Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 526350, 7 pages
http://dx.doi.org/10.1155/2013/526350

Research Article

Infinitely Many Periodic Solutions to Delay Differential Equations via Critical Point Theory

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Received 9 June 2013; Revised 8 November 2013; Accepted 14 November 2013

Academic Editor: Yuriy Rogovchenko

By the critical point theory, infinitely many 4σ-periodic solutions are obtained for the system of delay differential equations $\dot{x}(t) = -f(x(t-\sigma))$, where $\sigma \in (0,+\infty)$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$. It is shown that all the periodic solutions derived here are brought about by the time delay.

1. Introduction

This paper is concerned with the existence of periodic solutions to the system of delay differential equations

$$\dot{x}(t) = -f(x(t-\sigma)), \tag{1}$$

where $\sigma \in (0,+\infty)$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$.

Delay differential equations have widely been applied to describe the dynamics phenomena in both natural and manmade processes such as chemistry, physics, engineering, and economics. The existence of the periodic solutions for delay differential equations has been extensively investigated by using various methods, including fixed point theorems [1–5], Hopf bifurcation theorems [6–8], variational methods [9–14], the methods of differential inequalities [15–21], and other effective approaches (e.g., see [22–24]). In [25–31], the minimal periods of the periodic solutions to Lipschitzian differential equations are estimated through the Lipschitz constants (see Remark 4).

The use of variational methods in the study of 4σ-periodic solutions of system (1) having a variational structure was introduced in 2005 by Guo and Yu [9]. Assume that

(F_1) f is odd in x; that is, $f(-x) = -f(x)$, for all $x \in \mathbb{R}^n$;

(F_2) there exists $F \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $F_x(x) = f(x)$, for all $x \in \mathbb{R}^n$, where F_x denotes the gradient of F.

In [9], the authors obtained the multiplicity results for periodic solutions to (1) in the case that f is asymptotically linear. Later, the existence of the periodic solution of (1) was investigated by using Morse theory and Galerkin methods [10]. For the other relative investigations, we refer the reader to [11–14].

Many practical problems, such as nonlinear population growth models and control systems working with potentially explosive chemical reactions, can be transformed into the form of (1). For example, by the change of variables $y = a\tanh(ax)$, the following generalized food-limited population model

$$\dot{y}(t) = -\theta \operatorname{sign}(y(t-1)) |y(t-1)|^{\gamma} \left(a^2 - y^2(t)\right) \tag{2}$$

is transformed equivalently into (1) with $n = 1$, $f(x) = \theta a^{\gamma} \operatorname{sign}(x) |\tanh(ax)|^{\gamma}$, and $\sigma = 1$, where θ and a are positive numbers. When $\gamma = 1$, $f'(0) = \theta a^2$. It is known from [24] that, with the slope $f'(0)$ increasing and tending to infinity, the number of the periodic solutions of (2) increases and tends to infinity. Naturally, one would conjecture that when $0 < \gamma < 1$, (2) possesses infinitely many periodic solutions, since in this case $\lim_{x\to 0} f'(x) = +\infty$.

Motivated by the above observation, in this paper, we study the existence of infinitely many periodic solutions to the system (1) under the assumptions (F_1), (F_2), and

(F_3) there are $1 < \alpha, \beta < 2$ and $d_1, r_0 > 0$ such that

(i) $0 < (f(x), x) \leqslant \alpha F(x)$, for all $x \in B_{r_0} \setminus \{\mathbf{0}\}$;

(ii) $d_1 |f(x)|^{\beta'} \leqslant F(x)$, for all $x \in B_{r_0}$,

where $1/\beta + 1/\beta' = 1$, $B_{r_0} = \{x \in \mathbb{R}^n : |x| \leqslant r_0\}$.

Here and subsequently, $(\cdot, \cdot)$, $|\cdot|$ denote the inner product and the standard norm in $\mathbb{R}^n$, respectively, and the bold face $\mathbf{0}$ represents the coordinate origin of $\mathbf{R}^n$. The main result of this paper is stated as follows.

Theorem 1. *Assume that* (F_1)–(F_3) *hold. Then* (1) *possesses a sequence of nonconstant* 4σ*-periodic solutions* $\{x_m\}$ *satisfying* $\|x_m\|_\infty \to 0$ *as* $m \to \infty$.

Example 2. When $0 < \gamma < 1$, it is easy to check that $f(x) = \theta a \operatorname{sign}(x) |\tanh(ax)|^\gamma$ satisfies (F_1)–(F_3) with $\alpha = \beta = 1 + \gamma$; then (2) has a sequence of nonconstant 4-periodic solutions $\{x_m\}$ satisfying $\|x_m\|_\infty \to 0$ as $m \to \infty$.

Remark 3. Let us compare the result here with that in the case of ordinary differential equations (ODE). Without the time delay, (1) reduces to the following system of ODE

$$x'(t) = f(x(t)). \tag{3}$$

Let $x(t) = x(t; x_0)$ be the solution of (3) satisfying the initial condition $x(0) = x_0 \neq \mathbf{0}$. Then the derivative of the Lyapunov function $V(x) = |x|^2$ along $x(t)$ reads

$$\left.\frac{dV}{dt}\right|_{(3)} = (-f(x(t)), x(t)). \tag{4}$$

From (F_3)-(i), we see that $dV/dt|_{(3)} < 0$ for $0 < |x| < r_0$, which implies that there is no any periodic orbit of (3) across $B_{r_0} \setminus \{\mathbf{0}\}$; that is, the trivial solution is an isolated periodic solution. However, by the above theorem, with the time delay, the system (1) possesses infinitely many periodic solutions in any neighborhood of the origin.

Remark 4. Consider the following system of mth order functional differential equations:

$$x^{(m)}(t) = f(x(\tau(t))), \quad t \in \mathbb{R}, \tag{5}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies the Lipschitz condition and $\tau : \mathbb{R}^1 \mapsto \mathbb{R}^1$ is a measurable function. The lower bounds for the periods of the periodic solutions to (5) and their special forms are estimated in [25–31]. From this perspective, Theorem 1 complements the information in the case of non-Lipschitzian differential equations. For the unique solvability of the periodic problems on functional differential equations, we refer the reader to [1, 15–21].

The remainder of this paper is divided into two parts. In the next section, we state the preliminaries on the variational structure for (1). In the final section, the proof of Theorem 1 will be given via the $\mathbb{Z}_2$-genus theory, together with an approximating argument.

2. Preliminaries

Let $L^2(S^1, \mathbb{R}^n)$ denote the set of n-tuples of 2π-periodic functions which are square integrable. If $x \in L^2(S^1, \mathbb{R}^n)$, it has a Fourier expansion

$$x(t) = a_0 + \sum_{j \in \mathbb{N}} \left(a_j \cos jt + b_j \sin jt\right), \tag{6}$$

where $a_i, b_j \in \mathbb{R}^n$ and the series converges in the space $L^2(S^1, \mathbb{R}^n)$. For $x \in L^2(S^1, \mathbb{R}^n)$ with its expansion (6), set $H := \{x \in L^2(S^1, \mathbb{R}^n) \mid \|x\|_H < \infty\}$, where

$$\|x\|_H := |a_0|^2 + \sum_{j \in \mathbb{N}} (1 + j) \left(|a_j|^2 + |b_j|^2\right). \tag{7}$$

Then H, equipped with the norm $\|\cdot\|_H$, is a Sobolev space.

On the other hand, for $x \in H$ with its expansion (6), set

$$\|x\| := |a_0|^2 + \sum_{j \in \mathbb{N}} j \left(|a_j|^2 + |b_j|^2\right). \tag{8}$$

Then H possesses another norm $\|\cdot\|$ which is equivalent to $\|\cdot\|_H$. In the following, we always employ $\|\cdot\|$ as the norm of H. The associated inner product with $\|\cdot\|$ is denoted by $\langle \cdot, \cdot \rangle$.

Now set

$$E := \{x \in H \mid x(\cdot + \pi) = -x(\cdot)\}. \tag{9}$$

Then E is a closed subspace of H and the Fourier expansion of $x \in E$ reduces to

$$x(t) = \sum_{j=1}^{\infty} \left[a_{2j-1} \cos(2j-1)t + b_{2j-1} \sin(2j-1)t\right]. \tag{10}$$

Thus with $x_1, x_2 \in E$ being expanded as

$$x_i(t) = \sum_{j=1}^{\infty} \left[a_{2j-1}^{(i)} \cos(2j-1)t + b_{2j-1}^{(i)} \sin(2j-1)t\right], \quad i = 1, 2, \tag{11}$$

we have

$$\langle x_1, x_2 \rangle = \sum_{j=1}^{\infty} (2j-1) \left\{\left(a_{2j-1}^{(1)}, a_{2j-1}^{(2)}\right) + \left(b_{2j-1}^{(1)}, b_{2j-1}^{(2)}\right)\right\}. \tag{12}$$

For $x, y \in E$, we call y a weak derivative of x and denote it by $\dot{x} = y$ if

$$\int_0^{2\pi} \left(x(t), z'(t)\right) dt = -\int_0^{2\pi} \left(y(t), z(t)\right) dt, \quad \forall z \in C^\infty\left(S^1, \mathbb{R}^n\right). \tag{13}$$

Further, for $x \in C^\infty(S^1, \mathbb{R}^n) \cap E$ with its expansion (10), define

$$\begin{aligned} A(x) &:= \frac{1}{2} \int_0^{2\pi} \left(\dot{x}\left(t + \frac{\pi}{2}\right), x(t)\right) dt \\ &= \frac{1}{2} \sum_{j=1}^{\infty} (-1)^j (2j-1) \left(|a_{2j-1}|^2 + |b_{2j-1}|^2\right). \end{aligned} \tag{14}$$

Then it is easy to check that $|A(x)| \leqslant \|x\|^2$ for $x \in C^\infty(S^1, \mathbb{R}^n) \cap E$. Therefore A extends to all of E as a continuous quadratic form. This extension will still be denoted by A.

Let $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfy

$$\widetilde{F}(-x) = \widetilde{F}(x), \quad \left|\widetilde{F}(x)\right| \leqslant C_1 + C_2|x|^s, \quad x \in \mathbb{R}^n \tag{15}$$

for some $s \in [1, \infty)$. Define

$$\Phi(x) := \int_0^{2\pi} \widetilde{F}(x(t))\, dt, \quad x \in E \tag{16}$$

and $I(x) = A(x) + \Phi(x)$, $x \in E$. The following lemma is derived from [9, Lemma 2.2].

Lemma 5 (see [9]). *Let $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfy* (15). *Then $I \in C^1(E, \mathbb{R})$ and*

$$I'(x)\, y = A'(x)\, y + \int_0^{2\pi} \left(\widetilde{F}_x(x(t)), y(t)\right), \quad y \in E, \tag{17}$$

where

$$A'(x)\, y = \int_0^{2\pi} \left(\dot{x}\left(t + \frac{\pi}{2}\right), y(t)\right) dt, \quad y \in E. \tag{18}$$

Moreover, the existence of 2π-periodic solutions $x(t)$ for

$$x'(t) = -\widetilde{F}_x\left(x\left(t - \frac{\pi}{2}\right)\right) \tag{19}$$

satisfying $x \in E$ is equivalent to the existence of critical points of functional I.

Let $\{e_1, \ldots, e_n\}$ be the orthonormal basis of $\mathbb{R}^n$. For $k \in \mathbb{N}$, set

$$\begin{aligned} E_+(k) &:= \operatorname{span}\{\cos[(4k-1)t]\, e_i, \sin[(4k-1)t]\, e_i : \\ &\qquad i = 1, 2, \ldots, n\}, \\ E_-(k) &:= \operatorname{span}\{\cos[(4k-3)t]\, e_i, \sin[(4k-3)t]\, e_i : \\ &\qquad i = 1, 2, \ldots, n\}. \end{aligned} \tag{20}$$

For $l, m \in \mathbb{N} \cup \{+\infty\}$, define

$$V_l^{\pm} = \overline{\oplus_{k=1}^{l} E_{\pm}(k)}, \qquad V_l^m = V_l^- \oplus V_m^+, \tag{21}$$

where the closure is of E sense. Set $V^{\pm} := V_{+\infty}^{\pm}$; then $E = V^+ \oplus V^-$. In the rest of this paper, this decomposition will always be referred to when a point $x \in E$ is written as $x = x^+ + x^-$, where $x^{\pm} \in V^{\pm}$.

Remark 6. In view of (12), (14), and (18), we see that

$$A(x) = \frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right) \tag{22}$$

and that

$$A'(x)\, y = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle, \quad x, y \in E. \tag{23}$$

The following lemma is derived from [32, Lemma 2.1].

Lemma 7 (see [32]). *For each $s \in [1, \infty)$ there is $\gamma_s > 0$ such that*

$$\|x\|_s \leqslant \gamma_s m^{-1/s} \|x\| \tag{24}$$

for all $x \in (V_{m-1}^{m-1})^{\perp}$ with $m \geqslant 2$, the orthogonal complement in E, where (and below) $\|\cdot\|_s$ denotes the usual L^s-norm.

3. Proof of Theorem 1

Without loss of generality we assume that $\sigma = \pi/2$ since, under the change of variables $y(t) = x(2\sigma t/\pi)$, (1) can be transformed into the system

$$\dot{y}(t) = -\widetilde{f}\left(y\left(t - \frac{\pi}{2}\right)\right), \tag{1'}$$

where $\widetilde{f}(y) = (2\sigma/\pi) f(y)$ still satisfies (F_1–F_3) with f being replaced by $\widetilde{f}$.

Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\chi(s) = 0$ for $s \leqslant r_0/2$, $\chi(s) = 1$ for $s \geqslant r_0$, and $\chi'(s) > 0$ for $s \in (r_0/2, r_0)$. Define $\widetilde{F}: \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\widetilde{F}(x) := (1 - \chi(|x|)) F(x) + \chi(|x|) M_0 |x|^\alpha, \tag{25}$$

where $M_0 = \inf\{F(x)/r_0^\alpha : r_0/2 \leqslant |x| \leqslant r_0\}$.

Let $\alpha' > 0$ be such that $1/\alpha + 1/\alpha' = 1$. By (F_3) we get

$$0 < \left(\widetilde{F}_x(x), x\right) \leqslant \alpha \widetilde{F}(x), \quad \forall x \in \mathbb{R}^n, \tag{26}$$

$$\widetilde{F}(x) \geqslant \begin{cases} C_1\left|\widetilde{F}_x(x)\right|^{\beta'}, & |x| \leqslant 1, \\ C_1\left|\widetilde{F}_x(x)\right|^{\alpha'}, & |x| > 1, \end{cases} \tag{27}$$

where (and below) C_j's stand for positive constants.

Lemma 8. *Let $\widetilde{F}: \mathbb{R}^n \mapsto \mathbb{R}$ be defined by* (25); *then $1 < \beta < \alpha < 2$, $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$, and*

$$C_2|x|^\alpha \leqslant \widetilde{F}(x) \leqslant \begin{cases} C_3|x|^\beta, & |x| \leqslant 1, \\ C_3|x|^\alpha, & |x| > 1. \end{cases} \tag{28}$$

Proof. From (25), it is easy to see that $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$. Now we start to prove (28). Let M be such a constant that $|\ln \widetilde{F}(x) - \alpha \ln |x|| \leqslant M$ for $x \in S_1 \equiv \partial B_1$. For $x \in \mathbb{R}^n$, $|x| \leqslant 1$, set $x_0 = x/|x|$; then $x_0 \in S_1$. Define $g(t) = \ln \widetilde{F}(t x_0) - \alpha \ln |t x_0|$, $t \in (0, 1]$. Then, by (26),

$$g'(t) = \left(\frac{\widetilde{F}_x(t x_0)}{\widetilde{F}(t x_0)}, x_0\right) - \alpha \leqslant 0, \tag{29}$$

which implies that $g(|x|) \geqslant g(1)$; that is,

$$\ln \widetilde{F}(x) - \alpha \ln |x| \geqslant \ln \widetilde{F}(x_0) - \alpha \ln |x_0| \geqslant -M. \tag{30}$$

It follows that $\widetilde{F}(x) \geqslant e^{-M}|x|^\alpha$ for $|x| \leqslant 1$, which, combining with (25), leads to the inequality on the left hand of (28) with C_2 being chosen adequately.

Again, for $x \in \mathbb{R}^n$, $|x| \leqslant 1$, set $x_0 = x/|x|$ and define $h(t) = (\widetilde{F}(tx_0))^{1/\beta} - t|x_0|/(\beta C_1^{1/\beta'})$, $t \in [0,1]$. Then by the first inequality in (27),

$$\begin{aligned} h'(t) &= \frac{1}{\beta}\left(\frac{\left(\widetilde{F}_x(tx_0), x_0\right)}{\left(\widetilde{F}(tx_0)\right)^{1/\beta'}} - \frac{1}{C_1^{1/\beta'}}|x_0|\right) \\ &\leqslant \frac{|x_0|}{\beta}\left(\frac{\left|\widetilde{F}_x(tx_0)\right|}{\left(\widetilde{F}(tx_0)\right)^{1/\beta'}} - \frac{1}{C_1^{1/\beta'}}\right) < 0. \end{aligned} \tag{31}$$

Thus $h(|x|) \leqslant h(0) = 0$, which leads to $\widetilde{F}(x) \leqslant C_3'|x|^\beta$, where $C_3' = 1/(\beta C_1^{1/\beta'})^\beta$. In the same way, from the second inequality in (27), we can arrive at $\widetilde{F}(x) \leqslant C_3''|x|^\alpha$ for $|x| > 1$, where the constant C_3'' only depends on α and C_1. With $C_3 = \max\{C_3', C_3''\}$, the inequalities on the right hand of (28) hold. Thus we get (28), which implies that $C_2|x|^\alpha \leqslant C_3|x|^\beta$ for $|x| \leqslant 1$ and that $1 < \beta < \alpha < 2$. The proof is complete. □

Now we consider the functional

$$I(x) = A(x) + \int_0^{2\pi} \widetilde{F}(x)\,dt, \quad x \in E. \tag{32}$$

Lemma 9. *I satisfies (PS) condition; that is, every sequence $\{x_k\} \subset E$ such that $\{I(x_k)\}$ is bounded and $I'(x_k) \to 0$ as $k \to \infty$ has a convergent subsequence.*

Proof. By Lemma 5, for $x \in E$, $I'(x)$ is defined by

$$I'(x)y = A'(x)y + \int_0^{2\pi}\left(\widetilde{F}_x(x), y\right)dt, \quad \forall y \in E. \tag{33}$$

To verify that I satisfies (PS) condition, we suppose $|I(x_k)| \leqslant C_4$ and $I'(x_k) \to 0$ as $k \to \infty$. Note that, for large k, $|I'(x_k)x| \leqslant \|x\|$. Thus for large k and $x = x_k$, from (32) and (33),

$$\begin{aligned} C_4 + \|x\| &\geqslant I(x) - \frac{1}{2}I'(x)x \\ &= \int_0^{2\pi}\left[\widetilde{F}(x) - \frac{1}{2}\left(\widetilde{F}_x(x), x\right)\right]dt. \end{aligned} \tag{34}$$

Noticing that $1 < \beta < \alpha < 2$, we see from (25) that, for all $x \in \mathbb{R}^n$,

$$\left(\widetilde{F}_x(x), x\right) \geqslant C_5 \max\left\{|x|^\alpha, |x|^\beta\right\} - C_6, \tag{35}$$

which, combining with (26) and (34), implies

$$\begin{aligned} C_4 + \|x\| &\geqslant \left(\alpha^{-1} - 2^{-1}\right)\int_0^{2\pi}\left(\widetilde{F}_x(x), x\right)dt \\ &\geqslant C_7 \max\left\{\|x\|_\alpha^\alpha, \|x\|_\beta^\beta\right\} - C_8, \end{aligned} \tag{36}$$

$$\max\left\{\|x\|_\alpha^\alpha, \|x\|_\beta^\beta\right\} \leqslant C_9(\|x\| + 1). \tag{37}$$

Next for large k, taking $x = x_k$ and $\varsigma = x_k^+$ in

$$\left|\int_0^{2\pi}\left(\widetilde{F}_x(x), \varsigma\right)dt + A'(x)\varsigma\right| = \left|I'(x)\varsigma\right| \leqslant \|\varsigma\| \tag{38}$$

and using (23), (27), and (28) and the Hölder inequality ($1/\alpha + 1/\alpha' = 1$, $1/\beta + 1/\beta' = 1$), we get

$$\begin{aligned} \|x^+\|^2 &\leqslant \left|\int_0^{2\pi}\left(\widetilde{F}_x(x), x^+\right)dt\right| + \|x^+\| \\ &\leqslant C_{10}\left(\int_{|x(t)|>1}|x|^{\alpha/\alpha'}\left|x^+\right|dt + \int_{|x(t)|\leqslant 1}|x|^{\beta/\beta'}\left|x^+\right|dt\right) \\ &\quad + \|x^+\| \\ &\leqslant C_{10}\left(\|x\|_\alpha^{\alpha/\alpha'}\|x^+\|_\alpha + \|x\|_\beta^{\beta/\beta'}\|x^+\|_\beta\right) + \|x^+\| \\ &\leqslant C_{11}\left(\|x\|_\alpha^{\alpha/\alpha'} + \|x\|_\beta^{\beta/\beta'} + 1\right)\|x^+\|, \end{aligned} \tag{39}$$

where the last inequality holds since E is compactly embedded in $L^s(S^1, \mathbb{R}^n)$ for $s \geqslant 1$. It follows from (36) that

$$\|x^+\| \leqslant C_{12}\left(\|x\|^{1/\alpha'} + \|x\|^{1/\beta'} + 1\right). \tag{40}$$

Similarly, (40) works with x^+ being replaced by x^-. Combining these inequalities shows

$$\|x\| \leqslant C_{13}\left(\|x\|^{1/\alpha'} + \|x\|^{1/\beta'} + 1\right), \tag{41}$$

which implies that $\{x_k\}$ is bounded in E.

Let Φ be defined by (16). By [33, Proposition B.37], $\{\Phi'(x_k)\}$ is precompact in E. Moreover, from (23) and (33),

$$I'(x_k) = x_k^+ - x_k^- + \Phi'(x_k). \tag{42}$$

It follows that $\{x_k\}$ has a convergent subsequence. The proof is complete. □

Lemma 10. *For each $l \in \mathbb{N}$, there are $\rho_l > 0$, $a_l > 0$, and $0 < b_l \to 0$ such that*

(a) $I(x) \geqslant 0$, *for all* $x \in B_{\rho_l} \cap V_l^{+\infty}$ *and* $\inf I(\partial S_{\rho_l} \cap V_l^{+\infty}) \geqslant a_l$, *where* $B_{\rho_l} = \{x \in E : \|x\| \leqslant \rho_l\}$;

(b) $\sup I((V_{l-1}^{+\infty})^\perp) \leqslant b_l$.

Proof. Noticing that $V_l^{+\infty} = V_l^- \oplus V_{+\infty}^+$ and that $\dim(V_l^-) < \infty$, we have, for $x \in B_{\rho_l} \cap V_l^{+\infty}$,

$$\begin{aligned} \|x\|_\alpha &= \sup\left\{\int_{S^1}(x(t), y(t))\,dt \mid y \in L^{\alpha'}\left(S^1, \mathbb{R}^n\right), \|y\|_{\alpha'} = 1\right\} \\ &\geqslant \sup\left\{\int_{S^1}(x(t), y(t))\,dt \mid y \in V_l^-, \|y\|_{\alpha'} = 1\right\} \\ &= \sup\left\{\int_{S^1}(x^-(t), y(t))\,dt \mid y \in V_l^-, \|y\|_{\alpha'} = 1\right\} \\ &= \|x^-\|_\alpha \geqslant \eta_l\|x^-\|, \end{aligned} \tag{43}$$

where η_l is a positive constant depending on l. It follows by (28) that, for $x \in B_{\rho_l} \cap V_l^{+\infty}$,

$$\begin{aligned} I(x) &\geqslant \frac{1}{2}\|x^+\|^2 + C_2\eta_l\|x^-\|^\alpha - \frac{1}{2}\|x^-\|^2 \\ &= \frac{1}{2}\|x^+\|^2 + \left(C_2\eta_l - \frac{1}{2}\|x^-\|^{2-\alpha}\right)\|x^-\|^\alpha, \end{aligned} \tag{44}$$

which implies (a) by setting $\rho_l := \min\{1, (C_2\eta_l)^{1/(2-\alpha)}\}$ and $a_l = \rho_l^2(1 + C_2\eta_l)/2$.

Let $x \in (V_{l-1}^{+\infty})^\perp$. By (28) and Lemma 7,

$$\begin{aligned} I(x) &\leqslant C_3\left(\|x\|_\alpha^\alpha + \|x\|_\beta^\beta\right) - \frac{1}{2}\|x\|^2 \\ &\leqslant C_{16}l^{-1}\left(\|x\|^\alpha + \|x\|^\beta\right) - \frac{1}{2}\|x\|^2 \\ &\leqslant b_l := \sup_{s\geq 0} g(s), \end{aligned} \tag{45}$$

where $g(s) := C_{16}l^{-1}(s^\alpha + s^\beta) - s^2/2$. Noticing $1 < \beta \leqslant \alpha < 2$, one can see that $b_l \to 0$ as $l \to \infty$ and (b) follows. The proof is complete. □

In the following, let Σ denote the family of closed (in E) subsets of $E \setminus \{\mathbf{0}\}$ symmetric with respect to the origin, and

$$\gamma : \Sigma \longmapsto \mathbb{N} \cup \{0, \infty\}, \tag{46}$$

the $\mathbb{Z}_2$-genus map (see [33]). For $l, m \in \mathbb{N}$, set

$$\Sigma_l^m = \left\{A \in \Sigma : A \subset V_{+\infty}^m, \gamma(A) \geqslant n(l+m)\right\}, \tag{47}$$

and define

$$c_{l,m} = \sup_{A\in\Sigma_l^m} \inf_{x\in A} I(u). \tag{48}$$

Lemma 11. *For all $l, m \in \mathbb{N}$, $c_{l,m}$ is a critical value of I and*

$$a_l \leqslant c_{l,m} \leqslant b_l. \tag{49}$$

Proof. We first prove that (49) holds. For each $m \in \mathbb{N}$, let ρ_l be chosen as that in Lemma 10; then it follows by Lemma 10(a) that $\inf I(\partial S_{\rho_l} \cap V_l^{+\infty}) \geqslant a_l$. Denote $\widetilde{A} = \partial S_{\rho_l} \cap V_l^m$; then $\gamma(\widetilde{A}) = n(m+l)$ and $\widetilde{A} \in \Sigma_l^m$. Since $\widetilde{A} \subset \partial S_{\rho_l} \cap V_l^{+\infty}$, we have

$$c_{l,m} \geqslant \inf_{x\in\widetilde{A}} I(x) \geqslant \inf\, I\left(\partial S_{\rho_l} \cap V_l^{+\infty}\right) \geqslant a_l. \tag{50}$$

On the other hand, for every $A \in \Sigma_l^m$, by the property of genus, $A \cap (V_{l-1}^{+\infty})^\perp \neq \varnothing$, which, from Lemma 10(b), leads to $\inf_{x\in A} I(x) \leqslant b_l$ for every $A \in \Sigma_l^m$. Thus $c_{l,m} \leqslant b_l$ and (49) holds.

By (F_1) and (25), $\widetilde{F}(x)$ is even with respect to x, which implies that I is even. We claim that $c = c_{l,m}$ is a critical point of I. Otherwise, there exists $\epsilon > 0$, such that there is no any critical point in the interval $(c - \epsilon, c + \epsilon)$. By the definition of $c_{l,m}$, there exists $A \in \Sigma_l^m$, such that

$$\inf_{x\in A} I(x) > c - \epsilon. \tag{51}$$

For $a \in \mathbb{R}$, denote $I^a = \{x \in E : I(x) \geqslant a\}$. Use a positive rather than a negative gradient flow [33, Remark A.17], we get $\eta \in C([0,1] \times E, E)$ such that $\eta(1, \cdot)$ is odd and

$$\eta\left(1, I^{c-\epsilon}\right) \subset I^{c+\epsilon}. \tag{52}$$

Since $A \subset I^{c-\epsilon}$, we have $\eta(1, A) \subset I^{c+\epsilon}$; that is,

$$\inf_{x\in\eta(1,A)} I(x) \geqslant c + \epsilon. \tag{53}$$

On the other hand, by the property of genus, we know that $\gamma(\eta(1, A)) \in \Sigma_l^m$, which, by the definition of c, leads to

$$c \geqslant \inf_{x\in\eta(1,A)} I(x) \geqslant c + \epsilon. \tag{54}$$

This contradiction implies that $c_{l,m}$ is a critical value of I. The proof is complete. □

Now we are in a position to give the following proof.

Proof of Theorem 1. In view of Lemma 11, let $x_{l,m} \in V_{+\infty}^m$ be such that

$$I\left(x_{l,m}\right) = c_{l,m}, \qquad I'\left(x_{l,m}\right) = 0. \tag{55}$$

Then by (PS) condition, along a subsequence as $m \to \infty$, $x_{l,m} \to x_l \in E$ such that

$$a_l \leqslant I\left(x_l\right) \leqslant b_l, \qquad I'\left(x_l\right) = 0, \tag{56}$$

which implies that x_l is nonzero. Moreover, by Lemma 5,

$$x_l'(t) = -\widetilde{F}_x\left(x_l\left(t - \frac{\pi}{2}\right)\right). \tag{57}$$

We claim that, for sufficiently large l, x_l solves (1). In fact, from (26) and (56)

$$\begin{aligned} b_l &\geqslant I\left(x_l\right) = I\left(x_l\right) - \frac{1}{2}I'\left(x_l\right)x_l \\ &\geqslant \left(1 - \frac{\alpha}{2}\right)\int_0^{2\pi} \widetilde{F}\left(x_l\right)dt. \end{aligned} \tag{58}$$

By (27), (58), and Hölder inequality

$$\begin{aligned} \|x_l^+\|^2 &= \int_0^{2\pi}\left(\widetilde{F}_x\left(x_l\right), x_l^+\right)dt \\ &\leqslant \|x_l^+\|_\alpha\left(\int_0^{2\pi}\left|\widetilde{F}_x\left(x_l\right)\right|^{\alpha'}dt\right)^{1/\alpha'} \\ &\quad + \|x_l^+\|_\beta\left(\int_0^{2\pi}\left|\widetilde{F}_x\left(x_l\right)\right|^{\beta'}dt\right)^{1/\beta'} \\ &\leqslant C_{17}\|x_l^+\|\left(\int_0^{2\pi}\widetilde{F}\left(x_l\right)dt\right)^{1/\alpha'} \\ &\quad + C_{18}\|x_l^+\|\left(\int_0^{2\pi}\widetilde{F}\left(x_l\right)dt\right)^{1/\beta'} \\ &\leqslant \left(C_{17}b_l^{1/\alpha'} + C_{18}b_l^{1/\beta'}\right)\|x^+\|. \end{aligned} \tag{59}$$

Similarly, the above inequality works with x_l^+ replaced by x_l^-. These inequalities yield

$$\|x_l\| \le C_{19} b_l^{1/\alpha'} + C_{20} b_l^{1/\beta'}. \tag{60}$$

Since $b_l \to 0$ as $l \to \infty$, it follows that

$$\|x_l\| \longrightarrow 0 \quad \text{as } l \longrightarrow \infty. \tag{61}$$

Furthermore, from (27) and (57), we have

$$\int_0^{2\pi} |\dot{x}_l(t)|^2 dt = \int_0^{2\pi} \left|\widetilde{F}_x(x_l)\right|^2 dt \leqslant C_{21}\left(\int_0^{2\pi}\left[\widetilde{F}(x_l)\right]^{2/\beta'} dt + \int_0^{2\pi}\left[\widetilde{F}(x_l)\right]^{2/\alpha'} dt\right). \tag{62}$$

It follows from (58) that $\|\dot{x}_l\|_2 \to 0$ as $l \to \infty$. Recalling (61), we get

$$\|x_l\|_{W^{1,2}} \longrightarrow 0 \quad \text{as } l \longrightarrow \infty, \tag{63}$$

which implies that $\|x_l\|_\infty \to 0$ as $l \to 0$. Thus for m sufficiently large, $\|x_l\|_\infty < r_0/2$ and therefor $\widetilde{F}_x(x_l) = F_x(x_l)$. It follow from (57) that, for l sufficiently large, x_l solves (1). In addition, by (1) and (F_3)(i), the only constant solution of (1) is the trivial solution. Then (56) yields that x_l is nonconstant and the proof of Theorem 1 is complete. □

Acknowledgments

The authors are grateful to the referees' careful reviewing and helpful comments. The work was supported by BJJW (KZ201310028031 and KM2014).

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