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Research Article

Differences of Composition Operators Followed by Differentiation between Weighted Banach Spaces of Holomorphic Functions

Cui Chen, Ren-Yu Chen, and Ze-Hua Zhou

Department of Mathematics, Tianjin University, Tianjin 300072, China

Correspondence should be addressed to Ze-Hua Zhou; zehuazhoumath@aliyun.com

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We characterize the boundedness and compactness of differences of the composition operators followed by differentiation between weighted Banach spaces of holomorphic functions in the unit disk. As their corollaries, some related results on the differences of composition operators acting from weighted Banach spaces to weighted Bloch type spaces are also obtained.

1. Introduction

Let $H(\mathbb{D})$ and $S(\mathbb{D})$ denote the class of holomorphic functions and analytic self-maps on the unit disk $\mathbb{D}$ of the complex plane of $\mathbb{C}$, respectively. Let v be a strictly positive continuous and bounded function (weight) on $\mathbb{D}$.

The weighted Bloch space $\mathscr{B}_v$ is defined to be the collection of all $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_{\mathscr{B}_v} := \sup_{z\in\mathbb{D}} v(z)\left|f'(z)\right| < \infty. \tag{1}$$

Provided we identify the functions that differ by a constant, $\|\cdot\|_{\mathscr{B}_v}$ becomes a norm and $\mathscr{B}_v$ a Banach space.

The $H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$ endowed with the weighted sup-norm $\|\cdot\|_v$ is referred to as the weighted Banach space. In setting the so-called associated weight plays an important role.

For a weight v, its associated weight $\tilde{v}$ is defined as follows:

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{(H_v^\infty)'}}, \tag{2}$$

where δ_z denotes the point evaluation at z. By [1] the associated weight $\tilde{v}$ is continuous, $\tilde{v} \ge v > 0$, and for every $z \in \mathbb{D}$ we can find $g_z \in H_v^\infty$ with $\|g_z\|_v \le 1$ such that $g_z(z) = 1/\tilde{v}(z)$.

We say that a weight v is radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A positive continuous function v is called normal if there exist three positive numbers δ, t, s and $t > s$, such that for every $z \in \mathbb{D}$ with $|z| \in [\delta, 1)$,

$$\frac{v(|z|)}{(1-|z|)^s} \downarrow 0, \quad \frac{v(|z|)}{(1-|z|)^t} \uparrow \infty, \quad \text{as } |z| \to 1. \tag{3}$$

A radial, nonincreasing weight is called typical if $\lim_{|z|\to 1} v(z) = 0$. When studying the structure and isomorphism classes of the space H_v^∞, Lusky [2, 3] introduced the following condition (L1) (renamed after the author) for radial weights:

$$\inf_{n\in\mathbb{N}} \frac{v\left(1-2^{-n-1}\right)}{v(1-2^{-n})} > 0, \tag{L1}$$

which will play a great role in this paper. In case v is a radial weight, if it is also normal, then it satisfies the condition (L1). Moreover, the radial weights with (L1) are essential (e.g., see [4]); that is, we can find a constant $C > 0$ such that

$$v(z) \le \tilde{v}(z) \le Cv(z) \quad \text{for every } z \in \mathbb{D}. \tag{4}$$

Let $\varphi \in S(\mathbb{D})$; the composition operator C_φ induced by φ is defined by

$$\left(C_\varphi f\right)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}. \tag{5}$$

This operator has been studied for many years. Readers interested in this topic are referred to the books [5–7], which are excellent sources for the development of the theory of composition operators, and to the recent papers [8, 9] and the references therein.

By differentiation we are led to the linear operator $DC_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto (f' \circ \varphi)\varphi'$, which is regarded as the product of the composition operator and the differentiation operator denoted by $Df = f'$, $f \in H(\mathbb{D})$. The product operators have been studied, for example, in [10–16] and the references therein.

Recently, there has been an increasing interest in studying the compact difference of composition operators acting on different spaces of holomorphic functions. Some related results on differences of the composition operators or weighted composition operators on weighted Banach spaces of analytic functions, Bloch-type spaces, and weighted Bergman spaces can be found, for example [17–27]. More recently, Wolf [28, 29] characterized the boundedness and compactness of differences of composition operators between weighted Bergman spaces or weighted Bloch spaces and weighted Banach spaces of holomorphic functions in the unit disk. The same problems between standard weighted Bergman spaces were discussed by Saukko [30].

For each φ and ψ in $S(\mathbb{D})$, we are interested in the operators $DC_\varphi - DC_\psi$, and we characterize boundedness and compactness of the operators $DC_\varphi - DC_\psi$ between weighted Banach spaces of holomorphic functions in terms of the involved weights as well as the inducing maps. As a corollary we get a characterization of boundedness and compactness about the differences of composition operators $C_\varphi - C_\psi$ acting from weighted Banach spaces to weighted Bloch type spaces.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to another. And for each $\omega \in \mathbb{D}$, g_ω denotes a function in H_u^∞ with $\|g_\omega\|_u \le 1$ such that $|g_\omega(\omega)| = 1/\tilde{u}(\omega)$. The existence of this function is a consequence of Montel's theorem as can be seen in [1].

2. Background and Some Lemmas

Now let us state a couple of lemmas, which are used in the proofs of the main results in the next sections. The first lemma is taken from [14].

Lemma 1. *Let v be a radial weight satisfying condition $(L1)$. There is a constant $C > 0$ (depending only on the weight v) such that for all $f \in H_v^\infty$,*

$$\left|f'(z)\right| \le C \frac{\|f\|_v}{v(z)\left(1-|z|^2\right)}, \tag{6}$$

for every $z \in \mathbb{D}$.

In order to handle the differences, we need the pseudohyperbolic metric. Recall that for any point $a \in \mathbb{D}$, let $\varphi_a(z) = (z-a)/(1-\bar{a}z)$, $z \in \mathbb{D}$. It is well known that each φ_a is a homeomorphism of the closed unit disk $\overline{\mathbb{D}}$ onto itself. The pseudohyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(a,z) = \left|\varphi_a(z)\right|. \tag{7}$$

We know that $\rho(a,z)$ is invariant under automorphisms (see, e.g., [5]).

Lemma 2. *Let v be a radial weight satisfying condition $(L1)$. There is a constant $C > 0$ such that for all $f \in H_v^\infty$,*

$$\begin{aligned} &\left|v(z)\left(1-|z|^2\right)f'(z) - v(\omega)\left(1-|\omega|^2\right)f'(\omega)\right| \\ &\quad \le C\|f\|_v \rho(z,\omega), \end{aligned} \tag{8}$$

for all $z, \omega \in \mathbb{D}$.

Proof. For $f \in H_v^\infty$, let $u(z) = v(z)(1-|z|^2)$, by Lemma 1, we obtain $f' \in H_u^\infty$, so by Lemma 3.2 in [31] and Lemma 1, there is a constant $C > 0$ such that

$$\begin{aligned} \left|u(z)f'(z) - u(\omega)f'(\omega)\right| &\le C\left\|f'\right\|_u \rho(z,\omega) \\ &\le C\|f\|_v \rho(z,\omega) \end{aligned} \tag{9}$$

for each $z, \omega \in \mathbb{D}$. This completes the proof. □

Remark 3. From Lemma 2, it is not hard to see that for any $z, \omega \in r\mathbb{D} = \{z \in \mathbb{D} : |z|^2 < r < 1\}$, then

$$\begin{aligned} &\left|v(z)\left(1-|z|^2\right)f'(z) - v(\omega)\left(1-|\omega|^2\right)f'(\omega)\right| \\ &\quad \le C\|f_r\|_v \rho(z,\omega), \end{aligned} \tag{10}$$

for any $f \in H_v^\infty$, where $\|f_r\|_v = \sup_{z\in r\mathbb{D}} v(z)|f(z)|$.

The following result is well known (see, e.g., [32]).

Lemma 4. *Assume that v is a normal weight. Then for every $f \in H(\mathbb{D})$ the following asymptotic relationship holds:*

$$\sup_{z\in\mathbb{D}} v(z)\left|f(z)\right| \asymp \left|f(0)\right| + \sup_{z\in\mathbb{D}} v(z)(1-|z|)\left|f'(z)\right|. \tag{11}$$

Here and below we use the abbreviated notation $A \asymp B$ to mean $A/C \le B \le CA$ for some inessential constant $C > 0$.

The following lemma is the crucial criterion for compactness, and its proof is an easy modification of that of Proposition 3.11 of [5].

Lemma 5. *Suppose that $u, v \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$. Then the operator $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in H_u^∞ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, and then $\|(DC_\varphi - DC_\psi)f_n\| \to 0$, as $n \to \infty$.*

3. The Boundedness of $DC_\varphi - DC_\psi$

In this section we will characterize the boundedness of $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$. For this purpose, we consider the following three conditions:

$$\sup_{z\in\mathbb{D}} \frac{v(z)\left|\varphi'(z)\right|\rho(\varphi(z),\psi(z))}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)} < \infty; \tag{12}$$

$$\sup_{z\in\mathbb{D}} \frac{v(z)\left|\psi'(z)\right|\rho(\varphi(z),\psi(z))}{u(\psi(z))\left(1-|\psi(z)|^2\right)} < \infty; \tag{13}$$

$$\sup_{z\in\mathbb{D}} \left| \frac{v(z)\varphi'(z)}{\left(1-|\varphi(z)|^2\right)u(\varphi(z))} - \frac{v(z)\psi'(z)}{\left(1-|\psi(z)|^2\right)u(\psi(z))} \right| < \infty. \tag{14}$$

Theorem 6. *Suppose that v is an arbitrary weight and that u is a normal and radial weight. Then the following statements are equivalent.*

(i) $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ *is bounded.*

(ii) *The conditions* (12) *and* (14) *hold.*

(iii) *The conditions* (13) *and* (14) *hold.*

Proof. First, we prove the implication (i) $\Rightarrow$ (ii). Assume that $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ is bounded. Fixing $w \in \mathbb{D}$, we consider the function f_w defined by

$$f_w(z) = \int_0^z \varphi_{\psi(w)}(t)\, g_{\varphi(w)}(t)\, \frac{1-|\varphi(w)|^2}{\left(1-\overline{\varphi(w)}t\right)^2}dt, \quad z \in \mathbb{D}. \tag{15}$$

Next prove that $f_w \in H_u^\infty$. In fact,

$$f_w'(z) = \varphi_{\psi(w)}(z)\, g_{\varphi(w)}(z)\, \frac{1-|\varphi(w)|^2}{\left(1-\overline{\varphi(w)}z\right)^2}. \tag{16}$$

By Lemma 4 we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} u(z)\left|f_w(z)\right| &\asymp \left|f_w(0)\right| + \sup_{z\in\mathbb{D}} u(z)(1-|z|)\left|f_w'(z)\right| \\ &= \sup_{z\in\mathbb{D}} u(z)(1-|z|)\left|\varphi_{\psi(w)}(z)\right| \\ &\quad \times \left|g_{\varphi(w)}(z)\right| \frac{1-|\varphi(w)|^2}{\left|1-\overline{\varphi(w)}z\right|^2} \\ &\le C \sup_{z\in\mathbb{D}} u(z)\left|g_{\varphi(w)}(z)\right|,\end{aligned} \tag{17}$$

thus $f_w \in H_u^\infty$, and $\|f_w\|_u \le C$. Note that $f_w'(\varphi(w)) = \rho(\varphi(w),\psi(w))/\widetilde{u}(\varphi(w))(1-|\varphi(w)|^2)$, and $f_w'(\psi(w)) = 0$. So by the boundedness of $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$, it then follows that

$$\begin{aligned}\infty &> \left\|\left(DC_\varphi - DC_\psi\right)f_w\right\|_v \\ &= \sup_{z\in\mathbb{D}} v(z)\left|f_w'(\varphi(z))\varphi'(z) - f_w'(\psi(z))\psi'(z)\right| \\ &\ge v(w)\left|f_w'(\varphi(w))\varphi'(w) - f_w'(\psi(w))\psi'(w)\right| \\ &= \frac{v(w)\left|\varphi'(w)\right|}{\widetilde{u}(\varphi(w))\left(1-|\varphi(w)|^2\right)}\rho(\varphi(w),\psi(w)),\end{aligned} \tag{18}$$

for any $w \in \mathbb{D}$. Since $w \in \mathbb{D}$ is an arbitrary element, then from (18) and (4), we can obtain (12).

Next we prove (14). For given $w \in \mathbb{D}$, we consider the function

$$h_w(z) = \int_0^z g_{\psi(w)}(t)\, \frac{1-|\psi(w)|^2}{\left(1-\overline{\psi(w)}t\right)^2}dt, \quad z \in \mathbb{D}. \tag{19}$$

Like for f_w above, we can show that $h_w \in H_u^\infty$ with $h_w'(z) = g_{\psi(w)}(z)((1-|\psi(w)|^2)/(1-\overline{\psi(w)}z)^2)$. One sees that $h_w'(\psi(w)) = 1/\widetilde{u}(\psi(w))(1-|\psi(w)|^2)$. Then

$$\begin{aligned}\infty &> \left\|\left(DC_\varphi - DC_\psi\right)h_w\right\|_v \\ &\ge v(w)\left|h_w'(\varphi(w))\varphi'(w) - h_w'(\psi(w))\psi'(w)\right| \\ &= |I(w) + J(w)|,\end{aligned} \tag{20}$$

where

$$\begin{aligned} I(w) &= \frac{v(w)\varphi'(w)}{u(\varphi(w))\left(1-|\varphi(w)|^2\right)} \\ &\quad \times \left[u(\varphi(w))\left(1-|\varphi(w)|^2\right)h_w'(\varphi(w)) \right. \\ &\quad \left. - u(\psi(w))\left(1-|\psi(w)|^2\right)h_w'(\psi(w))\right], \\ J(w) &= u(\psi(w))\left(1-|\psi(w)|^2\right)h_w'(\psi(w)) \\ &\quad \times \left[\frac{v(w)\varphi'(w)}{u(\varphi(w))\left(1-|\varphi(w)|^2\right)} \right. \\ &\quad \left. - \frac{v(w)\psi'(w)}{u(\psi(w))\left(1-|\psi(w)|^2\right)}\right].\end{aligned} \tag{21}$$

By Lemma 2 and (12), we conclude that $|I(w)| < \infty$, which combines with (20), and we obtain that

$$\begin{aligned}|J(w)| = C&\left|\frac{v(w)\varphi'(w)}{u(\varphi(w))\left(1-|\varphi(w)|^2\right)}\right. \\ &\left. - \frac{v(w)\psi'(w)}{u(\psi(w))\left(1-|\psi(w)|^2\right)}\right| < \infty\end{aligned} \tag{22}$$

for all $w \in \mathbb{D}$; thus (14) holds.

(ii) ⇒ (iii). Assume that (12) and (14) hold, we need only to show that (13) holds. In fact,

$$\begin{aligned}
&\frac{v(z)\left|\psi'(z)\right|}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\rho(\varphi(z),\psi(z)) \\
&\quad\leq \frac{v(z)\left|\varphi'(z)\right|}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}\rho(\varphi(z),\psi(z)) \\
&\qquad+\left|\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}-\frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\right| \\
&\qquad\times\rho(\varphi(z),\psi(z)),
\end{aligned} \tag{23}$$

from which, using (12) and (14), the desired condition (13) holds.

(iii) ⇒ (i). Assume that (13) and (14) hold. By Lemmas 1 and 2, for any $f\in H_u^\infty$, we have

$$\begin{aligned}
&v(z)\left|\left(DC_\varphi-DC_\psi\right)f(z)\right| \\
&\quad= v(z)\left|f'(\varphi(z))\varphi'(z)-f'(\psi(z))\psi'(z)\right| \\
&\quad= \frac{v(z)\left|\psi'(z)\right|}{u(\psi(z))\left(1-|\psi(z)|^2\right)} \\
&\qquad\times\left|u(\psi(z))\left(1-|\psi(z)|^2\right)f'(\psi(z))\right. \\
&\qquad\qquad\left.-u(\varphi(z))\left(1-|\varphi(z)|^2\right)f'(\varphi(z))\right| \\
&\qquad+u(\varphi(z))\left(1-|\varphi(z)|^2\right)\left|f'(\varphi(z))\right| \\
&\qquad\times\left|\frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}-\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}\right| \\
&\quad\leq C\|f\|_u\frac{v(z)\left|\psi'(z)\right|}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\rho(\varphi(z),\psi(z)) \\
&\qquad+C\|f\|_u\leq C\|f\|_u,
\end{aligned} \tag{24}$$

from which it follows that $DC_\varphi-DC_\psi: H_u^\infty\to H_v^\infty$ is bounded. The whole proof is complete. □

Corollary 7. *Suppose that v is an arbitrary weight and that u is a normal and radial weight satisfying condition ($L1$). Then the following statements are equivalent.*

(i) $C_\varphi-C_\psi: H_u^\infty\to\mathscr{B}_v$ *is bounded.*

(ii) *The conditions* (12) *and* (14) *hold.*

(iii) *The conditions* (13) *and* (14) *hold.*

4. The Compactness of DC_φ–DC_ψ

In this section, we turn our attention to the question of compact difference. Here we consider the following conditions:

$$\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}\rho(\varphi(z),\psi(z))\longrightarrow 0 \quad \text{as } |\varphi(z)|\longrightarrow 1; \tag{25}$$

$$\frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\rho(\varphi(z),\psi(z))\longrightarrow 0 \quad \text{as } |\psi(z)|\longrightarrow 1; \tag{26}$$

$$\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}-\frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\longrightarrow 0 \quad \text{as } |\varphi(z)|\longrightarrow 1,\ |\psi(z)|\longrightarrow 1. \tag{27}$$

Theorem 8. *Suppose that v is an arbitrary weight and that u is a normal and radial weight. Then $DC_\varphi-DC_\psi: H_u^\infty\to H_v^\infty$ is compact if and only if $DC_\varphi-DC_\psi: H_u^\infty\to H_v^\infty$ is bounded and the conditions* (25)–(27) *hold.*

Proof. First we suppose that $DC_\varphi-DC_\psi: H_u^\infty\to H_v^\infty$ is bounded and the conditions (25)–(27) hold. Then the conditions (12)–(14) hold by Theorem 6. From (25)–(27), it follows that for any $\varepsilon>0$, there exists $0<r<1$ such that

$$\frac{v(z)\left|\varphi'(z)\right|}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}\rho(\varphi(z),\psi(z))\leq\varepsilon \quad \text{for } |\varphi(z)|>r, \tag{28}$$

$$\frac{v(z)\left|\psi'(z)\right|}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\rho(\varphi(z),\psi(z))\leq\varepsilon \quad \text{for } |\psi(z)|>r, \tag{29}$$

$$\left|\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)}-\frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\right|\leq\varepsilon \quad \text{for } |\varphi(z)|>r,\ |\psi(z)|>r. \tag{30}$$

Now, let $\{f_n\}$ be a sequence in H_u^∞ such that $\|f_n\|_u\leq L$ (constant) and $\{f_n\}\to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 5 we need only to show that $\|(DC_\varphi-DC_\psi)f_n\|_v\to 0$ as $n\to\infty$. A direct calculation shows that

$$v(z)\left|f_n'(\varphi(z))\varphi'(z)-f_n'(\psi(z))\psi'(z)\right|=\left|I_n(z)+J_n(z)\right|, \tag{31}$$

where

$$
\begin{aligned}
I_n(z) &= \frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)} \\
&\quad \times \left[u(\varphi(z))\left(1-|\varphi(z)|^2\right) f_n'(\varphi(z))\right. \\
&\quad \left. - u(\psi(z))\left(1-|\psi(z)|^2\right) f_n'(\psi(z))\right], \\
J_n(z) &= u(\psi(z))\left(1-|\psi(z)|^2\right) f_n'(\psi(z)) \\
&\quad \times \left[\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)} - \frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\right].
\end{aligned} \tag{32}
$$

We divide the argument into a few cases.

Case 1 ($|\varphi(z)| \le r$ *and* $|\psi(z)| \le r$). By the assumption, note that $\{f_n\}$ converges to zero uniformly on $E = \{w : |w| \le r\}$ as $n \to \infty$; using (14) and Cauchy's integral formula, it is easy to check that $J_n(z) \to 0$, $n \to \infty$ uniformly for all z with $|\psi(z)| \le r$.

On the other hand, it follows from Remark 3 after Lemma 2 and (12) that

$$
\begin{aligned}
|I_n(z)| &\le C \frac{v(z)|\varphi'(z)|}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)} \rho(\varphi(z), \psi(z)) \\
&\quad \times \sup_{|w|\le r} u(w)|f_n(w)| \le C\varepsilon.
\end{aligned} \tag{33}
$$

Case 2 ($|\varphi(z)| > r$ *and* $|\psi(z)| \le r$). As in the proof of Case 1, $J_n(z) \to 0$ uniformly as $n \to \infty$. On the other hand, using Lemma 2 and (28) we obtain $|I_n(z)| \le CL\varepsilon$.

Case 3 ($|\varphi(z)| > r$ *and* $|\psi(z)| > r$). For n sufficiently large, by Lemma 2 and (28) we obtain that $|I_n(z)| \le CL\varepsilon$. Meanwhile, $|J_n(z)| \le CL\varepsilon$ by Lemma 1 and (30).

Case 4 ($|\varphi(z)| \le r$ *and* $|\psi(z)| > r$). We rewrite

$$
\begin{aligned}
v(z)\left|f_n'(\varphi(z))\varphi'(z) - f_n'(\psi(z))\psi'(z)\right| \\
= |P_n(z) + Q_n(z)|,
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
P_n(z) &= \frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)} \\
&\quad \times \left[u(\varphi(z))\left(1-|\varphi(z)|^2\right) f_n'(\varphi(z))\right. \\
&\quad \left. - u(\psi(z))\left(1-|\psi(z)|^2\right) f_n'(\psi(z))\right], \\
Q_n(z) &= u(\varphi(z))\left(1-|\varphi(z)|^2\right) f_n'(\varphi(z)) \\
&\quad \times \left[\frac{v(z)\varphi'(z)}{u(\varphi(z))\left(1-|\varphi(z)|^2\right)} - \frac{v(z)\psi'(z)}{u(\psi(z))\left(1-|\psi(z)|^2\right)}\right].
\end{aligned} \tag{35}
$$

The desired result follows by an argument analogous to that in the proof of Case 2. Thus, together with the above cases, we conclude that

$$
\begin{aligned}
&\left\|\left(DC_\varphi - DC_\psi\right) f_n\right\|_v \\
&\quad = \sup_{z\in\mathbb{D}} v(z)\left|f_n'(\varphi(z))\varphi'(z) - f_n'(\psi(z))\psi'(z)\right| \le C\varepsilon,
\end{aligned} \tag{36}
$$

for sufficiently large n. Employing Lemma 5 we obtain the compactness of $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$.

For the converse direction, we suppose that $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ is compact. From which we can easily obtain the boundedness of $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$. Next we only need to show that (25)–(27) hold.

Let $\{z_n\}$ be a sequence of points in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Define the functions

$$
f_n(z) = \int_0^z \varphi_{\psi(z_n)}(t)\, g_{\varphi(z_n)}(t)\, \frac{1-|\varphi(z_n)|^2}{\left(1-\overline{\varphi(z_n)}t\right)^2} dt. \tag{37}
$$

Clearly, $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and $f_n \in H_u^\infty$ with $\|f_n\|_u \le L$ for all n. Moreover,

$$
\begin{aligned}
f_n'(\varphi(z_n)) &= \frac{\rho(\varphi(z_n), \psi(z_n))}{\tilde{u}(\varphi(z_n))\left(1-|\varphi(z_n)|^2\right)}, \\
f_n'(\psi(z_n)) &= 0.
\end{aligned} \tag{38}
$$

By the compactness of $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ and Lemma 5, it follows that $\|(DC_\varphi - DC_\psi) f_n\|_v \to 0$, $n \to \infty$. On the other hand, using (38) we have

$$
\begin{aligned}
&\left\|(DC_\varphi - DC_\psi) f_n\right\|_v \\
&\quad = \sup_{z\in\mathbb{D}} v(z)\left|f_n'(\varphi(z))\varphi'(z) - f_n'(\psi(z))\psi'(z)\right| \\
&\quad \ge v(z_n)\left|f_n'(\varphi(z_n))\varphi'(z_n) - f_n'(\psi(z_n))\psi'(z_n)\right| \\
&\quad = \frac{v(z_n)|\varphi'(z_n)|}{\tilde{u}(\varphi(z_n))\left(1-|\varphi(z_n)|^2\right)} \rho(\varphi(z_n), \psi(z_n)).
\end{aligned} \tag{39}
$$

Letting $n \to \infty$ in (39), it follows that (25) holds. The condition (26) holds for the similar arguments.

Now we need only to show the condition (27) holds. Assume that $\{z_n\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$ as $n \to \infty$. Define the function

$$h_n(z) = \int_0^z g_{\psi(z_n)}(t) \frac{1 - |\psi(z_n)|^2}{\left(1 - \overline{\psi(z_n)}t\right)^2} dt. \tag{40}$$

It is easy to check that $\{h_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and $h_n \in H_u^\infty$ with $\|h_n\|_u \le L$ for all $n \in N$. Note that $h_n'(z) = g_{\psi(z_n)}(z)((1 - |\psi(z_n)|^2)/(1 - \overline{\psi(z_n)}z)^2)$, then $h_n'(\psi(z_n)) = [\tilde{u}(\psi(z_n))(1 - |\psi(z_n)|^2)]^{-1}$, and it follows from Lemma 5 that $\|(DC_\varphi - DC_\psi)h_n\|_v \to 0$, $n \to \infty$. On the other hand we obtain that

$$\begin{aligned} &\left\|\left(DC_\varphi - DC_\psi\right)h_n\right\|_v \\ &\quad \ge v(z_n)\left|h_n'(\varphi(z_n))\varphi'(z_n) - h_n'(\psi(z_n))\psi'(z_n)\right| \\ &\quad = \left|I(z_n) + J(z_n)\right|, \end{aligned} \tag{41}$$

where

$$\begin{aligned} I(z_n) &= \frac{v(z_n)\varphi'(z_n)}{u(\varphi(z_n))\left(1 - |\varphi(z_n)|^2\right)} \\ &\quad \times \left[u(\varphi(z_n))\left(1 - |\varphi(z_n)|^2\right)h_n'(\varphi(z_n))\right. \\ &\quad\quad \left. - u(\psi(z_n))\left(1 - |\psi(z_n)|^2\right)h_n'(\psi(z_n))\right], \\ J(z_n) &= u(\psi(z_n))\left(1 - |\psi(z_n)|^2\right)h_n'(\psi(z_n)) \\ &\quad \times \left[\frac{v(z_n)\varphi'(z_n)}{u(\varphi(z_n))\left(1 - |\varphi(z_n)|^2\right)} - \frac{v(z_n)\psi'(z_n)}{u(\psi(z_n))\left(1 - |\psi(z_n)|^2\right)}\right] \\ &= C\left[\frac{v(z_n)\varphi'(z_n)}{u(\varphi(z_n))\left(1 - |\varphi(z_n)|^2\right)} - \frac{v(z_n)\psi'(z_n)}{u(\psi(z_n))\left(1 - |\psi(z_n)|^2\right)}\right]. \end{aligned} \tag{42}$$

By Lemma 2 and the condition (25) that has been proved, we get $I(z_n) \to 0$, $n \to \infty$. This combines with (41), and we obtain $J(z_n) \to 0$, $n \to \infty$. This shows that (27) is true. The whole proof is complete. □

Corollary 9. *Suppose that v is an arbitrary weight and that u is a normal and radial weight satisfying condition $(L1)$. Then $C_\varphi - C_\psi : H_u^\infty \to \mathscr{B}_v$ is compact if and only if $C_\varphi - C_\psi : H_u^\infty \to \mathscr{B}_v$ is bounded and the conditions* (25)–(27) *hold.*

5. Examples

In this final section we give an example of function u, v, φ, ψ for which the operator $DC_\varphi - DC_\psi$ between the weighted Banach spaces to show that the condition in Theorem 8 that $DC_\varphi - DC_\psi$ is bounded is necessary.

Example 1. In this example we will show that there exist weight u (normal, radial) and v, analytic self-maps on the unit disk φ, ψ such that the conditions (25)–(27) in Theorem 8 are satisfied while $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ is not compact.

Let

$$\varphi(z) = \frac{1}{M+1}\sum_{j=1}^{\infty}\frac{z^j}{j^2}, \tag{43}$$

and $\psi(z) = -\varphi(z)$, where $M = \sum_{j=1}^{\infty}(1/j^2)$.

Since for $|z| < 1$, we have $|\varphi(z)| < M/(M+1)$ so φ belongs to $S(\mathbb{D})$, as well as ψ. Moreover, $|\psi(z)|$ and $|\psi(z)|$ can never tend to 1 for any $z \in \mathbb{D}$, which means that conditions (25)–(27) hold trivially.

Now we will show that $DC_\varphi - DC_\psi : H_u^\infty \to H_v^\infty$ is not bounded, and then not compact. Let $z_k = 1 - 1/k$, and then it is easy to check that $\varphi(z_k) \to M/(M+1)$ and $\psi(z_k) \to -M/(M+1)$ as $k \to \infty$. So

$$\rho(\varphi(z_k), \psi(z_k)) \longrightarrow \frac{2(M/(M+1))}{1 + (M/(M+1))^2} > 0. \tag{44}$$

However, since $\varphi'(z) = (1/(M+1))\sum_{j=1}^{\infty}(z^{j-1}/j)$, then $|\varphi'(z_k)| \to \infty$ as $k \to \infty$. Thus choose $u(z) = 1 - |z|^2$ and $v(z) = u(\varphi(z))$, and we can obtain

$$\frac{v(z_k)\left|\varphi'(z_k)\right|\rho(\varphi(z_k), \psi(z_k))}{u(\varphi(z_k))\left(1 - |\varphi(z_k)|^2\right)} \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \tag{45}$$

Hence $DC_\varphi - DC_\psi$ does not map H_u^∞ boundedly into H_v^∞ by Theorem 6.

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