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Research Article

Solutions and Improved Perturbation Analysis for the Matrix Equation $X - A^* X^{-p} A = Q$ $(p > 0)$

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The nonlinear matrix equation $X - A^* X^{-p} A = Q$ with $p > 0$ is investigated. We consider two cases of this equation: the case $p \geq 1$ and the case $0 < p < 1$. In the case $p \geq 1$, a new sufficient condition for the existence of a unique positive definite solution for the matrix equation is obtained. A perturbation estimate for the positive definite solution is derived. Explicit expressions of the condition number for the positive definite solution are given. In the case $0 < p < 1$, a new sharper perturbation bound for the unique positive definite solution is derived. A new backward error of an approximate solution to the unique positive definite solution is obtained. The theoretical results are illustrated by numerical examples.

1. Introduction

In this paper, we consider the Hermitian positive definite solution of the nonlinear matrix equation

$$X - A^* X^{-p} A = Q, \tag{1}$$

where A, Q, and X are $n \times n$ complex matrices, Q is a positive definite matrix, and $p > 0$. This type of nonlinear matrix equations arises in the analysis of ladder networks, the dynamic programming, control theory, stochastic filtering, statistics, and many applications [1–7].

In the last few years, (1) was investigated in some special cases. For the nonlinear matrix equations $X - A^* X^{-1} A = Q$ [8–12], $X - A^* X^{-2} A = Q$ [13, 14], $X - A^* X^{-n} A = Q$ [15, 16], and $X^s - A^* X^{-t} A = Q$ [17], there were many contributions in the literature to the solvability, numerical solutions, and perturbation analysis. In addition, the related equations $X + A^* X^{-1} A = Q$ [9–11, 18–23], $X + A^* X^{-2} A = Q$ [13, 24, 25], $X + A^* X^{-n} A = Q$ [16, 26], $X^s + A^* X^{-t} A = Q$ [17, 27–30], $X + A^* X^{-q} A = Q$ [31–33], and $X \pm \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$ [34–36] were studied by many scholars.

In [31], a sufficient condition for the equation $X - A^* X^{-p} A = Q$ $(0 < p \leq 1)$ to have a unique positive definite solution was provided. When the coefficient matrix A is nonsingular, several sufficient conditions for the equation $X - A^* X^{-q} A = Q$ $(q \geq 1)$ to have a unique positive definite solution were given in [37]. When the coefficient matrix A is an arbitrary complex matrix, necessary conditions and sufficient conditions for the existence of positive definite solutions for the equation $X - A^* X^{-q} A = Q$ $(q \geq 1)$ were derived in [38]. Li and Zhang in [39] proved that there always exists a unique positive definite solution to the equation $X - A^* X^{-p} A = Q$ $(0 < p < 1)$. They also obtained a perturbation bound and a backward error of an approximate solution for the unique solution of the equation $X - A^* X^{-p} A = Q$ $(0 < p < 1)$.

As a continuation of the previous results, the rest of the paper is organized as follows. Section 2 gives some preliminary lemmas that will be needed to develop this work. In Section 3, a new sufficient condition for (1) with $p \geq 1$ having a unique positive definite solution is derived. In Section 4, a perturbation bound for the positive definite solution to (1) with $p \geq 1$ is given. In Section 5, applying the integral representation of matrix function, we also discuss the explicit expressions of condition number for the positive definite solution to (1) with $p \geq 1$. Furthermore, in Section 6, a new sharper perturbation bound for the unique positive definite solution to (1) with $0 < p < 1$ is given. In Section 7, a new backward error of an approximate solution to (1) with $0 < p < 1$ is obtained. Finally, several numerical examples are presented in Section 8.

We denote by $\mathscr{C}^{n\times n}$ the set of $n\times n$ complex matrices, by $\mathscr{H}^{n\times n}$ the set of $n\times n$ Hermitian matrices, by I the identity matrix, by $\|\cdot\|$ the spectral norm, by $\|\cdot\|_F$ the Frobenius norm and by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ the maximal and minimal eigenvalues of M, respectively. For $A = (a_{ij}) \in \mathscr{C}^{n\times n}$ with columns a_i and a matrix B, $A\otimes B = (a_{ij}B)$ is a Kronecker product, and vec A is a vector defined by vec $A = (a_1^T,\ldots,a_n^T)^T$. For $X, Y \in \mathscr{H}^{n\times n}$, we write $X \geq Y$ (resp., $X > Y$) if $X - Y$ is Hermitian positive semidefinite (resp., definite). Let $\overline{\kappa} = \lambda_{\max}(A^*A)$, $\underline{\kappa} = \lambda_{\min}(A^*A)$.

2. Preliminaries

In this section we quote some preliminary lemmas that we use later.

Lemma 1 (see [39, Lemma 3.2]). *For every positive definite matrix $X \in \mathscr{H}^{n\times n}$ and $0 < p < 1$, then*

(i) $X^{-p} = (\sin p\pi/\pi)\int_0^{\infty}(\lambda I + X)^{-1}\lambda^{-p}d\lambda$.

(ii) $X^{-p} = (\sin p\pi/p\pi)\int_0^{\infty}(\lambda I + X)^{-1}X(\lambda I + X)^{-1}\lambda^{-p}d\lambda$.

Lemma 2 (see [39, Theorem 2.5]). *There exists a unique positive definite solution X of $X - A^*X^{-p}A = Q$ $(0 < p < 1)$ and the iteration*

$$X_0 > 0, \qquad X_n = Q + A^*X_{n-1}^{-p}A, \quad n = 1, 2, \ldots \tag{2}$$

converges to X.

Lemma 3 (see [32, Lemma 2]). *(i) If $X \in \mathscr{H}^{n\times n}$, then $\|e^{-X}\| = e^{-\lambda_{\min}(X)}$.*

(ii) If $X \in \mathscr{H}^{n\times n}$ and $r > 0$, then $X^{-r} = (1/\Gamma(r))\int_0^{\infty} e^{-sX}s^{r-1}ds$.

(iii) If $A, B \in \mathscr{C}^{n\times n}$, then $e^{A+B} - e^A = \int_0^1 e^{(1-t)A}Be^{t(A+B)}dt$.

3. A Sufficient Condition for the Existence of a Unique Solution of $X - A^*X^{-p}A = Q$ $(p \geq 1)$

In this section, we derive a new sufficient condition for the existence of a unique solution of $X - A^*X^{-p}A = Q$ $(p \geq 1)$ beginning with the lemma.

Lemma 4 (see [38, Theorem 5, Remark 4]). *If*

$$\beta > (p\overline{\kappa})^{1/(p+1)}, \tag{3}$$

then (1) has a unique positive definite solution $X \in [\beta I, \alpha I]$, where α and β are, respectively, positive solutions of the following equations:

$$(x - \lambda_{\max}(Q))\left(\lambda_{\min}(Q) + \frac{\underline{\kappa}}{x^p}\right)^p = \overline{\kappa}, \tag{4}$$

$$(x - \lambda_{\min}(Q))\left(\lambda_{\max}(Q) + \frac{\overline{\kappa}}{x^p}\right)^p = \underline{\kappa}. \tag{5}$$

Furthermore,

$$\lambda_{\min}(Q) \leq \beta \leq \alpha. \tag{6}$$

Theorem 5. *If*

$$\left((p\overline{\kappa})^{1/(p+1)} - \lambda_{\min}(Q)\right)\left(\lambda_{\max}(Q) + \frac{\overline{\kappa}}{(p\overline{\kappa})^{p/(p+1)}}\right)^p < \underline{\kappa}, \tag{7}$$

$$\frac{(\lambda_{\min}(Q)\,p)^{p/(p+1)}\lambda_{\max}^{1/(p+1)}(Q)}{(\overline{\kappa})^{1/(p+1)}} + 1 - p > 0, \tag{8}$$

then (1) has a unique positive definite solution.

Proof. We first prove $\beta > (p\overline{\kappa})^{1/(p+1)}$, where β is the positive solution to (5). Let

$$f(x) = (x - \lambda_{\min}(Q))\left(\lambda_{\max}(Q) + \frac{\overline{\kappa}}{x^p}\right)^p - \underline{\kappa}. \tag{9}$$

By computation, we obtain

$$\begin{aligned} f'(x) &= \frac{\overline{\kappa}}{x^p}\left(\lambda_{\max}(Q) + \frac{\overline{\kappa}}{x^p}\right)^{p-1} \\ &\quad\times\left(\frac{\lambda_{\max}(Q)}{\overline{\kappa}}x^p + p^2\lambda_{\min}(Q)\,x^{-1} + 1 - p^2\right). \end{aligned} \tag{10}$$

Define

$$g(x) = \frac{\lambda_{\max}(Q)}{\overline{\kappa}}x^p + p^2\lambda_{\min}(Q)\,x^{-1} + 1 - p^2. \tag{11}$$

Then $g(x)$ is decreasing on $[0, \lambda_{\min}(Q)p\overline{\kappa}/(\lambda_{\max}(Q))^{1/(p+1)}]$ and increasing on $[\lambda_{\min}(Q)p\overline{\kappa}/(\lambda_{\max}(Q))^{1/(p+1)}, +\infty)$, which implies that

$$\begin{aligned} g_{\min} &= g\left(\left(\frac{\lambda_{\min}(Q)\,p\overline{\kappa}}{\lambda_{\max}(Q)}\right)^{1/(p+1)}\right) \\ &= (1+p)\left(\frac{(\lambda_{\min}(Q)\,p)^{p/(p+1)}\lambda_{\max}^{1/(p+1)}(Q)}{(\overline{\kappa})^{1/(p+1)}} + 1 - p\right). \end{aligned} \tag{12}$$

According to the condition (8), it follows that $g_{\min} > 0$. Note that

$$f'(x) = \frac{\overline{\kappa}}{x^p}\left(\lambda_{\max}(Q) + \frac{\overline{\kappa}}{x^p}\right)^{p-1} g(x), \tag{13}$$

which implies that $f(x)$ is increasing on $(0, +\infty)$. Considering the condition (7), one sees that $f((p\overline{\kappa})^{1/(p+1)}) < 0$. Combining that and the definition of β in Lemma 4, we obtain $\beta > (p\overline{\kappa})^{1/(p+1)}$. By Lemma 4, (1) has a unique positive definite solution. □

4. Perturbation Bound for $X - A^*X^{-p}A = Q$ $(p \geq 1)$

Li and Zhang in [39] proved that there always exists a unique positive definite solution to the equation

$X - A^*X^{-p}A = Q$ $(0 < p < 1)$. They also obtained a perturbation bound for the unique solution. But their approach becomes invalid for the case of $p \geq 1$. Since the equation $X - A^*X^{-p}A = Q$ $(p \geq 1)$ does not always have a unique positive definite solution, there are two difficulties for a perturbation analysis of the equation $X - A^*X^{-p}A = Q$ $(p \geq 1)$. One difficulty is how to find some reasonable restrictions on the coefficient matrices of perturbed equation ensuring that this equation has a unique positive definite solution. The other difficulty is how to find an expression of ΔX which is easy to handle.

Assume that the coefficient matrix A is perturbed to $\widetilde{A} = \Delta A + A$. Let $\widetilde{X} = \Delta X + X$ with $\Delta X \in \mathscr{H}^{n\times n}$ satisfying the perturbed equation

$$\widetilde{X} - \widetilde{A}^*\widetilde{X}^{-p}\widetilde{A} = Q, \quad p \geq 1. \tag{14}$$

In the following, we derive a perturbation estimate for the positive definite solution to the matrix equation $X - A^*X^{-p}A = Q$ $(p \geq 1)$ beginning with the lemma.

Lemma 6 (see [38, Corollary 1. Remark 4]). *If*

$$p\|A\|^2 < \lambda_{\min}^{p+1}(Q), \tag{15}$$

then (1) *has a unique positive definite solution* X. *Moreover,* $X \geq \lambda_{\min}(Q)I$.

Theorem 7. *If*

$$\|A\| < \sqrt{\frac{\lambda_{\min}^{p+1}(Q)}{p}}, \qquad \|\Delta A\| < \sqrt{\frac{\lambda_{\min}^{p+1}(Q)}{p}} - \|A\|, \tag{16}$$

then

$$X - A^*X^{-p}A = Q, \qquad \widetilde{X} - \widetilde{A}^*\widetilde{X}^{-p}\widetilde{A} = Q \tag{17}$$

have unique positive definite solutions X *and* $\widetilde{X}$, *respectively. Furthermore,*

$$\frac{\left\|\widetilde{X} - X\right\|}{\|X\|} \leq \frac{(2\|A\| + \|\Delta A\|)}{\lambda_{\min}^{p+1}(Q) - p\|A\|^2}\|\Delta A\| \equiv \varrho. \tag{18}$$

Proof. By (16), it follows that $\|\widetilde{A}\| \leq \|A\| + \|\Delta A\| \leq \sqrt{\lambda_{\min}^{(p+1)}(Q)/p}$. According to Lemma 6, the condition (16) ensures that (1) and (14) have unique positive definite solutions X and $\widetilde{X}$, respectively. Furthermore, we obtain that

$$X \geq \lambda_{\min}(Q)I, \qquad \widetilde{X} \geq \lambda_{\min}(Q)I. \tag{19}$$

Subtracting (14) from (1) gives

$$\begin{aligned}\Delta X &= \widetilde{A}^*\widetilde{X}^{-p}\widetilde{A} - A^*X^{-p}A \\ &= A^*\left(\widetilde{X}^{-p} - X^{-p}\right)A + \Delta A^*\widetilde{X}^{-p}A + \widetilde{A}^*\widetilde{X}^{-p}\Delta A.\end{aligned} \tag{20}$$

By Lemma 3 and inequalities in (19), we have

$$\begin{aligned}&\left\|\Delta X + A^*X^{-p}A - A^*\widetilde{X}^{-p}A\right\| \\ &= \left\|\Delta X + A^*\frac{1}{\Gamma(p)}\int_0^\infty\left(e^{-sX} - e^{-s\widetilde{X}}\right)s^{p-1}ds A\right\| \\ &= \left\|\Delta X + A^*\frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-(1-t)s\widetilde{X}}\Delta X e^{-tsX}dts^p ds A\right\| \\ &\geq \|\Delta X\| - \frac{\|A\|^2\|\Delta X\|}{\Gamma(p)}\int_0^\infty\int_0^1\left\|e^{-(1-t)s\widetilde{X}}\right\|\left\|e^{-tsX}\right\|dts^p ds \\ &\geq \|\Delta X\| - \frac{\|A\|^2\|\Delta X\|}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-(1-t)s\lambda_{\min}(\widetilde{X})}e^{-ts\lambda_{\min}(X)}dts^p ds \\ &\geq \|\Delta X\| - \frac{\|A\|^2\|\Delta X\|}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-(1-t)s\lambda_{\min}(Q)}e^{-ts\lambda_{\min}(Q)}dts^p ds \\ &= \|\Delta X\| - \frac{\|A\|^2\|\Delta X\|}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-s\lambda_{\min}(Q)}dts^p ds \\ &= \|\Delta X\| - \frac{\Gamma(p+1)}{\Gamma(p)}\cdot\frac{\|A\|^2\|\Delta X\|}{\lambda_{\min}^{p+1}(Q)} \\ &= \frac{\lambda_{\min}^{p+1}(Q) - p\|A\|^2}{\lambda_{\min}^{p+1}(Q)}\|\Delta X\|.\end{aligned} \tag{21}$$

Noting (16), we have

$$\lambda_{\min}^{p+1}(Q) - p\|A\|^2 > 0. \tag{22}$$

Combining (20) and (21), one sees that

$$\begin{aligned}\frac{\lambda_{\min}^{p+1}(Q) - p\|A\|^2}{\lambda_{\min}^{p+1}(Q)}\|\Delta X\| &\leq \left\|\Delta A^*\widetilde{X}^{-p}A + \widetilde{A}^*\widetilde{X}^{-p}\Delta A\right\| \\ &\leq (\|\Delta A\| + 2\|A\|)\|\Delta A\|\left\|\widetilde{X}^{-p}\right\| \\ &\leq (\|\Delta A\| + 2\|A\|)\|\Delta A\|\lambda_{\min}^{-p}(Q),\end{aligned} \tag{23}$$

which implies that

$$\frac{\|\Delta X\|}{\|X\|} \leq \frac{(\|\Delta A\| + 2\|A\|)}{\lambda_{\min}^{p+1}(Q) - p\|A\|^2}\|\Delta A\|. \tag{24}$$

□

5. Condition Number for $X - A^*X^{-p}A = Q$ $(p \geq 1)$

A condition number is a measurement of the sensitivity of the positive definite stabilizing solutions to small changes in the coefficient matrices. In this section, we apply the theory of condition number developed by Rice [40] to derive explicit expressions of the condition number for the matrix equation $X - A^* X^{-p} A = Q$ $(p \geq 1)$.

Here we consider the perturbed equation

$$\widetilde{X} - \widetilde{A}^* \widetilde{X}^{-p} \widetilde{A} = \widetilde{Q}, \quad p \geq 1, \tag{25}$$

where $\widetilde{A}$ and $\widetilde{Q}$ are small perturbations of A and Q in (1), respectively.

Suppose that $p\|A\|^2 < \lambda_{\min}^{p+1}(Q)$ and $p\|\widetilde{A}\|^2 < \lambda_{\min}^{p+1}(\widetilde{Q})$. According to Lemma 6, (1) and (25) have unique positive definite solutions X and $\widetilde{X}$, respectively. Let $\Delta X = \widetilde{X} - X$, $\Delta Q = \widetilde{Q} - Q$, and $\Delta A = \widetilde{A} - A$.

Subtracting (25) from (1) gives

$$\begin{aligned}
\Delta X &= \widetilde{A}^* \widetilde{X}^{-p} \widetilde{A} - A^* X^{-p} A + \Delta Q \\
&= A^* \left(\widetilde{X}^{-p} - X^{-p}\right) A + \Delta A^* \widetilde{X}^{-p} A + \widetilde{A}^* \widetilde{X}^{-p} \Delta A + \Delta Q \\
&= -A^* \frac{1}{\Gamma(p)} \int_0^\infty \left(e^{-sX} - e^{-s\widetilde{X}}\right) s^{p-1} ds A \\
&\quad + \Delta A^* \widetilde{X}^{-p} A + \widetilde{A}^* \widetilde{X}^{-p} \Delta A + \Delta Q \\
&= -A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)s\widetilde{X}} \left(\widetilde{X} - X\right) e^{-tsX} dt s^p ds A \\
&\quad + \Delta A^* \widetilde{X}^{-p} A + \widetilde{A}^* \widetilde{X}^{-p} \Delta A + \Delta Q \\
&= -A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 \left(e^{-(1-t)s\widetilde{X}} - e^{-(1-t)sX}\right) \\
&\qquad \times \Delta X e^{-tsX} dt s^p ds A + \Delta Q \\
&\quad - A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)sX} \Delta X e^{-tsX} dt s^p ds A \\
&\quad - \left(\widetilde{A}^* X^{-p} \Delta A - \widetilde{A}^* (X + \Delta X)^{-p} \Delta A\right) \\
&\quad + \widetilde{A}^* X^{-p} \Delta A - \left(\Delta A^* X^{-p} A - \Delta A^* (X + \Delta X)^{-p} A\right) \\
&\quad + \Delta A^* X^{-p} A \\
&= A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 \int_0^1 e^{-(1-m)(1-t)sX} \Delta X e^{-m(1-t)s\widetilde{X}} \\
&\qquad \times \Delta X e^{-tsX} dm (1-t) dt s^{p+1} ds A \\
&\qquad + \Delta Q \\
&\quad - A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)sX} \Delta X e^{-tsX} dt s^p ds A \\
&\quad + \Delta A^* X^{-p} \Delta A + A^* X^{-p} \Delta A + \Delta A^* X^{-p} A \\
&\quad - \widetilde{A}^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)s(X+\Delta X)} \Delta X e^{-tsX} dt s^p ds \Delta A \\
&\quad - \Delta A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)s(X+\Delta X)} \Delta X e^{-tsX} dt s^p ds A.
\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}
&\Delta X + A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)sX} \Delta X e^{-tsX} dt s^p ds A \\
&\quad = E + h(\Delta X),
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
B &= X^{-p} A, \\
E &= \Delta Q + \left(B^* \Delta A + \Delta A^* B\right) + \Delta A^* X^{-p} \Delta A, \\
&h(\Delta X) \\
&\quad = A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 \int_0^1 e^{-(1-m)(1-t)sX} \\
&\qquad \times \Delta X e^{-m(1-t)s\widetilde{X}} \Delta X e^{-tsX} \\
&\qquad \times dm (1-t) dt s^{p+1} ds A \\
&\quad - \widetilde{A}^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)s(X+\Delta X)} \Delta X e^{-tsX} dt s^p ds \Delta A \\
&\quad - \Delta A^* \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 e^{-(1-t)s(X+\Delta X)} \Delta X e^{-tsX} dt s^p ds A.
\end{aligned} \tag{28}$$

Lemma 8. *If*

$$p\|A\|^2 < \lambda_{\min}^{p+1}(Q), \tag{29}$$

then the linear operator $\mathbf{V} : \mathscr{H}^{n\times n} \to \mathscr{H}^{n\times n}$ *defined by*

$$\mathbf{V}W = W + \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 A^* e^{-(1-t)sX} W e^{-tsX} A dt s^p ds, \quad W \in \mathscr{H}^{n\times n} \tag{30}$$

is invertible.

Proof. Define the operator $\mathbf{R} : \mathscr{H}^{n\times n} \to \mathscr{H}^{n\times n}$ by

$$\mathbf{R}Z = \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 A^* e^{-(1-t)sX} Z e^{-tsX} A dt s^p ds, \quad Z \in \mathscr{H}^{n\times n}, \tag{31}$$

it follows that

$$\mathbf{V}W = W + \mathbf{R}W. \tag{32}$$

Then, $\mathbf{V}$ is invertible if and only if $I + \mathbf{R}$ is invertible.

According to Lemma 3 and the condition (29), we have

$$\begin{aligned}
&\|\mathbf{R}W\| \\
&\leq \|A\|^2\|W\| \frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 \left\|e^{-(1-t)sX}\right\|\left\|e^{-tsX}\right\| dts^p ds \\
&= \|A\|^2\|W\| \frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-(1-t)s\lambda_{\min}(X)}e^{-ts\lambda_{\min}(X)} dts^p ds \\
&\leq \|A\|^2\|W\| \frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 e^{-(1-t)s\lambda_{\min}(Q)}e^{-ts\lambda_{\min}(Q)} dts^p ds \\
&= \|A\|^2\|W\| \frac{1}{\Gamma(p)}\int_0^\infty e^{-s\lambda_{\min}(Q)} s^p ds \\
&= \frac{p\|A\|^2}{\lambda_{\min}^{p+1}(Q)}\|W\| < \|W\|,
\end{aligned} \tag{33}$$

which implies that $\|\mathbf{R}\| < 1$ and $I+\mathbf{R}$ is invertible. Therefore, the operator $\mathbf{V}$ is invertible. □

Thus, we can rewrite (27) as

$$\begin{aligned}
\Delta X = {} & \mathbf{V}^{-1}\Delta Q + \mathbf{V}^{-1}(B^*\Delta A + \Delta A^* B) \\
& + \mathbf{V}^{-1}(\Delta A^* X^{-p}\Delta A) + \mathbf{V}^{-1}(h(\Delta X)),
\end{aligned} \tag{34}$$

$$\begin{aligned}
\Delta X = {} & \mathbf{V}^{-1}\Delta Q + \mathbf{V}^{-1}(B^*\Delta A + \Delta A^* B) \\
& + O\left(\|(\Delta A, \Delta Q)\|_F^2\right), \quad (\Delta A, \Delta Q) \longrightarrow 0.
\end{aligned} \tag{35}$$

By the theory of condition number developed by Rice [40], we define the condition number of the Hermitian positive definite solution X to the matrix equation $X - A^*X^{-p}A = Q$ $(p \geq 1)$ by

$$c(X) = \lim_{\delta\to 0} \sup_{\|(\Delta A/\eta, \Delta Q/\rho)\|_F \leq \delta} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{36}$$

where ξ, η, and ρ are positive parameters. Taking $\xi = \eta = \rho = 1$ in (36) gives the absolute condition number $c_{\text{abs}}(X)$, and taking $\xi = \|X\|_F$, $\eta = \|A\|_F$, and $\rho = \|Q\|_F$ in (36) gives the relative condition number $c_{\text{rel}}(X)$.

Substituting (35) into (36), we get

$$\begin{aligned}
c(X) &= \frac{1}{\xi} \max_{\substack{(\Delta A/\eta, \Delta Q/\rho)\neq 0 \\ \Delta A\in\mathscr{C}^{n\times n}, \Delta Q\in\mathscr{H}^{n\times n}}} \frac{\left\|\mathbf{V}^{-1}(\Delta Q + B^*\Delta A + \Delta A^* B)\right\|_F}{\|(\Delta A/\eta, \Delta Q/\rho)\|_F} \\
&= \frac{1}{\xi} \max_{\substack{(E,H)\neq 0 \\ E\in\mathscr{C}^{n\times n}, H\in\mathscr{H}^{n\times n}}} \frac{\left\|\mathbf{V}^{-1}(\rho H + \eta(B^*E + E^*B))\right\|_F}{\|(E,H)\|_F}.
\end{aligned} \tag{37}$$

Let V be the matrix representation of the linear operator $\mathbf{V}$. It follows from Lemma 4.3.2. in [41] that

$$\operatorname{vec}(\mathbf{V}W) = V \cdot \operatorname{vec} W. \tag{38}$$

By Lemma 4.3.1. in [41], we have

$$\begin{aligned}
&\operatorname{vec}(\mathbf{V}W) \\
&= \left(I\otimes I + \frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 \left(e^{-tsX}A\right)^T \otimes \left(A^* e^{-(1-t)sX}\right) dts^p ds\right) \\
&\quad \cdot \operatorname{vec} W.
\end{aligned} \tag{39}$$

Then,

$$\begin{aligned}
V = I\otimes I + \frac{1}{\Gamma(p)}\int_0^\infty\int_0^1 & \left(e^{-tsX}A\right)^T \\
& \otimes \left(A^* e^{-(1-t)sX}\right) dts^p ds.
\end{aligned} \tag{40}$$

Let

$$\begin{gathered}
V^{-1} = S + i\Sigma, \\
V^{-1}(I\otimes B^*) = V^{-1}\left(I\otimes (X^{-p}A)^*\right) = U_1 + i\Omega_1, \\
V^{-1}\left(B^T\otimes I\right)\Pi = V^{-1}\left(\left(X^{-p}A\right)^T\otimes I\right)\Pi = U_2 + i\Omega_2, \\
S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, \qquad U_c = \begin{bmatrix} U_1 + U_2 & \Omega_2 - \Omega_1 \\ \Omega_1 + \Omega_2 & U_1 - U_2 \end{bmatrix},
\end{gathered} \tag{41}$$

$$\begin{gathered}
\operatorname{vec} H = x + \mathbf{i}y, \quad \operatorname{vec} E = a + \mathbf{i}b, \\
g = \left(x^T, y^T, a^T, b^T\right)^T,
\end{gathered} \tag{42}$$

where $x, y, a, b \in \mathscr{R}^{n^2}$, $S, \Sigma, U_1, U_2, \Omega_1, \Omega_2 \in \mathscr{R}^{n^2\times n^2}$, $M = (E, H)$, $\mathbf{i} = \sqrt{-1}$, Π is the vec-permutation matrix, that is,

$$\operatorname{vec} E^T = \Pi \operatorname{vec} E. \tag{43}$$

Furthermore, we obtain that

$$c(X) = \frac{1}{\xi}\max_{M\neq 0} \frac{\left\|\mathbf{V}^{-1}(\rho H + \eta(B^*E + E^*B))\right\|_F}{\|(E,H)\|_F} \tag{44}$$

$$\begin{aligned}
&= \frac{1}{\xi}\max_{M\neq 0} \Big\|\rho V^{-1}\operatorname{vec} H + \eta V^{-1} \\
&\qquad \times \left(\left(I\otimes B^*\right)\operatorname{vec} E + \left(B^T\otimes I\right)\operatorname{vec} E^*\right)\Big\|\Big) \\
&\qquad \times (\|\operatorname{vec}(E,H)\|)^{-1} \\
&\qquad = \frac{1}{\xi}\max_{M\neq 0} \big\|\rho(S + \mathbf{i}\Sigma)(x + \mathbf{i}y) \\
&\qquad + \eta\left[\left(U_1 + \mathbf{i}\Omega_1\right)(a + \mathbf{i}b)\right. \\
&\qquad \left. + \left(U_2 + \mathbf{i}\Omega_2\right)(a - \mathbf{i}b)\right]\big\| \\
&\qquad \times (\|\operatorname{vec}(E,H)\|)^{-1} \\
&\qquad = \frac{1}{\xi}\max_{g\neq 0}\frac{\|(\rho\, S_c, \eta U_c)\, g\|}{\|g\|} \\
&\qquad = \frac{1}{\xi}\, \left\|(\rho S_c, \eta U_c)\right\|, \quad E\in\mathscr{C}^{n\times n},\ H\in\mathscr{H}^{n\times n}.
\end{aligned} \tag{45}$$

Then, we have the following theorem.

Theorem 9. *If $p\|A\|^2 < \lambda_{\min}^{p+1}(Q)$, then the condition number $c(X)$ defined by* (36) *has the explicit expression*

$$c(X) = \frac{1}{\xi}\left\|\left(\rho S_c, \eta U_c\right)\right\|, \tag{46}$$

where the matrices S_c and U_c are defined by (40)-(41).

Remark 10. From (46) we have the relative condition number

$$c_{\mathrm{rel}}(X) = \frac{\left\|\left(\|Q\|_F S_c,\ \|A\|_F U_c\right)\right\|}{\|X\|_F}. \tag{47}$$

5.1. The Real Case. In this subsection, we consider the real case, that is, where all the coefficients matrices A, Q of the matrix equation $X - A^* X^{-p} A = Q$ $(p \geq 1)$ are real. In such a case the corresponding solution X is also real. Similar arguments as in Theorem 9 give the following theorem.

Theorem 11. *Let A, Q be real, $c(X)$ the condition number defined by* (36). *If $p\|A\|^2 < \lambda_{\min}^{p+1}(Q)$, then $c(X)$ has the explicit expression*

$$c(X) = \frac{1}{\xi}\left\|\left(\rho S_r, \eta U_r\right)\right\|, \tag{48}$$

where

$$\begin{aligned} S_r &= \left(I \otimes I + \frac{1}{\Gamma(p)} \int_0^\infty \int_0^1 \left(e^{-tsX} A\right)^T \right. \\ &\qquad \left. \otimes \left(A^T e^{-(1-t)sX}\right) dt s^p ds\right)^{-1}, \\ U_r &= S_r\left[I \otimes \left(A^T X^{-p}\right) + \left(\left(A^T X^{-p}\right) \otimes I\right)\Pi\right]. \end{aligned} \tag{49}$$

Proof. Let

$$\begin{aligned} S_r &= V^{-1}, \\ U_r &= V^{-1}\left(\left(I \otimes B^T\right) + \left(B^T \otimes I\right)\Pi\right) \\ &= S_r\left(\left(I \otimes \left(X^{-p}A\right)^T\right) + \left(\left(X^{-p}A\right)^T \otimes I\right)\Pi\right) \\ \operatorname{vec} H &= x, \quad \operatorname{vec} E = a, \quad g = \left(x^T, a^T\right)^T, \end{aligned} \tag{50}$$

where $x, a, \in \mathscr{R}^{n^2}$, $M = (E, H)$, $\mathbf{i} = \sqrt{-1}$, Π is the vec-permutation matrix, that is,

$$\operatorname{vec} E^T = \Pi \operatorname{vec} E. \tag{51}$$

It follows from (44) that

$$\begin{aligned} c(X) &= \frac{1}{\xi}\max_{M \neq 0} \frac{\left\|\mathbf{V}^{-1}\left(\rho H + \eta\left(B^T E + E^T B\right)\right)\right\|_F}{\|(E, H)\|_F} \\ &= \frac{1}{\xi}\max_{M \neq 0} \left\|\rho V^{-1} \operatorname{vec} H \right. \\ &\qquad \left. + \eta V^{-1}\left(\left(I \otimes B^T\right)\operatorname{vec} E + \left(B^T \otimes I\right)\operatorname{vec} E^T\right)\right\| \\ &\qquad \times \left(\|\operatorname{vec}(E, H)\|\right)^{-1} \\ &= \frac{1}{\xi}\max_{g \neq 0} \frac{\left\|\left(\rho S_r, \eta U_r\right) g\right\|}{\|g\|} \\ &= \frac{1}{\xi}\left\|\left(\rho S_r, \eta U_r\right)\right\|. \end{aligned} \tag{52}$$

□

Remark 12. In the real case the relative condition number is given by

$$c_{\mathrm{rel}}(X) = \frac{\left\|\left(\|Q\|_F S_r, \|A\|_F U_r\right)\right\|}{\|X\|_F}. \tag{53}$$

6. New Perturbation Bound for $X - A^* X^{-p} A = Q$ $(0 < p < 1)$

Here, we consider the perturbed equation

$$\widetilde{X} - \widetilde{A}^* \widetilde{X}^{-p} \widetilde{A} = \widetilde{Q}, \quad 0 < p < 1, \tag{54}$$

where $\widetilde{A}$ and $\widetilde{Q}$ are small perturbations of A and Q in (1), respectively. We assume that X and $\widetilde{X}$ are the solutions of (1) and (54), respectively. Let $\Delta X = \widetilde{X} - X$, $\Delta Q = \widetilde{Q} - Q$, and $\Delta A = \widetilde{A} - A$.

In this section, we develop a new perturbation bound for the solution of (1) which is sharper than that in [39, Theorem 3.1].

Subtracting (1) from (54), using Lemma 1, we have

$$\begin{aligned} \Delta X &+ \frac{\sin p\pi}{\pi} \int_0^\infty \left[(\lambda I + X)^{-1} A\right]^* \\ &\times \Delta X \left[(\lambda I + X)^{-1} A\right] \lambda^{-p} d\lambda = E + h(\Delta X), \end{aligned} \tag{55}$$

where

$$\begin{aligned} B &= X^{-p} A, \\ E &= \Delta Q + \left(B^* \Delta A + \Delta A^* B\right) + \Delta A^* X^{-p} \Delta A, \\ h(\Delta X) &= \frac{\sin p\pi}{\pi} A^* \int_0^\infty (\lambda I + X)^{-1} \\ &\qquad \times \Delta X (\lambda I + X + \Delta X)^{-1} \\ &\qquad \times \Delta X (\lambda I + X)^{-1} \lambda^{-p} d\lambda A \end{aligned}$$

$$-\frac{\sin p\pi}{\pi}\widetilde{A}^*\int_0^\infty(\lambda I+X)^{-1}\Delta X(\lambda I+X+\Delta X)^{-1}\lambda^{-p}d\lambda\Delta A$$
$$-\frac{\sin p\pi}{\pi}\Delta A^*\int_0^\infty(\lambda I+X)^{-1}\Delta X(\lambda I+X+\Delta X)^{-1}\lambda^{-p}d\lambda A. \quad (56)$$

By Lemma 5.1 in [39], the linear operator $\mathbf{L}:\mathscr{H}^{n\times n}\to\mathscr{H}^{n\times n}$ defined by

$$\mathbf{L}W = W+\frac{\sin p\pi}{\pi}\int_0^\infty\left[(\lambda I+X)^{-1}A\right]^* \times W\left[(\lambda I+X)^{-1}A\right]\lambda^{-p}d\lambda, \quad W\in\mathscr{H}^{n\times n} \quad (57)$$

is invertible.

We also define operator $\mathbf{P}:\mathscr{C}^{n\times n}\to\mathscr{H}^{n\times n}$ by

$$\mathbf{P}Z=\mathbf{L}^{-1}\left(B^*Z+Z^*B\right), \quad Z\in\mathscr{C}^{n\times n}, \quad i=1,2,\ldots,m. \quad (58)$$

Thus, we can rewrite (55) as

$$\Delta X=\mathbf{L}^{-1}\Delta Q+\mathbf{P}\Delta A+\mathbf{L}^{-1}\left(\Delta A^* X^{-p}\Delta A\right)+\mathbf{L}^{-1}\left(h\left(\Delta X\right)\right). \quad (59)$$

Define

$$\left\|\mathbf{L}^{-1}\right\|=\max_{\substack{W\in\mathscr{H}^{n\times n}\\ \|W\|=1}}\left\|\mathbf{L}^{-1}W\right\|, \qquad \|\mathbf{P}\|=\max_{\substack{Z\in\mathscr{C}^{n\times n}\\ \|Z\|=1}}\|\mathbf{P}Z\|. \quad (60)$$

Now we denote

$$\begin{aligned} &l=\left\|\mathbf{L}^{-1}\right\|^{-1}, \qquad \zeta=\left\|X^{-1}\right\|, \qquad \xi=\left\|X^{-p}\right\|,\\ &n=\|\mathbf{P}\|, \qquad \eta=p\xi\|A\|^2,\\ &\epsilon=\frac{1}{l}\|\Delta Q\|+n\|\Delta A\|+\frac{\xi}{l}\|\Delta A\|^2,\\ &\sigma=\frac{p}{l}\zeta\xi\left(2\|A\|+\|\Delta A\|\right)\|\Delta A\|. \end{aligned} \quad (61)$$

Theorem 13. *If*

$$\sigma<1, \qquad \epsilon<\frac{l(1-\sigma)^2}{\zeta\left(l+l\sigma+2\eta+2\sqrt{(l\sigma+\eta)(\eta+l)}\right)}, \quad (62)$$

then

$$\left\|\widetilde{X}-X\right\| \le \frac{2l\epsilon}{l(1+\zeta\epsilon-\sigma)+\sqrt{l^2(1+\zeta\epsilon-\sigma)^2-4l\zeta\epsilon(l+\eta)}}\equiv\mu_* \quad (63)$$

Proof. Let

$$f(\Delta X)=\mathbf{L}^{-1}\Delta Q+\mathbf{P}\Delta A+\mathbf{L}^{-1}\left(\Delta A^* X^{-p}\Delta A\right)+\mathbf{L}^{-1}\left(h\left(\Delta X\right)\right). \quad (64)$$

Obviously, $f:\mathscr{H}^{n\times n}\to\mathscr{H}^{n\times n}$ is continuous. The condition (62) ensures that the quadratic equation

$$\zeta(l+\eta)x^2-l(1+\zeta\epsilon-\sigma)x+l\epsilon=0 \quad (65)$$

in x has two positive real roots. The smaller one is

$$\mu_*=\frac{2l\epsilon}{l(1+\zeta\epsilon-\sigma)+\sqrt{l^2(1+\zeta\epsilon-\sigma)^2-4l\zeta\epsilon(l+\eta)}}. \quad (66)$$

Define $\Omega=\{\Delta X\in\mathscr{H}^{n\times n}:\|\Delta X\|\le\mu_*\}$. Then for any $\Delta X\in\Omega$, by (62), we have

$$\begin{aligned} &\left\|X^{-1}\Delta X\right\|\\ &\le\left\|X^{-1}\right\|\|\Delta X\|\\ &\le\zeta\mu_*\le\zeta\cdot\frac{2\epsilon}{1+\zeta\epsilon-\sigma}=1+\frac{\zeta\epsilon+\sigma-1}{1+\zeta\epsilon-\sigma}\\ &\le 1+\left(\zeta\cdot l(1-\sigma)^2/\zeta\left(l+l\sigma+2\eta+2\sqrt{(l\sigma+\eta)(\eta+l)}\right)\right.\\ &\qquad\left.+\sigma-1\right)\times(1+\zeta\epsilon-\sigma)^{-1}\\ &\le 1+\frac{l(1-\sigma)^2+(\sigma-1)(l+l\sigma+2\eta)}{(l+l\sigma+2\eta)(1+\zeta\epsilon-\sigma)}\\ &=1+\frac{-2(1-\sigma)(l\sigma+\eta)}{(l\sigma+l+2\eta)(1+\zeta\epsilon-\sigma)}<1. \end{aligned} \quad (67)$$

It follows that $I-X^{-1}\Delta X$ is nonsingular and

$$\left\|I-X^{-1}\Delta X\right\|\le\frac{1}{1-\left\|X^{-1}\Delta X\right\|}\le\frac{1}{1-\zeta\|\Delta X\|}. \quad (68)$$

Therefore

$$\begin{aligned} &\|f(\Delta X)\|\\ &\le\frac{1}{l}\|\Delta Q\|+n\|\Delta A\|\\ &\quad+\frac{\xi}{l}\left\|\Delta A_i\right\|^2+\frac{p}{l}\zeta\xi\|A\|^2\frac{\|\Delta X\|^2}{1-\zeta\|\Delta X\|}\\ &\quad+\frac{p}{l}\zeta\xi\left(2\|A\|+\|\Delta A\|\right)\|\Delta A\|\cdot\frac{\|\Delta X\|}{1-\zeta\|\Delta X\|}\\ &\le\epsilon+\frac{\sigma\|\Delta X\|}{1-\zeta\|\Delta X\|}+\frac{\eta\zeta\|\Delta X\|^2}{l\left(1-\zeta\|\Delta X\|\right)}\\ &\le\epsilon+\frac{\sigma\mu_*}{1-\zeta\mu_*}+\frac{\eta\zeta\mu_*^2}{l\left(1-\zeta\mu_*\right)}=\mu_*, \end{aligned} \quad (69)$$

for $\Delta X \in \Omega$, in which the last equality is due to the fact that μ_* is a solution to (65). That is $f(\Omega) \subseteq \Omega$. According to Schauder fixed point theorem, there exists $\Delta X_* \in \Omega$ such that $f(\Delta X_*) = \Delta X_*$. It follows that $X + \Delta X_*$ is a Hermitian solution of (54). By Lemma 2, we know that the solution of (54) is unique. Then $\Delta X_* = \widetilde{X} - X$ and $\|\widetilde{X} - X\| \le \mu_*$. □

7. New Backward Error for $X - A^* X^{-p} A = Q$ $(0 < p < 1)$

In this section, we evaluate a new backward error estimate for an approximate solution to the unique solution, which is sharper than that in [39, Theorem 4.1], .

Theorem 14. *Let $\widetilde{X} > 0$ be an approximation to the solution X of* (1). *If $\|\widetilde{X}^{-p/2}A\|^2\|\widetilde{X}^{-1}\| < 1$ and the residual $R(\widetilde{X}) \equiv Q + A^*\widetilde{X}^{-p}A - \widetilde{X}$ satisfies*

$$\left\|R\left(\widetilde{X}\right)\right\| \le \frac{\theta_1}{2}\min\left\{1, \frac{\theta_1}{2\lambda_{\min}\left(\widetilde{X}\right)}\right\}, \quad \text{where}$$
$$\theta_1 = \left(1 - \left\|\widetilde{X}^{-p/2}A\right\|^2\left\|\widetilde{X}^{-1}\right\|\right)\lambda_{\min}\left(\widetilde{X}\right) + \left\|R\left(\widetilde{X}\right)\right\| > 0, \tag{70}$$

then

$$\left\|\widetilde{X} - X\right\| \le \theta\left\|R\left(\widetilde{X}\right)\right\|, \quad \text{where}$$
$$\theta = \frac{2\lambda_{\min}\left(\widetilde{X}\right)}{\theta_1 + \sqrt{\theta_1^2 - 4\lambda_{\min}\left(\widetilde{X}\right)\left\|R\left(\widetilde{X}\right)\right\|}}. \tag{71}$$

To prove the above theorem, we first verify the following lemma.

Lemma 15. *For every positive definite matrix $X \in \mathscr{H}^{n\times n}$, $0 < p < 1$, if $X + \Delta X \ge (1/\nu)I > 0$, then*

$$\begin{aligned}&\left\|A^*\left((X+\Delta X)^{-p} - X^{-p}\right)A\right\| \\ &\quad \le p\left(\|\Delta X\| + \nu\|\Delta X\|^2\right)\left\|X^{-p/2}A\right\|^2\left\|X^{-1}\right\|.\end{aligned} \tag{72}$$

Proof. It follows from Lemma 1 that

$$\begin{aligned}&\left\|A^*\left((X+\Delta X)^{-p} - X^{-p}\right)A\right\| \\ &= \left\|A^*\left(\frac{\sin p\pi}{\pi}\int_0^\infty\left((\lambda I + X + \Delta X)^{-1}(\lambda I + X)^{-1}\right)\times\lambda^{-p}d\lambda\right)A\right\| \\ &\le \frac{\sin p\pi}{\pi}\left(\left\|A^*\int_0^\infty(\lambda I + X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda\,A\right\|\right) \\ &\quad + \frac{\sin p\pi}{\pi}\left(\left\|A^*\int_0^\infty(\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\times\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda\,A\right\|\right).\end{aligned} \tag{73}$$

Note that $\Delta X \le \|\Delta X\| I$, $X + \Delta X \ge (1/\nu)I > 0$, and $X^{-p} = (\sin p\pi/p\pi)\int_0^\infty(\lambda I + X)^{-1}X(\lambda I + X)^{-1}\lambda^{-p}\,d\lambda$, we have

$$\begin{aligned}&\left\|A^*\int_0^\infty(\lambda I + X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda\,A\right\| \\ &= \left\|A^*\int_0^\infty(\lambda I + X)^{-1}XX^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda\,A\right\| \\ &\le \left\|A^*\int_0^\infty(\lambda I + X)^{-1}X(\lambda I + X)^{-1}\lambda^{-p}d\lambda\,A\right\|\left\|X^{-1}\right\|\|\Delta X\| \\ &= \frac{p\pi}{\sin(p\pi)}\cdot\left\|A^*X^{-p}A\right\|\left\|X^{-1}\right\|\|\Delta X\|,\end{aligned} \tag{74}$$

$$\begin{aligned}&\left\|A^*\int_0^\infty(\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\times\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda A\right\| \\ &= \left\|A^*\int_0^\infty(\lambda I + X)^{-1}XX^{-1}\times\Delta X(\lambda I + X + \Delta X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-p}d\lambda A\right\| \\ &\le \left\|A^*\left(\int_0^\infty(\lambda I + X)^{-1}X(\lambda I + X)^{-1}\lambda^{-p}d\lambda A\right)\right\|\cdot\nu\|\Delta X\|^2\left\|X^{-1}\right\| \\ &= \frac{p\pi}{\sin(p\pi)}\cdot\left\|A^*X^{-p}A\right\|\cdot\nu\|\Delta X\|^2\left\|X^{-1}\right\|.\end{aligned} \tag{75}$$

A combination of (73)–(75) gives

$$\begin{aligned}&\left\|A^*\left((X+\Delta X)^{-p} - X^{-p}\right)A\right\| \\ &\quad\le p\left\|A^*X^{-p}A\right\|\left\|X^{-1}\right\|\|\Delta X\| \\ &\qquad + p\left\|A^*X^{-p}A\right\|\nu\|\Delta X\|^2\left\|X^{-1}\right\| \\ &\quad = p\left(\|\Delta X\| + \nu\|\Delta X\|^2\right)\left\|X^{-p/2}A\right\|^2\left\|X^{-1}\right\|.\end{aligned} \tag{76}$$

Here, we have used the result $\|A^*X^{-p}A\| = \|(X^{-p/2}A)^*(X^{-p/2}A)\| = \|X^{-p/2}A\|^2$ to derive the last equality (refer to [42, Problem 11. Page 312]). □

Proof of Theorem 14. Let

$$\Psi = \left\{\Delta X \in \mathscr{H}^{n\times n} : \|\Delta X\| \le \theta\left\|R\left(\widetilde{X}\right)\right\|\right\}. \tag{77}$$

Table 1: Assumptions check for Example 1 with different values of j.

j	4	5	6	7
ass_1	0.0455	0.0456	0.0456	0.0456
ass_2	0.9999	1.0000	1.0000	1.0000
ass_3	0.3957	0.3959	0.3959	0.3959

Obviously, Ψ is a nonempty bounded convex closed set. Let

$$g(\Delta X) = A^*\left(\left(\widetilde{X}+\Delta X\right)^{-p} - \widetilde{X}^{-p}\right)A + R\left(\widetilde{X}\right). \tag{78}$$

Evidently $g : \Psi \mapsto \mathscr{H}^{n\times n}$ is continuous. The condition (70) ensures that the equation

$$\begin{aligned} &y^2 - \left[\left(1 - \left\|\widetilde{X}^{-p/2}A\right\|^2\left\|\widetilde{X}^{-1}\right\|\right)\lambda_{\min}\left(\widetilde{X}\right) + \left\|R\left(\widetilde{X}\right)\right\|\right]y \\ &\quad + \lambda_{\min}\left(\widetilde{X}\right)\left\|R\left(\widetilde{X}\right)\right\| = 0 \end{aligned} \tag{79}$$

in y has two positive real roots. The smaller one is $y_* = 2\lambda_{\min}(\widetilde{X})\|R(\widetilde{X})\|/\theta_1 + \sqrt{\theta_1^2 - 4\lambda_{\min}(\widetilde{X})\|R(\widetilde{X})\|}$, where $\theta_1 = (1 - \|\widetilde{X}^{-p/2}A\|^2\|\widetilde{X}^{-1}\|)\lambda_{\min}(\widetilde{X}) + \|R(\widetilde{X})\|$.

We will prove that $g(\Psi) \subseteq \Psi$. For every $\Delta X \in \Psi$, we have

$$\Delta X \geq -\theta\left\|R\left(\widetilde{X}\right)\right\| I. \tag{80}$$

Hence,

$$\widetilde{X} + \Delta X \geq \widetilde{X} - \theta\left\|R\left(\widetilde{X}\right)\right\| I \geq \left(\lambda_{\min}\left(\widetilde{X}\right) - \theta\left\|R\left(\widetilde{X}\right)\right\|\right) I. \tag{81}$$

Using (70) and (71), one sees that

$$\begin{aligned} \theta\left\|R\left(\widetilde{X}\right)\right\| &= \frac{2\lambda_{\min}\left(\widetilde{X}\right)\left\|R\left(\widetilde{X}\right)\right\|}{\theta_1 + \sqrt{\theta_1^2 - 4\lambda_{\min}\left(\widetilde{X}\right)\left\|R\left(\widetilde{X}\right)\right\|}} \\ &< \frac{2\lambda_{\min}\left(\widetilde{X}\right)\left\|R\left(\widetilde{X}\right)\right\|}{\theta_1} < \lambda_{\min}\left(\widetilde{X}\right). \end{aligned} \tag{82}$$

Therefore, $(\lambda_{\min}(\widetilde{X}) - \theta\|R(\widetilde{X})\|)I > 0$.

According to (72), we obtain

$$\begin{aligned} &\|g(\Delta X)\| \\ &\quad \leq p\left(\|\Delta X\| + \frac{\|\Delta X\|^2}{\lambda_{\min}\left(\widetilde{X}\right) - \theta\left\|R\left(\widetilde{X}\right)\right\|}\right) \\ &\qquad \times \left\|\widetilde{X}^{-p/2}A\right\|^2\left\|\widetilde{X}^{-1}\right\| + \left\|R\left(\widetilde{X}\right)\right\| \\ &\quad \leq \left(\theta\left\|R\left(\widetilde{X}\right)\right\| + \frac{\left(\theta\left\|R\left(\widetilde{X}\right)\right\|\right)^2}{\lambda_{\min}\left(\widetilde{X}\right) - \theta\left\|R\left(\widetilde{X}\right)\right\|}\right) \\ &\qquad \times \left(p\left\|\widetilde{X}^{-p/2}A\right\|^2\left\|\widetilde{X}^{-1}\right\|\right) + \left\|R\left(\widetilde{X}\right)\right\| \\ &\quad = \theta\left\|R\left(\widetilde{X}\right)\right\|, \end{aligned} \tag{83}$$

for $\Delta X \in \Psi$, in which the last equality is due to the fact that $\theta\|R(\widetilde{X})\|$ is a solution to (79). That is $g(\Psi) \subseteq \Psi$. By Brouwer fixed point theorem, there exists a $\Delta X \in \Psi$ such that $g(\Delta X) = \Delta X$. Hence $\widetilde{X}+\Delta X$ is a solution of (1). Moreover, by Lemma 2, we know that the solution X of (1) is unique. Then

$$\left\|\widetilde{X} - X\right\| = \|\Delta X\| \leq \theta\left\|R\left(\widetilde{X}\right)\right\|. \tag{84}$$

□

8. Numerical Examples

To illustrate the theoretical results of the previous sections, in this section four simple examples are given, which were carried out using MATLAB 7.1. For the stopping criterion we take $\varepsilon_{k+1}(X) = \|X_k - A^* X_k^{-p} A - Q\| < 1.0e-10$.

Example 1. We consider the matrix equation

$$X - A^* X^{-1/3} A = I, \tag{85}$$

where

$$A = \frac{A_0}{\|A_0\|}, \qquad A_0 = \begin{pmatrix} 2 & 0.95 \\ 0 & 1 \end{pmatrix}. \tag{86}$$

Suppose that the coefficient matrix A is perturbed to $\widetilde{A} = A + \Delta A$, where

$$\Delta A = \frac{10^{-j}}{\|C^T + C\|}\left(C^T + C\right) \tag{87}$$

and C is a random matrix generated by MATLAB function randn.

We compare our own result $\mu_*/\|X\| \triangleq \text{err}_2$ in Theorem 13 with the perturbation bound $\xi_* \triangleq \text{err}_1$ proposed in [39, Theorem 3.1].

The assumption in [39, Theorem 3.1] is

$$\text{ass}_1 = \sqrt{\|A\|^2 + \zeta} - \|A\| - \|\Delta A\| > 0. \tag{88}$$

The assumptions in Theorem 13 are

$$\begin{aligned} &\text{ass}_2 = 1 - \sigma > 0, \\ &\text{ass}_3 = \frac{l(1-\sigma)^2}{\zeta\left(l + \sigma l + 2\eta + 2\sqrt{(l\sigma + \eta)(\eta + l)}\right)} - \epsilon > 0. \end{aligned} \tag{89}$$

By computation, we list them in Table 1.

The results listed in Table 1 show that the assumptions in Theorem 3.1 [39] and Theorem 13 are satisfied.

By Theorem 3.1 in [39] and Theorem 13, we can compute the relative perturbation bounds $\text{err}_1, \text{err}_2$, respectively. These results averaged as the geometric mean of 10 randomly perturbed runs. Some results are listed in Table 2.

The results listed in Table 2 show that the perturbation bound err_2 given by Theorem 13 is fairly sharp, while the bound err_1 given by Theorem 3.1 in [39] is conservative.

Table 2: Perturbation bounds for Example 1 with different values of j.

j	4	5	6	7
$\dfrac{\lVert\widetilde{X}-X\rVert}{\lVert X\rVert}$	6.8119×10^{-5}	4.2332×10^{-6}	4.3287×10^{-7}	5.5767×10^{-8}
err_1	2.6003×10^{-4}	2.1375×10^{-5}	1.9229×10^{-6}	2.7300×10^{-7}
err_2	8.8966×10^{-5}	6.5825×10^{-6}	7.2867×10^{-7}	9.3455×10^{-8}

Table 3: Results for Example 2 with different values of k.

k	4	5	6	7
$\lVert\widetilde{X}_k-X\rVert$	6.2131×10^{-6}	1.5830×10^{-7}	8.2486×10^{-9}	6.0132×10^{-10}
$\nu_*\lVert R(\widetilde{X}_k)\rVert$	2.5930×10^{-5}	6.6257×10^{-7}	3.5697×10^{-8}	2.4646×10^{-9}
κ_1	4.1734	4.1856	4.3277	4.0986
$\theta\lVert R(\widetilde{X}_k)\rVert$	7.0053×10^{-6}	1.7900×10^{-7}	9.6440×10^{-9}	6.6583×10^{-10}
κ_2	1.1275	1.1308	1.1692	1.1073

Example 2. Consider the equation

$$X - A^{*}X^{-3/4}A = Q, \tag{90}$$

for

$$A = \begin{pmatrix} 0.2 & -0.2 \\ 0.1 & 0.1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.8939 & 0.2987 \\ 0.1991 & 0.6614 \end{pmatrix}. \tag{91}$$

Choose $\widetilde{X}_0 = 3Q$. Let the approximate solution $\widetilde{X}_k$ be given with the iterative method (2), where k is the iteration number. Assume that the solution X of (1) is unknown.

We compare our own result with the backward error proposed in Theorem 4.1 [39].

The residual $R(\widetilde{X}_k) \equiv Q + A^{*}\widetilde{X}_k^{-p}A - \widetilde{X}_k$ satisfies the conditions in Theorem 4.1 [39] and in Theorem 14.

By Theorem 4.1 in [39], we can compute the backward error bound

$$\begin{gathered} \left\lVert \widetilde{X}_k - X \right\rVert \le \nu_* \left\lVert R\left(\widetilde{X}_k\right)\right\rVert, \quad \textit{where} \\ \nu_* = \frac{2\,\lVert \widetilde{X}_k \rVert \lVert \widetilde{X}_k^{-1} \rVert}{1-(3/4)\quad \left\lVert \widetilde{X}_k^{-3/8} A \widetilde{X}_k^{-1/2}\right\rVert^2}. \end{gathered} \tag{92}$$

By Theorem 14, we can compute the new backward error bound

$$\begin{gathered} \left\lVert \widetilde{X}_k - X \right\rVert \le \theta \left\lVert R\left(\widetilde{X}_k\right)\right\rVert, \quad \textit{where} \\ \theta = \frac{2\lambda_{\min}\left(\widetilde{X}_k\right)}{\theta_1 + \sqrt{\theta_1^2 - 4\lambda_{\min}\left(\widetilde{X}_k\right)\left\lVert R\left(\widetilde{X}_k\right)\right\rVert}}, \\ \theta_1 = \left(1 - \left\lVert \widetilde{X}_k^{-3/8} A\right\rVert^2 \left\lVert \widetilde{X}_k^{-1}\right\rVert\right)\lambda_{\min}\left(\widetilde{X}_k\right) + \left\lVert R\left(\widetilde{X}_k\right)\right\rVert. \end{gathered} \tag{93}$$

Let

$$\kappa_1 = \frac{\nu_*\left\lVert R\left(\widetilde{X}_k\right)\right\rVert}{\left\lVert \widetilde{X}_k - X\right\rVert}, \quad \kappa_2 = \frac{\theta\left\lVert R\left(\widetilde{X}_k\right)\right\rVert}{\left\lVert \widetilde{X}_k - X\right\rVert}. \tag{94}$$

Some results are shown in Table 3.

From the results listed in Table 3 we see that the new backward error bound $\theta\lVert R(\widetilde{X}_k)\rVert$ is sharper and closer to the actual error than the backward error bound $\nu_*\lVert R(\widetilde{X}_k)\rVert$ in [39]. Moreover, we see that the backward error $\theta\lVert R(\widetilde{X})\rVert$ for an approximate solution $\widetilde{X}$ seems to be independent of the conditioning of the solution X.

Example 3. We consider the matrix equation

$$X - A^{*}X^{-3}A = 5I, \tag{95}$$

where

$$A = \frac{A_0}{\lVert A_0\rVert}, \qquad A_0 = \begin{pmatrix} 2 & 0.95 \\ 0 & 1 \end{pmatrix}. \tag{96}$$

We now consider the perturbation bounds for the solution X when the coefficient matrix A is perturbed to $\widetilde{A} = A + \Delta A$, where

$$\Delta A = \frac{10^{-j}}{\lVert C^T + C\rVert}\left(C^T + C\right) \tag{97}$$

and C is a random matrix generated by MATLAB function randn.

The conditions in Theorem 7 are satisfied.

By Theorem 7, we can compute the relative perturbation bound ϱ with different values of j. These results averaged as the geometric mean of 10 randomly perturbed runs. Some results are listed in Table 4.

The results listed in Table 4 show that the perturbation bound ϱ given by Theorem 7 is fairly sharp.

Example 4. Consider the matrix equation $X - A^{*}X^{-3}A = Q$, where

$$A = \begin{pmatrix} 0.5 & 0.55 - 10^{-k} \\ 1 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}. \tag{98}$$

By Remark 12, we can compute the relative condition number $c_{\mathrm{rel}}(X)$. Some results are listed in Table 5.

Table 5 shows that the unique positive definite solution X is well conditioned.

TABLE 4: Results for **Example 3** with different values of j.

j	4	5	6	7
$\frac{\|\widetilde{X}-X\|}{\|X\|}$	1.1892×10^{-7}	2.1101×10^{-8}	2.4085×10^{-9}	1.6847×10^{-10}
ϱ	2.0791×10^{-7}	3.5353×10^{-8}	3.9573×10^{-9}	3.2580×10^{-10}

TABLE 5: Results for **Example 4** with different values of k.

k	1	3	5	7	9
$c_{\rm rel}(X)$	1.2510	1.0991	1.0009	1.0009	1.0009

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