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Research Article

Variational Approach to Impulsive Differential Equations with Singular Nonlinearities

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We discuss the existence of periodic solutions for nonautonomous second order differential equations with singular nonlinearities. Simple sufficient conditions that enable us to obtain many distinct periodic solutions are provided. Our approach is based on a variational method.

1. Introduction

Differential equations with impulsive effects appear naturally in the description of many evolution processes whose states experience sudden changes at certain times, called impulse moments. There is an extensive bibliography about the subject. For recent references, see [1].

Variational methods have been successfully employed to investigate regular second order differential equations with impulsive effects; See, for instance, [2–8]. In particular, the paper [8] considers the existence of n distinct pairs of nontrivial solutions. However, very few papers have used variational methods to investigate the case of impulsive second order boundary value problems with singular nonlinearities. In fact, it seems that the work [9] is the first paper along this line. Singular boundary value problems without impulses have attracted the attention of many researchers; see [10] for details and references. This paper is devoted to the study of the existence and multiplicity of periodic solutions for impulsive second order differential equations with singular nonlinearities. More specifically, we consider the following impulsive problem:

$$\begin{gathered} u''(t) + \lambda f(t, u(t)) = 0, \quad t \neq t_j,\ t \in I, \\ -\Delta u'(t) = I_j\left(u\left(t_j\right)\right), \quad j = 1, 2, \ldots, p, \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{gathered} \tag{1}$$

where I denotes the real interval $[0, T]$, with $T > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, λ is a positive parameter,

$$\Delta u'(t) = u'\left(t_j^+\right) - u'\left(t_j^-\right), \tag{2}$$

and $f : I \times \mathbb{R} \to \mathbb{R}$ presents a singularity with respect to its second argument at $u = 0$.

Throughout this paper we will use the following notations. $L^p(I)$ is the classical Lebesgue space of measurable functions $u : I \to \mathbb{R}$ such that $|u(\cdot)|^p$ is integrable, and for $u \in L^p(I)$ we define its norm by

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}. \tag{3}$$

Let $\|u\|_\infty = \sup\{|u(t)|; t \in I\}$ denote the norm of $u \in C(I)$, the space of real-valued continuous functions. $W^{1;2}(I)$ is the classical Sobolev space of functions $u \in L^2(I)$ with their distributional derivatives $u' \in L^2(I)$. We set $H_T^1 = \{u \in W^{1,2}(I); u(0) = u(T)\}$ and for $u \in H_T^1$ we define its norm by

$$\|u\|_{H_T^1} = \left(\left\|u'\right\|_{L^2} + \|u\|_{L^2}\right)^{1/2}. \tag{4}$$

Wirtinger inequality

$$\|u\|_{L^2} \leq \frac{T}{2\pi}\left\|u'\right\|_{L^2} \tag{5}$$

implies that we can consider on H_T^1 the norm

$$\|u\| = \left\|u'\right\|_{L^2}. \tag{6}$$

H_T^1 endowed with the norm $\|\cdot\|$ is a reflexive Banach space.

We introduce the following assumptions on the nonlinearity.

(H1)

(i) $f : I \times (0;+\infty) \to \mathbb{R}$ is an L^1-Carathéodory function;

(ii) $\lim_{u\to 0^+} f(t,u) = -\infty$ for almost every $t \in I$;

(iii) there exist $u_0 \geq 1$, $a, b > 0$, and $\gamma \in (0,1)$ such that $f(t,u_0) = 0$, and for almost all $t \in I$ and all $u \in (u_0, +\infty)$

$$\left|f(t,u)\right| \leq a + bu^{\gamma}; \tag{7}$$

(iv) $\inf_{u>u_0+1}\int_0^T F(t,u)dt > 0$, with $F(t;u) = \int_{u_0}^u f(t,s)ds$;

(v) $(\partial F/\partial t)(t,u)$ exists and is nonpositive for almost all $t \in I$ and all $u \in \mathbb{R}$;

(vi) $\lim_{u\to+\infty}\int_0^T F(t,u)dt = +\infty$.

The jump functions I_j, $j = 1, 2, \ldots, p$, satisfy the following:

(H2)

(i) $I_j : \mathbb{R} \to \mathbb{R}$ is odd, continuous, and bounded;

(ii) $\int_0^u I_j(s)ds \leq 0$.

Definition 1. A solution of (1) is a function $u \in C(I)$ such that for every $j = 1, \ldots, p$,

$$u_j = u\Big|_{[t_j, t_{j+1}]} \tag{8}$$

is absolutely continuous with its derivatives u_j' and $u_j'' \in L^2(t_j, t_{j+1})$ and satisfies the differential equation in (1) for $t \in I \setminus \{t_1, t_2, \ldots, t_p\}$; the limits $u'(t_j^-)$ and $u'(t_j^+)$, $j = 1, 2, \ldots, p$, exist; the impulsive conditions and the boundary conditions in (1) hold.

2. The Main Result

In this section we state and prove our main result.

Theorem 2. *Suppose that conditions (H1) and (H2) hold. Then for any $n \in \mathbb{N}$, there exists λ_n such that for $\lambda > \lambda_n$ problem* (1) *has infinitely many distinct nontrivial solutions.*

Proof. To give the proof of the main result, we first modify problem (1) to another one which is not singular.

For $\xi \in (0, \min(1, u_0/4))$, define a function f_ξ on $I \times [0, +\infty)$ by

$$f_\xi(t,u) = \begin{cases} \dfrac{f(t,\xi)}{\sin 2\pi(\xi/u_0)} \sin 2\pi \dfrac{u}{u_0}, & 0 \leq u \leq u_0, \\ f(t,u), & u > u_0. \end{cases} \tag{9}$$

Finally, we require $f_\xi(t,-u) = -f_\xi(t,u)$ for all $u \in \mathbb{R}$ and almost $t \in I$, to make f_ξ odd. Then f_ξ satisfies the following:

(j) f_ξ is an L^1-Carathéodory function and is odd in u;

(jj) there exist $a > 0, b > 0, b' > 0$, and $\gamma \in (0,1)$ such that $|f_\xi(t,u)| \leq a + bu^\gamma$ for all $u \in (-\infty, -u_0) \cup (u_0, +\infty)$ and almost $t \in I$ and $|f_\xi(t,u)| \leq b'u$ for all $u \in (-u_0, u_0)$ and almost all $t \in I$;

(jjj) for almost all $t \in I$ and all $u \in \mathbb{R}$, $F_\xi(t,u) := \int_{u_0}^u f_\xi(t,s)ds$ is such that

$$\begin{aligned} F_\xi(t,0) &= -\int_0^{u_0} f_\xi(t,s)\,ds = -\frac{f(t,\xi)}{\sin(2\pi\xi/u_0)}\int_0^{u_0}\sin\frac{2\pi}{u_0}s\,ds \\ &= \left(\frac{f(t,\xi)}{\sin(2\pi\xi/u_0)}\frac{u_0}{2\pi}\right)\int_0^{2\pi} -\sin\zeta\,d\zeta = 0, \\ &\qquad \text{for almost all } t \in I, \end{aligned}$$

$$\begin{aligned} F_\xi(t,-u) &= \int_{u_0}^{-u} f_\xi(t,s)\,ds \\ &= -\int_{-u}^{u_0} f_\xi(t,s)\,ds \\ &= -\left(\int_{-u}^{0} f_\xi(t,s)\,ds + \int_0^{u_0} f_\xi(t,s)\,ds\right) \\ &= -\left(\int_{-u}^{0} f_\xi(t,s)\,ds\right) = \int_0^{-u} f_\xi(t,s)\,ds \\ &= -\int_0^{u} f_\xi(t,-\zeta)\,d\zeta = \int_0^{u} f_\xi(t,\zeta)\,d\zeta \\ &= \int_0^{u_0} f_\xi(t,s)\,ds + \int_{u_0}^{u} f_\xi(t,s)\,ds = F_\xi(t,u), \\ &\qquad \text{for almost all } t \in I, \text{ all } u \in \mathbb{R}, \end{aligned}$$

$$\inf_{u>u_0+1}\int_0^T F_\xi(t,u)\,dt = \inf_{u>u_0+1}\int_0^T F(t,u)\,dt > 0,$$
$$F_\xi(t,u) < 0, \text{ for } 0<u<u_0, \text{for almost all } t\in I. \tag{10}$$

We study the modified problem

$$\begin{aligned} &u''(t) + \lambda f_\xi(t,u(t)) = 0, \quad t\neq t_j,\ t\in I,\\ &-\Delta u'(t) = I_j\left(u\left(t_j\right)\right), \quad j=1,2,\ldots,p,\\ &u(0)=u(T), \qquad u'(0)=u'(T). \end{aligned} \tag{11}$$

Consider the functional $\varphi_\xi : H_T^1 \to \mathbb{R}$, defined by

$$\varphi_\xi(u) = \frac{1}{2}\|u\|^2 - \lambda\int_0^T F_\xi(t,u(t))\,dt - \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds. \tag{12}$$

Clearly φ_ξ is an even functional, $\varphi_\xi(0)=0$, and is Frechet differentiable, whose Frechet derivative at the point $u\in H_T^1$ is the functional $\varphi_\xi'(u)\in(H_T^1)^*$ given by

$$\varphi_\xi'(u)\cdot v = \int_0^T u'(t)\cdot v'(t)\,dt - \lambda\int_0^T f_\xi(t,u(t))\cdot v(t)\,dt - \sum_{j=1}^{p} I_j\left(u\left(t_j\right)\right)v\left(t_j\right). \tag{13}$$

Obviously, φ_ξ' is continuous. If $u\in H_T^1$ is a critical point of the functional φ_ξ, then u is a solution of problem (11).

First, we show that φ_ξ is bounded from below. Define a subset Ω of H_T^1 as follows:

$$\Omega = \left\{u\in H_T^1, \min(u) > 1+u_0\right\} \cup \left\{u\in H_T^1, \max(u) < -\left(1+u_0\right)\right\}, \tag{14}$$

noticing that

$$\begin{aligned} \partial\Omega = &\left\{u\in H_T^1,\ u(t)\ge 1+u_0\ \forall t\in I,\ \exists t_u\in I \text{ with } u\left(t_u\right)=1+u_0\right\}\\ &\cup\left\{u\in H_T^1,\ u(t)\le -\left(1+u_0\right)\ \forall t\in I,\ \exists t_u\in I \text{ with } u\left(t_u\right) = -\left(1+u_0\right)\right\}. \end{aligned} \tag{15}$$

For $u\in\Omega$, we have

$$\begin{aligned} \varphi_\xi(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_0^T F_\xi(t,u(t))\,dt - \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds\\ &\ge \frac{1}{2}\|u\|^2 - \lambda\int_0^T\left(a|u(t)| + b|u(t)|^{\gamma+1}\right)dt\\ &\ge \frac{1}{2}\|u\|^2 - \lambda a T^{3/2}\|u\| - b\lambda T^{(3+\gamma)/2}\|u\|^{\gamma+1} > -\infty. \end{aligned} \tag{16}$$

If $u\notin\Omega$, we use the partition of the interval $I = I_1\cup I_2\cup I_3$, where

$$\begin{aligned} I_1 &= \left\{t\in I;\ u(t)\in\left[-u_0,u_0\right]\right\},\\ I_2 &= \left\{t\in I;\ u(t)\in\left[u_0,\ 1+u_0\right]\cup\left[-\left(u_0+1\right),\ -u_0\right]\right\},\\ I_3 &= \left\{t\in I;\ u(t)\in\left(-\infty,-\left(1+u_0\right]\cup\left[1+u_0,-\infty\right)\right\}. \end{aligned} \tag{17}$$

By (H2) and (jjj), we have

$$\begin{aligned} \varphi_\xi(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_0^T F_\xi(t,u(t))\,dt - \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds \ge \frac{1}{2}\|u\|^2\\ &\quad - \lambda\int_{I_3}\left(a|u(t)| + b|u(t)|^{\gamma+1}\right)dt\\ &\quad - \lambda\int_{I_2} F_\xi(t,u(t))\,dt - \lambda\int_{I_1} F_\xi(t,u(t))\,dt. \end{aligned} \tag{18}$$

From (jjj), $F_\xi(t,u(t)) < 0$ on I_1, which implies

$$\begin{aligned} \varphi_\xi(u) &\ge \frac{1}{2}\|u\|^2 - \lambda\int_{I_3} a|u(t)|\\ &\quad + b|u(t)|^{\gamma+1}dt - \lambda\int_{I_2} F_\xi(t,u(t))\,dt. \end{aligned} \tag{19}$$

If $t\in I_2$, then $u(t)\in[u_0,u_0+1]\cup[-(u_0+1),-u_0]$. This means that $u(t)$ is bounded uniformly in $t\in I_2$. Since f_ξ is a Caratheodory function, it follows that F_ξ is bounded by some positive constant C. Hence,

$$\begin{aligned} \varphi_\xi(u) &\ge \frac{1}{2}\|u\|^2 - \lambda a T^{3/2}\|u\|\\ &\quad - b\lambda\cdot T^{(3+\gamma)/2}\|u\|^{\gamma+1} - \lambda C > -\infty. \end{aligned} \tag{20}$$

This shows that φ_ξ is bounded from below.

Remark 3. There exists $m>0$ such that $\inf_{u\in\partial\Omega}\varphi_\xi(u)\ge -m$, whenever $\xi\in(0,\min(1,u_0/4))$.

Our next step is to show that φ_ξ satisfies the Palais-Smale condition. For this purpose, let $(u_k)_k$ be a sequence in H_T^1 such that $(\varphi(u_k))_k$ is bounded and $\lim_{k\to+\infty}\varphi'(u_k)=0$. Then, there exists $M > 0$ such that $|\varphi_\xi(u_k)| \le M$. In view of (16), if $u_k \in \Omega$, we have

$$M \ge \frac{1}{2}\|u_k\|^2 - \lambda a' T^{3/2}\|u_k\| - b\lambda \cdot T^{(3+\gamma)/2}\|u_k\|^{\gamma+1}, \quad (21)$$

and if $u_k \notin \Omega$, we can proceed as above to show that

$$\begin{aligned}\frac{1}{2}\|u_k\|^2 - \lambda a T^{3/2}\|u_k\| - b\lambda \cdot T^{(3+\gamma)/2}\|u_k\|^{\gamma+1} - \lambda C \\ \le \varphi_\xi(u_k) \le |\varphi_\xi(u_k)| \le M.\end{aligned} \quad (22)$$

This shows that $\|u_k\|$ is bounded. So that, $(u_k)_{k\in\mathbb{N}}$ is bounded in H_T^1. From the reflexivity of H_T^1, we may extract from $(u_k)_{k\in\mathbb{N}}$ a weakly convergent subsequence, which we label the same; that is, $u_k \rightharpoonup u$ in H_T^1. Since the injection of H_T^1 into $C(I)$, with its natural norm, is continuous, it follows that $u_k \rightharpoonup u$ in $C(I)$ and by Banach-Steinhaus theorem, the subsequence $(u_k)_{k\in\mathbb{N}}$ is bounded in H_T^1 and hence, in $C(I)$. Moreover, the subsequence $(u_k)_k$ is uniformly equicontinuous since, for $0 \le s \le t \le T$, we have

$$\begin{aligned}|u_k(t) - u_k(s)| &\le \int_s^t |u_k'(\tau)|\, d\tau \\ &\le (t-s)^{1/2}\left(\int_s^t |u_k'(\tau)|^2 d\tau\right)^{1/2} \\ &\le (t-s)^{1/2}\|u_k\| \le C(t-s)^{1/2}.\end{aligned} \quad (23)$$

By Ascoli-Arzela theorem the subsequence $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C(I)$. By the uniqueness of the weak limit in $C(I)$, every uniformly convergent subsequence of $(u_k)_{k\in\mathbb{N}}$ converges to u. Thus, $(u_k)_{k\in\mathbb{N}}$ converges uniformly on I to u.

Next, we will verify that $(u_k)_{k\in\mathbb{N}}$ strongly converges to u in H_T^1. By (13), we have

$$\begin{aligned}&(\varphi_\xi'(u_k) - \varphi_\xi'(u))(u_k - u) \\ &= \|u_k - u\|^2 \\ &\quad -\lambda \int_0^T [f_\xi(t, u_k(t)) - f_\xi(t, u(t))](u_k(t) - u(t))\, dt \\ &\quad - \sum_{j=1}^p [I_j(u_k(t_j)) - I_j(u(t_j))](u_k(t_j) - u(t_j)).\end{aligned} \quad (24)$$

The uniform convergence of $(u_k)_{k\in\mathbb{N}}$ to u in $C(I)$ implies

$$\begin{aligned}&\lambda \int_0^T [f_\xi(t, u_k(t)) - f_\xi(t, u(t))](u_k(t) - u(t))\, dt \longrightarrow 0, \\ &\qquad \text{as } k \longrightarrow \infty, \\ &\sum_{j=1}^p [I_j(u_k(t_j)) - I_j(u(t_j))](u_k(t_j) - u(t_j)) \longrightarrow 0, \\ &\qquad \text{as } k \longrightarrow \infty.\end{aligned} \quad (25)$$

Since $\varphi'(u_k) \to 0$ and $u_k \rightharpoonup u$, as $k \to \infty$,

$$\lim_{k\to\infty} (\varphi_\xi'(u_k) - \varphi_\xi'(u))(u_k - u) = 0. \quad (26)$$

In view of (24), (25), and (26), we obtain

$$\lim_{k\to\infty} (u_k - u) = 0. \quad (27)$$

Thus, φ satisfies the Palais-Smale condition.

Let $v_1, v_2, \dots, v_n$ denote the eigenfunctions of $-u'' = \lambda u$, $u(0) = u(T)$, $u'(0) = u'(T)$, corresponding to the eigenvalues $\mu_1, \mu_2, \dots, \mu_n$. We normalize $v_1, v_2, \dots, v_n$, so that

$$\|v_k\| = 1 = \mu_k \int_0^T v_k(t)^2 dt, \qquad 1 \le k \le n. \quad (28)$$

For $r > u_0 + 1$, we set

$$K_n(r) = \left\{\sum_{m=1}^n c_m v_m;\ \sum_{m=1}^n c_m^2 = r^2\right\}, \quad (29)$$

and $E = K_n(r) \cap \Omega$. Then, for any $u \in E$, we have for almost every $t \in I$, $u(t) \ge u_0 + 1$, or $u(t) \le -(u_0 + 1)$, and

$$\begin{aligned}F_\xi(t, -u(t)) = F_\xi(t, u(t)) &= \int_{u_0}^{u(t)} f_\xi(t, s)\, ds \\ &= \int_{u_0}^{u(t)} f(t, s)\, ds = F(t, u(t)).\end{aligned} \quad (30)$$

It follows from (H1)(iv) that, for any $u \in E$,

$$\int_0^T F_\xi(t, u(t))\, dt > 0. \quad (31)$$

Let

$$\alpha_n = \inf_{u\in E}\int_0^T F_\xi(t, u(t))\, dt, \qquad \beta_n = \inf_{u\in E}\sum_{j=1}^p \int_0^{u(t_j)} I_j(s)\, ds. \quad (32)$$

It is clear that

$$\alpha_n > 0, \qquad \beta_n \le 0. \quad (33)$$

Letting

$$\lambda_n = \left(\frac{1}{2}r^2 - \beta_n\right) \cdot \alpha_n^{-1} > 0, \quad (34)$$

we see that when $\lambda > \lambda_n$ we have, for any $u \in E$,

$$\varphi_\xi(u) \le \frac{1}{2}r^2 - \lambda\alpha_n - \beta_n < \frac{1}{2}r^2 - \lambda_n\alpha_n - \beta_n = 0. \quad (35)$$

On the other hand, there is a constant $\rho \in (0, u_0)$ such that $B_\rho = \{u \in H_T^1; \|u\| < \rho\}$ is not contained in Ω. So $F_\xi(t, u) < 0$ from (jjj). Hence, using (H2)(ii) we obtain, for $u \in \partial B_\rho \cap X$, where $X = [\operatorname{span}\{v_1, v_2, \dots, v_n\}]^\perp$,

$$\begin{aligned}\varphi_\xi(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^T F_\xi(t, u(t))\, dt \\ - \sum_{j=1}^p \int_0^{u_k(t_j)} I_j(s)\, ds > \frac{1}{2}\rho^2 > 0.\end{aligned} \quad (36)$$

Remark 4. Sobolev inequality implies that $E \subset K_n(r) \setminus B_{u_0}$.

Now, we apply Theorem 9.12 in [11] to conclude that φ_ξ possesses infinitely many distinct pairs of nontrivial critical points. That is, problem (11) has infinitely many distinct pairs of distinct nontrivial solutions.

Finally, we must prove that there exists $\xi_0 \in (0, \min(1, u_0/4))$ with the property that, for every $\xi \in (\xi_0, \min(1, u_0/4))$, any positive solution u of (11) satisfying $\varphi_\xi(u) \geq -m$ is such that $\min u \geq u_0$ and hence u is a solution of (1). We proceed by contradiction. Assume, on the contrary, that there are sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ such that

$$\begin{gathered} \xi_n \leq \frac{1}{n}; \\ u_n \text{ is a positive solution of (11) with } \xi = \xi_n, \\ \varphi_{\xi_n}(u_n) \geq -m, \\ \min u_n < u_0. \end{gathered} \tag{37}$$

First, we have

$$\begin{aligned} \lambda \int_0^T f_{\xi_n}(t, u_n(t))\, dt &= -\sum_{j=0}^{p} \int_{t_j^+}^{t_{j+1}^-} u_n''(s)\, ds \\ &= -\sum_{j=0}^{p} u_n'(t_{j+1}^-) - u_n'(t_j^+) \\ &= \left(u_n'(0) - u_n'(t_1^-)\right) \\ &\quad + \cdots + \left(u_n'(t_p) - u_n'(T)\right) \\ &= \sum_{j=1}^{p} I_j(u_n(t_j)). \end{aligned} \tag{38}$$

Then, by (H1)(i), there exists $c_1 > 0$ such that

$$\left\| f_{\xi_n}(\cdot, u_n(\cdot)) \right\|_{L^1} \leq c_1, \tag{39}$$

which implies that

$$\left\| u_n' \right\|_{L^\infty} \leq c_2, \tag{40}$$

for some constant $c_2 > 0$. Since $\varphi_{\xi_n}(u_n) \geq -m$, it follows that there must exist two constants R_1 and R_2, with $0 < R_1 < R_2$, such that $\max_{t\in I} u_n(t) \in [R_1, R_2]$; otherwise, u_n would tend uniformly to $+\infty$ and, in this case, $\varphi_{\xi_n}(u_n)$ would go to $-\infty$ (because of (H1)(vi) and (40)) and this contradicts $\varphi_{\xi_n}(u_n) \geq -m$. Also, $\min u_n < u_0$ implies that there exists an integer $k_n > 1$ such that $\min u_n \leq u_0 - (u_0/k_n)$. Since u_n is continuous, there exists τ_n^1 such that $u_n(\tau_n^1) = u_0 - (u_0/k_n)$. Let $\tau_n^2 \in I$ be such that, for n large enough,

$$u_n(\tau_n^1) = u_0 - \frac{u_0}{k_n} < R_1 = u_n(\tau_n^2). \tag{41}$$

Multiplying the differential equation in (11) by u_n' and integrating the resulting equation on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$, we get

$$\begin{aligned} J :=& \int_{\tau_n^1}^{\tau_n^2} u_n''(t)\, u_n'(t)\, dt \\ &+ \lambda \int_{\tau_n^1}^{\tau_n^2} f_{\xi_n}(t, u_n(t))\, u_n'(t)\, dt = 0 \\ =& \sum_{t_j\in[\tau_n^1, \tau_n^2]} \left[\int_{t_j^+}^{t_{j+1}^-} u_n''(t) \cdot u_n'(t)\, dt \right. \\ & \left. + \lambda \int_{t_j^+}^{t_{j+1}^-} f_{\xi_n}(t, u_n(t))\, u_n'(t)\, dt \right]. \end{aligned} \tag{42}$$

It is clear that

$$J = J_1 + \sum_{t_j\in[\tau_n^1, \tau_n^2]} \frac{1}{2}\left(u_n'(t_{j+1}^-)^2 - u_n'(t_j^+)^2 \right), \tag{43}$$

where

$$\begin{aligned} J_1 &= \lambda \sum_{t_j\in[\tau_n^1, \tau_n^2]} \int_{t_j^+}^{t_{j+1}^-} f_{\xi_n}(t, u_n(t))\, u_n'(t)\, dt \\ &= \lambda \sum_{t_j\in[\tau_n^1, \tau_n^2]} \int_{u_n(t_k)}^{u_n(t_{k+1})} f_{\xi_n}(t, s)\, ds. \end{aligned} \tag{44}$$

Now, $J = 0$ and $\|u_n'\|_{L^\infty} \leq c_2$ imply that J_1 is bounded. Since

$$f_{\xi_n}(t, u_n(t))\, u_n'(t) = \frac{d}{dt}\left[F_{\xi_n}(t, u_n(t)) \right] - D_1 F_{\xi_n}(t, u_n(t)), \tag{45}$$

it follows that

$$\begin{aligned} J_1 &= \lambda \sum_{t_k\in(\tau_n^1, \tau_n^2)} \int_{t_k^+}^{t_{k+1}^-} f_{\xi_n}(t, u_n(t))\, u_n'(t)\, dt \\ &= \lambda \sum_{t_k\in(\tau_n^1, \tau_n^2)} \int_{u_n(t_k)}^{u_n(t_{k+1})} f_{\xi_n}(t, s)\, ds \\ &= \lambda F_{\xi_n}(\tau_n^2, R_1) - \lambda F_{\xi_n}\left(\tau_n^1, u_0 - \frac{u_0}{k_n}\right) \\ &\quad - \lambda \int_{\tau_n^1}^{\tau_n^2} D_1 F_{\xi_n}(t, u_n(t))\, dt \\ &\geq \lambda F_{\xi_n}(\tau_n^2, R_1) - \lambda F_{\xi_n}\left(\tau_n^1, u_0 - \frac{u_0}{k_n}\right) \\ &= \lambda F_{\xi_n}(\tau_n^2, R_1) - \int_{u_0}^{u_0-(u_0/k_n)} f_{\xi_n}(\tau_n^1, s)\, ds. \end{aligned} \tag{46}$$

Now, we have that

$$\int_{u_0}^{u_0-(u_0/k_n)} f_{\xi_n}\left(\tau_n^1, s\right) ds$$
$$= \frac{f\left(\tau_n^1, \xi_n\right)}{\sin\left(2\pi\xi_n/u_0\right)} \int_{u_0}^{u_0-(u_0/k_n)} \sin\frac{2\pi}{u_0} s ds \tag{47}$$
$$= \frac{f\left(\tau_n^1, \xi_n\right)}{\sin\left(2\pi\xi_n/u_0\right)} \left(\frac{u_0}{2\pi}\right)\left[1-\cos\left(1-\frac{1}{k_n}\right)\right].$$

It follows from (H1)(ii) that

$$F_{\xi_n}\left(\tau_n^1, u_0 - \frac{u_0}{k_n}\right)$$
$$= \int_{u_0}^{u_0-(u_0/k_n)} f_{\xi_n}\left(\tau_n^1, s\right) ds \longrightarrow -\infty \quad \text{as } n \longrightarrow +\infty. \tag{48}$$

This implies that J_1 is not bounded. We arrive at a contradiction. This completes the proof of our main result. □

3. Example

Consider the boundary value problem

$$\begin{gathered} u''(t) + \lambda(t+1+\sin t)\left(\sqrt[3]{u(t)}+3\right)\cdot\ln\sqrt[3]{u(t)} = 0, \\ t \neq t_j,\ t \in I, \\ -\Delta u'(t) = -\sin\left(u\left(t_j\right)\right), \quad j = 1,2,3, \\ u(0) = u(T), \qquad u'(0) = u'(T). \end{gathered} \tag{49}$$

(i) $f : [0, 2\pi] \times (0; +\infty) \to \mathbb{R}$, given by $f(t,u) = (t + 1 + \sin t)(\sqrt[3]{u}+3)\cdot\ln\sqrt[3]{u}$, satisfies (H1) with $T = 2\pi$, $u_0 = 1$; $a = 0$; $b = 8(\pi+1)$; $\gamma = 2/3$.

(ii) For $u > 1$, $\ln\sqrt[3]{u} \le \sqrt[3]{u} \Rightarrow (t+1+\sin t)(\sqrt[3]{u}+3)\cdot\ln\sqrt[3]{u} \le 8(\pi+1)u^{2/3}$.

(iii) $I_j(s) = -\sin(s)$ satisfies (H2).

We consider $r \ge e$ in the definition of $K_n(r)$. We have

$$\begin{aligned} \int_1^u (3+\sqrt[3]{s})\cdot\ln\sqrt[3]{s} ds &= \int_1^u \ln s ds + 3\int_1^{u^{1/3}} x^3\cdot\ln x dx \\ &= u\ln u - u + 3\frac{\left(\sqrt[3]{u}\right)^4}{16}\left(4\ln\sqrt[3]{u}-1\right) \\ &\quad + \frac{19}{16}. \end{aligned} \tag{50}$$

Then

$$\begin{aligned} &\inf_{u>2}\int_0^{2\pi} F(t,u)\cdot dt > 0, \qquad \lim_{u\to+\infty}\int_0^{2\pi} F(t,u) dt = +\infty, \\ &\alpha_n = \inf_{u>2}\int_0^{2\pi} F_\xi(t,u(t)) dt \\ &\quad = \inf_{u>2}\int_0^{2\pi}\left((t+1+\sin t)\cdot\int_1^{u(t)}(3+\sqrt[3]{s})\cdot\ln\sqrt[3]{s} ds\right) dt \\ &\quad \ge \inf_{u>2}\int_0^{2\pi}\left(u\ln u - u + 3\frac{\left(\sqrt[3]{u}\right)^4}{16}\left(4\ln\sqrt[3]{u}-1\right)+\frac{19}{16}\right) \\ &\qquad \times (t) dt \\ &\quad \ge 2\pi\inf_{u>2}\left[\left(u\ln u - u + \frac{19}{16}\right) + 3\frac{\left(\sqrt[3]{u}\right)^4}{16}\left(4\ln\sqrt[3]{u}-1\right)\right] \\ &\quad \ge \pi. \end{aligned} \tag{51}$$

Also,

$$\begin{aligned} \alpha_n &= \inf_{u>2}\int_0^{2\pi} F_\xi(t,u(t))\cdot dt \\ &= \inf_{u>2}\int_0^{2\pi}\left((t+1+\sin t)\cdot\int_1^u(3+\sqrt[3]{s})\cdot\ln\sqrt[3]{s} ds\right) dt \\ &\le (2\pi+1)\inf_{u>2}\int_0^T\left[\left(u\ln u - u + \frac{19}{16}\right)\right. \\ &\qquad \left. + 3\frac{\left(\sqrt[3]{u}\right)^4}{16}\left(4\ln\sqrt[3]{u}-1\right)\right] dt \\ &\le (2\pi+1)\cdot\int_0^T \frac{3}{4}u^2(t) dt \\ &\le (2\pi+1)\frac{3}{4}r^2\sum_{k=1}^n\frac{1}{\mu_k} \le \frac{3}{2}r^2\sum_{k=1}^n\frac{1}{\mu_k}, \end{aligned} \tag{52}$$

where $\sum_{k=1}^n (1/\mu_k) = \sum_{k=1}^n (2\pi/2\pi k)^2 < \sum_{k=1}^n (1/k^2) < n$.

It is clear that

$$\begin{aligned} &\beta_n = \inf_{u>2} -\sum_{j=1}^3\int_0^{u(t_j)}\sin s ds = \inf_{u>2}\sum_{j=1}^3\left(\cos\left(u\left(t_j\right)\right)-1\right) \le 0, \\ &\beta_n = \inf_{u>2} -\sum_{j=1}^3\int_0^{u(t_j)}\sin s ds \ge \inf_{u>2} -\sum_{j=1}^3\int_0^{u(t_j)} s ds \\ &\quad \ge \inf_{u>2} -\sum_{j=1}^3\frac{\left|u\left(t_j\right)\right|^2}{2} ds > -\frac{3}{2}r^2. \end{aligned} \tag{53}$$

Then

$$\lambda_n = \left(\frac{1}{2}r^2 - \beta_n\right)\alpha_n^{-1} < \frac{2}{\pi}r^2,$$
$$\lambda_n = \left(\frac{1}{2}r^2 - \beta_n\right)\cdot\alpha_n^{-1} > \left(\frac{1}{2}r^2\right)\frac{2}{3}\frac{1}{r^2 n} = \frac{1}{3n}. \tag{54}$$

Applying our main result, we see that when $\lambda > \lambda_n$, for any $n \in \mathbb{N}^*$, problem (49) has infinitely many distinct nontrivial solutions.

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