)\gamma_n}{\gamma_n + t\beta_n}\frac{1}{n+1}\sum_{i=0}^{n}T^i x. \tag{13}$$

That is,

$$x_{t,n} = \frac{(1-\alpha_n)t}{\gamma_n + t\beta_n}u + \frac{(1-t)\gamma_n}{\gamma_n + t\beta_n}\frac{1}{n+1}\sum_{i=0}^{n}T^i x_{t,n}. \tag{14}$$

From Lemma 4, for fixed n, we have

$$\lim_{t\to 0} x_{t,n} = p \in F(T), \tag{15}$$

which is the unique fixed point.

Theorem 8. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \phi$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in (0,1) satisfying conditions (i)–(iii). Then, for given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by* (5). *Then the sequence $\{x_n\}$ defined by* (5) *converges strongly to a fixed point of T.*

Proof. Taking a fixed point $p \in F(T)$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \alpha_n\|u-p\| + \beta_n\|x_n - p\| \\ &\quad + \gamma_n\left\|\frac{1}{n+1}\sum_{i=0}^{n}\left(T^i x_n - T^i p\right)\right\| \\ &\leq \alpha_n\|u-p\| + \beta_n\|x_n-p\| + \gamma_n\|x_n-p\| \\ &= \alpha_n\|u-p\| + (1-\alpha_n)\|x_n-p\| \\ &\leq \max\{\|u-p\|, \|x_n-p\|\}.\end{aligned} \tag{16}$$

By induction, we get that $\{x_n\}$ is bounded. We observe that (14) can be rewritten as follows:

$$\begin{gathered} x_{t,n} = tu + (1-t)\left[\frac{\beta_n}{1-\alpha_n}x_{t,n} + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}T^i x_{t,n}\right], \\ \lim_{t\to 0} x_{t,n} = p \in F(T), \quad \forall n \geq 0.\end{gathered} \tag{17}$$

By (14), we have

$$\begin{aligned} x_{t,n} - x_n &= t(u - x_n) \\ &\quad + (1-t)\left[\frac{\beta_n}{1-\alpha_n}(x_{t,n} - x_n)\right. \\ &\quad \left. + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left(T^i x_{t,n} - x_n\right)\right].\end{aligned} \tag{18}$$

Applying Lemma 6 to (18), we have

$$\begin{aligned} &\|x_{t,n} - x_n\| \\ &\leq (1-t)^2\left\|\frac{\beta_n}{1-\alpha_n}(x_{t,n}-x_n)\right. \\ &\qquad \left. + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left(T^i x_{t,n} - x_n\right)\right\|^2 \\ &\quad + 2t\langle u - x_n, j(x_{t,n} - x_n)\rangle \\ &\leq (1-t)^2\left[\frac{\beta_n}{1-\alpha_n}\|x_{t,n}-x_n\|\right. \\ &\qquad + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_{t,n} - T^i x_n\right\| \\ &\qquad \left. + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|\right]^2 \\ &\quad + 2t\|x_{t,n}-x_n\|^2 + 2t\langle u - x_{t,n}, j(x_{t,n}-x_n)\rangle \\ &\leq (1-t)^2\left[\|x_{t,n}-x_n\| + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|\right]^2 \\ &\quad + 2t\|x_{t,n}-x_n\|^2 + 2t\langle u - x_{t,n}, j(x_{t,n}-x_n)\rangle \\ &= \left(1+t^2\right)\|x_{t,n}-x_n\|^2 \\ &\quad + (1-t)^2\left(\frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|\right)^2 \\ &\quad + 2(1-t)^2\frac{\gamma_n}{1-\alpha_n}\|x_{t,n}-x_n\|\left(\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|\right) \\ &\quad + 2t\langle u - x_{t,n}, j(x_{t,n}-x_n)\rangle \\ &\leq \left(1+t^2\right)\|x_{t,n}-x_n\|^2 + \frac{\gamma_n}{1-\alpha_n}Q \\ &\quad + 2t\langle u - x_{t,n}, j(x_{t,n}-x_n)\rangle,\end{aligned} \tag{19}$$

where Q is some constant such that

$$\begin{aligned}\sup&\left\{\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|\right. \\ &\left. +2\|x_{t,n}-x_n\|\frac{1}{1+n}\sum_{i=0}^{n}\left\|T^i x_n - x_n\right\|, n \geq 0\right\} \leq Q.\end{aligned} \tag{20}$$

Hence, we get

$$\langle u - x_{t,n}, j(x_n - x_{t,n})\rangle \leq \frac{t}{2}Q_1 + \frac{1}{2t}\left(\frac{\gamma_n}{1-\alpha_n}Q\right), \tag{21}$$

where Q_1 is a constant such that

$$\sup \{\|x_{t,n} - x_n\|, n \geq 0, 0 < t < 1\} \leq Q_1. \tag{22}$$

It follows that

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle u - x_{t,n}, j(x_n - x_{t,n})\rangle \leq 0. \tag{23}$$

By the fact that the order of $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ is changeable, we have

$$\limsup_{n\to\infty} \langle u - p, j(x_n - p)\rangle \leq 0. \tag{24}$$

Finally, we prove $x_n \to p$. Indeed, applying Lemma 6 to (5), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \left\|\alpha_n(u-p) + \beta_n(x_n - p) + \gamma_n\left(\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n - p\right)\right\|^2 \\
&\quad \leq \left\|\beta_n(x_n - p) + \gamma_n \frac{1}{n+1}\sum_{i=0}^{n}\left(T^i x_n - T^i p\right)\right\|^2 \\
&\qquad + 2\alpha_n \langle u - p, j(x_{n+1} - p)\rangle \\
&\quad \leq (\beta_n\|x_n - p\| + \gamma_n\|x_n - p\|)^2 \\
&\qquad + 2\alpha_n \langle u - p, j(x_{n+1} - p)\rangle \\
&\quad \leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle u - p, j(x_{n+1} - p)\rangle .
\end{aligned} \tag{25}$$

Hence, by Lemma 7, we have that $x_n \to p$ as $n \to \infty$.

The proof is complete. □

Now we will give the second main result. In order to prove the strong convergence of viscosity iterative (6), we assume that $z_{t,n}$ is the unique fixed point of the following contractive mapping $T_{t,n}$ given by

$$T_{t,n}x = \frac{(1-\alpha_n)t}{\gamma_n + t\beta_n} f(x) + \frac{(1-t)\gamma_n}{\gamma_n + t\beta_n}\frac{1}{n+1}\sum_{i=0}^{n} T^i x. \tag{26}$$

That is,

$$z_{t,n} = \frac{(1-\alpha_n)t}{\gamma_n + t\beta_n} f(z_{t,n}) + \frac{(1-t)\gamma_n}{\gamma_n + t\beta_n}\frac{1}{n+1}\sum_{i=0}^{n} T^i z_{t,n}. \tag{27}$$

From Lemma 5, for fixed n, we have

$$\lim_{t\to 0} z_{t,n} = p \in F(T), \tag{28}$$

which is the unique fixed point.

Theorem 9. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \phi$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in (0,1) satisfying the following control conditions (i–iii). Then, for given $x_0 \in C$ arbitrarily, the sequence $\{x_n\}$ defined by (6) converges strongly to a fixed point of T.*

Proof. Taking a fixed point $q \in F(T)$, we have

$$\begin{aligned}
&\|x_{n+1} - q\| \\
&\quad \leq \alpha_n\|f(x_n) - q\| + \beta_n\|x_n - q\| \\
&\qquad + \gamma_n\left\|\frac{1}{1+n}\sum_{i=0}^{n}\left(T^i x_n - T^i q\right)\right\| \\
&\quad \leq \alpha_n(\|f(x_n) - f(q)\| + \|f(q) - q\|) \\
&\qquad + \beta_n\|x_n - q\| + \gamma_n\|x_n - q\| \\
&\quad \leq \alpha_n\beta\|x_n - q\| + \alpha_n\|f(q) - q\| \\
&\qquad + (1-\alpha_n)\|x_n - q\| \\
&\quad = \alpha_n\|f(q) - q\| + (1-(1-\beta)\alpha_n)\|x_n - q\| \\
&\quad \leq \max\left\{\frac{1}{1-\beta}\|f(q) - q\|, \|x_n - q\|\right\}.
\end{aligned} \tag{29}$$

By induction,

$$\begin{aligned}
\|x_n - q\| \leq \max\left\{\frac{1}{1-\beta}\|f(q) - q\|, \|x_0 - q\|\right\}, \\
n \geq 0
\end{aligned} \tag{30}$$

and $\{x_n\}$ is bounded so are $\{T^i x_n\}$ and $\{f(x_n)\}$.

We observe that (27) can be rewritten as

$$\begin{aligned}
z_{t,n} &= tf(z_{t,n}) \\
&\quad + (1-t)\left[\frac{\beta_n}{1-\alpha_n} z_{t,n} + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n} T^i z_{t,n}\right],
\end{aligned} \tag{31}$$

and $\lim_{t\to 0} z_{t,n} = q \in F(T)$, for all $n \geq 0$.

From (31), we have

$$\begin{aligned}
z_{t,n} - x_n &= t(f(z_{t,n} - x_n)) \\
&\quad + (1-t)\left[\frac{\beta_n}{1-\alpha_n}(z_{t,n} - x_n)\right. \\
&\quad \left. + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left(T^i z_{t,n} - x_n\right)\right].
\end{aligned} \tag{32}$$

Applying Lemma 6 to (32), we get

$$\begin{aligned}
&\|z_{t,n} - x_n\|^2 \\
&\quad \leq (1-t)^2\left\|\frac{\beta_n}{1-\alpha_n}(z_{t,n} - x_n)\right. \\
&\qquad \left. + \frac{\gamma_n}{1-\alpha_n}\frac{1}{1+n}\sum_{i=0}^{n}\left(T^i z_{t,n} - x_n\right)\right\|^2 \\
&\qquad + 2t\langle f(z_{t,n}) - x_n, j(z_{t,n} - x_n)\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-t)^2 \left\| \frac{\beta_n}{1-\alpha_n} (z_{t,n} - x_n) \right. \\
&\qquad + \frac{\gamma_n}{1-\alpha_n} \frac{1}{1+n} \sum_{i=0}^{n} \left(T^i z_{t,n} - T^i x_n \right) \\
&\qquad \left. + \frac{\gamma_n}{1-\alpha_n} \frac{1}{1+n} \sum_{i=0}^{n} \left(T^i x_n - x_n \right) \right\|^2 \\
&\quad + 2t \|z_{t,n} - x_n\|^2 + 2t \langle f(z_{t,n}) - z_{t,n}, j(z_{t,n} - x_n) \rangle \\
&\leq (1-t)^2 \left[\|z_{t,n} - x_n\| + \frac{\gamma_n}{1-\alpha_n} \frac{1}{1+n} \sum_{i=0}^{n} \left\| T^i x_n - x_n \right\| \right]^2 \\
&\quad + 2t \|z_{t,n} - x_n\|^2 + 2t \langle f(z_{t,n}) - z_{t,n}, j(z_{t,n} - x_n) \rangle \\
&= \left(1 + t^2\right) \|z_{t,n} - x_n\|^2 \\
&\quad + (1-t)^2 \left(\frac{\gamma_n}{1-\alpha_n} \frac{1}{1+n} \sum_{i=0}^{n} \left\| T^i x_n - x_n \right\| \right)^2 \\
&\quad + 2(1-t)^2 \frac{\gamma_n}{1-\alpha_n} \frac{1}{1+n} \sum_{i=0}^{n} \left\| T^i x_n - x_n \right\| \|z_{t,n} - x_n\| \\
&\quad + 2t \langle f(z_{t,n}) - z_{t,n}, j(z_{t,n} - x_n) \rangle \\
&\leq \left(1 + t^2\right) \|z_{t,n} - x_n\|^2 + \frac{\gamma_n}{1-\alpha_n} M \\
&\quad + 2t \langle f(z_{t,n}) - z_{t,n}, j(z_{t,n} - x_n) \rangle,
\end{aligned} \tag{33}$$

where M is some constant such that

$$\sup \left\{ \frac{1}{1+n} \sum_{i=0}^{n} \left(\left\| T^i x_n - x_n \right\|^2 + 2 \left\| T^i x_n - x_n \right\| \|z_{t,n} - x_n\| \right), n \geq 0 \right\} \leq M. \tag{34}$$

Hence, we have

$$\langle f(z_{t,n}) - z_{t,n}, j(x_n - z_{t,n}) \rangle \leq \frac{t}{2} M_1 + \frac{1}{2t} \left(\frac{\gamma_n}{1-\alpha_n} M \right), \tag{35}$$

where M_1 is also a constant such that

$$M_1 \geq \sup \left\{ \|z_{t,n} - x_n\|^2, n \geq 0,\ 0 < t < 1 \right\}. \tag{36}$$

It follows that

$$\limsup_{t \to 0} \limsup_{n \to \infty} \langle f(z_{t,n}) - z_{t,n}, j(x_n - z_{t,n}) \rangle \leq 0. \tag{37}$$

Since the order of $\limsup_{t \to 0}$ and $\limsup_{n \to \infty}$ is exchangeable, hence

$$\limsup_{n \to \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0. \tag{38}$$

Finally, we prove that $x_n \to q$. Indeed, applying Lemma 6 to (6), we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \left\| \alpha_n (f(x_n) - q) + \beta_n (x_n - q) + \gamma_n \frac{1}{1+n} \sum_{i=0}^{n} \left(T^i x_n - T^i q \right) \right\|^2 \\
&\quad \leq \left\| \beta_n (x_n - q) + \gamma_n \frac{1}{1+n} \sum_{i=0}^{n} \left(T^i x_n - T^i q \right) \right\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q) \rangle \\
&\quad \leq [\beta_n \|x_n - q\| + \gamma_n \|x_n - q\|]^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - f(q), j(x_{n+1} - q) \rangle \\
&\qquad + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\quad \leq (1 - \alpha_n)^2 \|x_n - q\|^2 \\
&\qquad + \alpha_n \left(\|f(x_n) - f(q)\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\qquad + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle.
\end{aligned} \tag{39}$$

Therefore, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \frac{1 - 2\alpha_n + \beta^2 \alpha_n + \alpha_n^2}{1 - \alpha_n} \|x_n - q\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n} \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\quad \leq \left(1 - \frac{1 - \beta^2}{1 - \alpha_n} \alpha_n \right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n} \|x_n - q\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n} \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\quad \leq (1 - \widetilde{\alpha_n}) \|x_n - q\|^2 + \widetilde{\alpha_n} \widetilde{\beta_n}.
\end{aligned} \tag{40}$$

Put

$$\begin{gathered}
\widetilde{\alpha_n} = \frac{1 - \beta^2}{1 - \alpha_n} \alpha_n, \\
\widetilde{\beta_n} = \frac{M_1}{1 - \beta^2} \alpha_n + \frac{2}{1 - \beta^2} \langle f(q) - q, j(x_{n+1} - q) \rangle.
\end{gathered} \tag{41}$$

It follows that

$$\|x_{n+1} - q\|^2 \leq (1 - \widetilde{\alpha_n}) \|x_n - q\|^2 + \widetilde{\alpha_n} \widetilde{\beta_n}. \tag{42}$$

It is easily seen from (ii) and (38) that

$$\widetilde{\alpha_n} \longrightarrow 0, \qquad \sum_{n=0}^{\infty} \widetilde{\alpha_n} = \infty, \qquad \limsup_{n \to \infty} \widetilde{\beta_n} \leq 0. \tag{43}$$

Hence, applying Lemma 7 to (42), we have that $x_n \to q$ as $n \to \infty$.

The proof is complete. □

Remark 10. Our main result extends the main result of Yao et al. to Cesàro means and viscosity iteration method. Our results are new and the proofs are simple and different from many others.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. B. Baillon, "Un theoreme de type ergodique pour les contractions non lineairs dans un espaces de Hilbert," *Comptes Rendus de l'Académie des Sciences A*, vol. 280, no. 22, pp. 1511–1514, 1975.

[2] R. E. Bruck, "A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces," *Israel Journal of Mathematics*, vol. 32, no. 2-3, pp. 107–116, 1979.

[3] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[4] Y. Song and R. Chen, "Viscosity approximate methods to Cesàro means for non-expansive mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1120–1128, 2007.

[5] Y. Yao, Y.-C. Liou, and H. Zhou, "Strong convergence of an iterative method for nonexpansive mappings with new control conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 6, pp. 2332–2336, 2009.

[6] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and its Applications, Yokohama Publishers, Yokohama, Japan, 2000 (Japanese).

[7] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.

[8] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.