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Research Article

Some Integrals Involving q-Laguerre Polynomials and Applications

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The integrals involving multivariate q-Laguerre polynomials and then auxiliary ones are studied. In addition, the representations of q-Hermite polynomials by q-Laguerre polynomials and their related integrals are given. At last, some generalized integrals associated with generalized q-Hermite polynomials are deduced.

Dedicated to Srinivasa Ramanujan on the occasion of his 125th birth anniversary

1. Introduction

The q-Laguerre polynomials are important q-orthogonal polynomials whose applications and generalizations arise in many applications such as quantum group (oscillator algebra, etc.), q-harmonic oscillator, and coding theory. For example, covariant oscillator algebra can be expressed by q-Laguerre polynomials [1]. The q-deformed radial Schrödinger is analyzed by q-Laguerre polynomials [2]. The q-Laguerre polynomials are the eigenvectors of an $su_q(1 \mid q)$-representation by [3]. For more information, please refer to [1–5].

The q-Laguerre polynomials are defined by [6, equation (1.0.1)]

$$\mathscr{L}_n^{(\alpha)}(x;q) = \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} \left(xq^{n+\alpha+1}\right)^k}{(q^{\alpha+1};q)_k (q;q)_k}, \quad (1)$$

which belong to the Askey scheme of basic hypergeometric orthogonal polynomials and according to Koekoek and Swarttouw [7, equation (3.21.1)]. The case of x in (1) replaced by $(1-q)x$ is studied by Moak [8, equation (2.3)].

In this paper, we first define the auxiliary q-Laguerre polynomials as follows:

$$\mathscr{M}_n^{(\alpha)}(x;q) = q^{-n\alpha} \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k x^k}{(q^{\alpha+1};q)_k (q;q)_k}. \quad (2)$$

It is easy to see the validity of the following:

$$\mathscr{L}_n^{(\alpha)}((1-q)x;q) = \mathscr{M}_n^{(\alpha)}\left(\left(1-q^{-1}\right)x;q^{-1}\right), \quad (3)$$

$$\lim_{q\to 1-} \mathscr{L}_n^{(\alpha)}((1-q)x;q) = \lim_{q\to 1+} \mathscr{M}_n^{(\alpha)}((1-q)x;q) = L_n^{(\alpha)}(x), \quad (4)$$

where the classical Laguerre polynomials are defined by [9, page 201]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k!\,(n-k)!(1+\alpha)_k}. \quad (5)$$

For more information about classical Laguerre polynomials, please refer to [9–15] and the references therein.

The well-known orthogonality of q-Laguerre polynomials reads the following.

Proposition 1 (see [6, equation (2.0.7)] and [8, equation (2.4)]). *For $\alpha > -1$ and for $m, n \in \mathbb{N}$, one has*

$$\int_0^\infty \mathscr{L}_m^{(\alpha)}(x;q)\,\mathscr{L}_n^{(\alpha)}(x;q)\,\frac{x^\alpha}{(-x;q)_\infty}dx = -q^{-n}\frac{(q^{\alpha+1};q)_n(q^{-\alpha};q)_\infty}{(q;q)_n(q;q)_\infty}\frac{\pi\delta_{m,n}}{\sin\pi\alpha}. \tag{6}$$

Hahn [16] discovered the previous q-extensions of the Laguerre polynomials, although he said little about them. Moak [8] found that the q-Laguerre polynomials are orthogonal with respect to the discrete measures (Dirac measure). Koekoek and Meijer [17–19] studied systematically the inner product of q-Laguerre polynomials. Ismail and Rahman [20] studied the indeterminate Hamburger moment problems related to q-Laguerre polynomials. For more information, please refer to [6–8, 16–21] and the references therein.

In this paper, we first generalize Proposition 1 and the auxiliary ones as follows.

Theorem 2. *For $\Re\{\mu\} > -1$ and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &\mathscr{L}_m^{(\alpha)}(xy;q)\,\mathscr{L}_n^{(\beta)}(xz;q)\,\frac{x^\mu}{(-x;q)_\infty}dx \\ &= (1-q)^{1+\mu}\frac{\pi\csc(-\mu\pi)}{\Gamma_q(-\mu)} \\ &\times \sum_{k=0}^{\min\{m,n\}}\begin{bmatrix} m+\alpha \\ m-k\end{bmatrix}\begin{bmatrix} n+\beta \\ n-k\end{bmatrix}\begin{bmatrix} k+\mu \\ k\end{bmatrix} q^{(\alpha+\beta-2\mu-1)k}y^k z^k \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1}\end{matrix};\ yq^{\alpha-\mu+m-k}\right] \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\mu+1+k} \\ q^{\beta+k+1}\end{matrix};\ zq^{\beta-\mu+n-k}\right].\end{aligned} \tag{7}$$

Theorem 3. *For $\Re\{\mu\} > -1$ and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &\mathscr{M}_m^{(\alpha)}(xy;q)\,\mathscr{M}_n^{(\beta)}(xz;q)\,x^\mu(x;q)_\infty dx \\ &= q^{\binom{\mu+2}{2}-\alpha m-\beta n}(1-q)^{1+\mu}\frac{\pi\csc(-\mu\pi)}{\Gamma_q(-\mu)} \\ &\times \sum_{k=0}^{\min\{m,n\}}\begin{bmatrix} m+\alpha \\ m-k\end{bmatrix}\begin{bmatrix} n+\beta \\ n-k\end{bmatrix}\begin{bmatrix} k+\mu \\ k\end{bmatrix} \\ &\times q^{(2k+\mu-m-n+1)k}y^k z^k \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1}\end{matrix};\ yq\right]{}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\mu+1+k} \\ q^{\beta+k+1}\end{matrix};\ zq\right].\end{aligned} \tag{8}$$

Corollary 4 (see [15, equation (14)]). *For $\Re\{\gamma\} > -1$, $\Re\{\sigma\} > 0$, and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &x^\gamma e^{-\sigma x}L_m^{(\alpha)}(\lambda x)\,L_n^{(\beta)}(\mu x)\,dx \\ &= \frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}}\sum_{k=0}^{\min\{m,n\}}\binom{m+\alpha}{m-k} \\ &\times \binom{n+\beta}{n-k}\binom{k+\gamma}{k}\left(\frac{\lambda\mu}{\sigma^2}\right)^k \\ &\times {}_2F_1\left[\begin{matrix} -m+k, \gamma+k+1 \\ \alpha+k+1\end{matrix};\ \frac{\lambda}{\sigma}\right] \\ &\times {}_2F_1\left[\begin{matrix} -n+k, \gamma+k+1 \\ \beta+k+1\end{matrix};\ \frac{\mu}{\sigma}\right].\end{aligned} \tag{9}$$

Remark 5. Theorems 2 and 3 reduce to Proposition 1 and formula (41), respectively, if letting $y = z = 1$ and $\alpha = \beta = \mu$, and become Corollary 4 by setting $q \to 1$ and taking $(\mu, x, y, z) = (\gamma, \sigma x, \lambda/\sigma, \mu/\sigma)$.

The discrete q-Hermite polynomials $h_n(x;q)$ and $\tilde{h}_n(x;q)$ are defined by [7, pages 90-91]

$$\begin{aligned}h_n(x;q) &= x^n{}_2\phi_0\left[\begin{matrix} q^{-n}, q^{-n+1} \\ -\end{matrix};\ q^2, \frac{q^{2n-1}}{x^2}\right] \\ &\triangleq q^{\binom{n}{2}}\mathscr{G}_n\left(x\sqrt{1-q^2};q^2\right), \\ \tilde{h}_n(x;q) &= x^n{}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-n+1} \\ 0\end{matrix};\ q^2, \frac{q^2}{x^2}\right] \\ &\triangleq q^{-\binom{n}{2}}\mathscr{H}_n\left(x\sqrt{1-q^2};q^2\right),\end{aligned} \tag{10}$$

which are equivalent to Al-Salam-Carlitz polynomials with $a = -1$ (please refer to [22, page 53] also), and the relation between them is $h_n(ix;q^{-1}) = i^n\tilde{h}_n(x;q)$. For more information about the Al-Salam-Carlitz polynomials and the discrete q-Hermite polynomials, please refer to [7, 22–30] and the references therein.

In this paper, we also define new q-Hermite polynomials $\mathscr{H}_n(x;q)$ and $\mathscr{G}_n(x;q)$, whose names come from the facts

$$\begin{aligned}\lim_{q\to 1}&(1-q)^{-n/2}\mathscr{H}_n((1-q)x;q) \\ &= \lim_{q\to 1}(1-q)^{-n/2}\mathscr{G}_n((1-q)x;q) = 2^{-n}H_n(x),\end{aligned} \tag{11}$$

$$\begin{aligned}\lim_{\substack{x\to(1-q)x,\\ t\to\sqrt{(1-q)}t,\ q\to 1}}&\sum_{n=0}^\infty\frac{t^n}{(q^{1/2};q^{1/2})_n}\mathscr{H}_n(x;q) \\ &= \exp\left(2xt-t^2\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \\
&= \lim_{\substack{x \to (1-q)x, \\ t \to \sqrt{(1-q)}t,\ q \to 1}} \sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/4}}{(q^{1/2}; q^{1/2})_n} \mathscr{G}_n(x; q),
\end{aligned}
\tag{12}
$$

then we deduce the representations of $\mathscr{H}_n(x; q)$ and $\mathscr{G}_n(x; q)$ by q-Laguerre polynomials; see Theorems 15 and 16.

As an application, using the orthogonality of q-Laguerre polynomials (6), and (41), and combining the expressions of q-Hermite polynomials (52) and (54), we can obtain the following results immediately.

Theorem 6. *For $\alpha > -1$ and $j \le n \in \mathbb{N}$, one has*

$$
\begin{aligned}
&\int_0^{\infty} \mathscr{H}_n(x; q) \mathscr{L}_j^{(\alpha)}(x; q) \frac{x^{\alpha}}{(-x; q)_{\infty}} dx \\
&= -\frac{(q^{\alpha+1}, q; q)_n (q^{\alpha+1}; q)_j (q^{-\alpha}; q)_{\infty} (1-q)^{-n/2}}{(q; q)_j (q; q)_{\infty}} \\
&\quad \times \frac{\pi C(n, j)}{\sin \pi\alpha},
\end{aligned}
\tag{13}
$$

where $C(n, k)$ is defined by (52).

Theorem 7. *For $\alpha > -1$ and $j \le n \in \mathbb{N}$, one has*

$$
\begin{aligned}
&\int_0^{\infty} \mathscr{G}_n(x; q) \mathscr{M}_j^{(\alpha)}(x; q) x^{\alpha} (x; q)_{\infty} dx \\
&= -q^{-(1/2)\binom{n}{2} + \binom{\alpha+2}{2} + j - \alpha j} \\
&\quad \times \frac{(q^{\alpha+1}, q; q)_n (q^{\alpha+1}; q)_j (q^{-\alpha}; q)_{\infty} (1-q)^{-n/2}}{(q; q)_j (q; q)_{\infty}} \\
&\quad \times \frac{\pi D(n, j)}{\sin \pi\alpha},
\end{aligned}
\tag{14}
$$

where $D(n, k)$ is defined by (54).

The generalized Hermite polynomials were introduced by Szegö [31], (see also [23, equation (1.1)]) as follows:

$$
\begin{aligned}
H_{2n}^{(\mu)}(x) &= (-1)^n 2^{2n} n! L_n^{(\mu-1/2)}(x^2), \\
H_{2n+1}^{(\mu)}(x) &= (-1)^n 2^{2n+1} n! x L_n^{(\mu+1/2)}(x^2).
\end{aligned}
\tag{15}
$$

The authors [23, equation (2.7)] defined the following generalized q-Hermite polynomials:

$$
\begin{aligned}
\mathscr{H}_{2n}^{(\mu)}(x; q) &= (-1)^n (q; q)_n \mathscr{L}_n^{(\mu-1/2)}(x^2; q), \\
\mathscr{H}_{2n+1}^{(\mu)}(x; q) &= (-1)^n (q; q)_n x \mathscr{L}_n^{(\mu+1/2)}(x^2; q)
\end{aligned}
\tag{16}
$$

and deduced their orthogonal relations; see Proposition 19 below.

In this paper, we continue to define the auxiliary polynomials according to (16) as follows:

$$
\begin{aligned}
\mathscr{G}_{2n}^{(\mu)}(x; q) &= q^{-\binom{n+1}{2}} (q; q)_n \mathscr{M}_n^{(\mu-1/2)}(x^2; q), \\
\mathscr{G}_{2n+1}^{(\mu)}(x; q) &= q^{-\binom{n+1}{2}} (q; q)_n x \mathscr{M}_n^{(\mu+1/2)}(x^2; q).
\end{aligned}
\tag{17}
$$

With the aid of (15)–(17) and (4), one readily verifies that

$$
\begin{aligned}
&\lim_{q \to 1} (1-q)^{-n/2} \mathscr{H}_n^{(\mu)}\left(\sqrt{1-q}x; q\right) \\
&= \lim_{q \to 1} (1-q)^{-n/2} \mathscr{G}_n^{(\mu)}\left(\sqrt{1-q}x; q\right) = 2^{-n} H_n^{(\mu)}(x).
\end{aligned}
\tag{18}
$$

As another application of this paper, we gain the general q-Laguerre polynomials of several variables by Theorems 2 and 3, and we also deduce the orthogonal polynomials of $\mathscr{G}_n^{(\mu)}(x; q)$. For more details of the results, see Theorems 20 and 21 and Corollary 23.

The structure of this paper is organized as follows. In Section 2, we show how to prove the integrals involving q-Laguerre polynomials of several variables. In Section 3, we represent discrete q-Hermite polynomials by q-Laguerre polynomials and their related integral results. In Section 4, we study the general integrals of q-Hermite polynomials involving several variables.

2. Notations and Proof of Theorems 2 and 3

Throughout this paper, we follow the notations and terminology in [32] and assume that $0 < q < 1$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{R}$ is rational number. The q-series and its compact factorials are defined [32, page 6], respectively, by

$$
\begin{aligned}
(a; q)_0 &= 1, \qquad (a; q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \\
(a; q)_{\infty} &= \prod_{k=0}^{\infty} \left(1 - aq^k\right),
\end{aligned}
\tag{19}
$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is a nonnegative integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ is given by

$$
\begin{aligned}
&{}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[(-1)^n q^{\binom{n}{2}}\right]^{s+1-r}.
\end{aligned}
\tag{20}
$$

For convergence of the infinite series in (20), $|q| < 1$ and $|z| < \infty$ when $r \le s$, or $|q| < 1$ and $|z| < 1$ when $r = s + 1$, provided that no zeros appear in the denominator. Letting $(a_i, b_i) = (q^{a_i}, q^{b_i})$ and setting $q \to 1$, (20) reduces to the classical Gauss' hypergeometric series

$$
{}_rF_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n! (b_1)_n \cdots (b_s)_n} z^n,
\tag{21}
$$

where Pochhammer symbol $(z)_n$ is defined by $(z)_n = z(z+1)\cdots(z+n-1) = \Gamma(z+n)/\Gamma(z)$.

The q-analogue of the gamma function is defined by (see [32, equation (1.10.1)]) as follows:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{22}$$

The q-Chu-Vandermonde formula [32, equations (II.6) and (II.7)] reads that

$$\begin{gathered} {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} ; q, q\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n, \\ {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} ; q, \frac{cq^n}{b}\right] = \frac{(c/b;q)_n}{(c;q)_n}. \end{gathered} \tag{23}$$

The ${}_3\phi_2$ transformations [32, equations (III.12) and (III.13)] stats that

$$\begin{aligned} {}_3\phi_2\left[\begin{matrix} q^{-n}, a, c \\ b, d \end{matrix} ; q, q\right] &= \frac{a^n(d/a;q)_n}{(d;q)_n} \\ &\quad \times {}_3\phi_2\left[\begin{matrix} q^{-n}, a, \dfrac{b}{c} \\ b, \dfrac{aq^{1-n}}{d} \end{matrix} ; q, \frac{cq}{d}\right], \\ {}_3\phi_2\left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, \frac{deq^n}{bc}\right] &= \frac{(e/c;q)_n}{(e;q)_n} \\ &\quad \times {}_3\phi_2\left[\begin{matrix} q^{-n}, c, \dfrac{d}{b} \\ d, \dfrac{cq^{1-n}}{e} \end{matrix} ; q, \frac{cq}{d}\right]. \end{aligned} \tag{24}$$

The q-analogue of the Pfaff-Kummer transformation [32, equation (III.4)] is as follows:

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} ; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left[\begin{matrix} a, \dfrac{c}{b} \\ c, az \end{matrix} ; q, bz\right]. \tag{25}$$

The Ramanujan beta integral is stated as follows [33, equation (2.8)]:

$$\begin{gathered} \int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \frac{(a,q^{1-x};q)_\infty}{(q,aq^{-x};q)_\infty}\frac{\pi}{\sin \pi x}, \\ (0<a<q^x, x>0). \end{gathered} \tag{26}$$

Lemma 8 (see [33, equation (4.2)]). *One has*

$$\begin{aligned} \int_0^\infty \frac{x^\alpha}{(-x;q)_\infty}dx &= \frac{\Gamma(-\alpha)\Gamma(\alpha+1)(1-q)^{1+\alpha}}{\Gamma_q(-\alpha)} \\ &= -\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin \pi\alpha}, \end{aligned} \tag{27}$$

$$\int_0^\infty x^\alpha (x;q)_\infty dx = -q^{\binom{\alpha+2}{2}}\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin \pi\alpha}. \tag{28}$$

Proof. Taking $(a,t) = (-1/a, at)$ in (26), then letting $a \to 0$, we obtain (28) immediately. The proof is complete. □

Lemma 9. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$\mathscr{L}_n^{(\alpha)}(xy;q) = \sum_{k=0}^{n}\frac{(q^{\alpha+1};q)_n (y;q)_{n-k} y^k \mathscr{L}_k^{(\alpha)}(x;q)}{(q;q)_{n-k}(q^{\alpha+1};q)_k}, \tag{29}$$

$$\begin{aligned} \mathscr{M}_n^{(\alpha)}(xy;q) &= \sum_{k=0}^{n}\frac{(-1)^{n-k}(q^{\alpha+1};q)_n (1/y;q)_{n-k} y^n \mathscr{M}_k^{(\alpha)}(x;q)}{(q;q)_{n-k}(q^{\alpha+1};q)_k} \\ &\quad \times q^{\binom{k}{2}+\alpha k-\binom{n}{2}-\alpha n}. \end{aligned} \tag{30}$$

Proof. Letting $\gamma = 0$ in [6, Proposition 4.1],

$$\begin{aligned} &\sum_{n=0}^\infty \frac{(\gamma t;q)_n q^{\alpha n+n^2}}{(\gamma t, q^{\alpha+1}, q;q)_n}(-xt)^n \\ &\quad = \frac{(t;q)_\infty}{(\gamma t;q)_\infty}\sum_{n=0}^\infty \frac{(\gamma;q)_n t^n}{(q^{\alpha+1};q)_n}\mathscr{L}_n^{(\alpha)}(x;q), \end{aligned} \tag{31}$$

then replacing x by xy, we have

$$\begin{aligned} &\sum_{n=0}^\infty \frac{\mathscr{L}_n^{(\alpha)}(xy;q)}{(q^{\alpha+1};q)_n}t^n \\ &\quad = \frac{(yt;q)_\infty}{(t;q)_\infty}\frac{1}{(yt;q)_\infty}\sum_{n=0}^\infty \frac{q^{\alpha n+n^2}}{(q^{\alpha+1},q;q)_n}(-xyt)^n \\ &\quad = \sum_{k=0}^\infty \frac{(y;q)_k}{(q;q)_k}t^k \sum_{n=k}^\infty \frac{L_{n-k}^{(\alpha)}(x;q)(yt)^{n-k}}{(q^{\alpha+1};q)_{n-k}}. \end{aligned} \tag{32}$$

Comparing the coefficients of t^n on both sides of (32) yields (29). Similar to (32), by the definition (1), we have

$$\begin{aligned} &\sum_{n=0}^\infty \frac{(\gamma;q)_n (tq^\alpha)^n}{(q^{\alpha+1};q)_n}\mathscr{M}_n^{(\alpha)}(x;q) \\ &\quad = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^\infty \frac{(\gamma;q)_n x^n}{(q/t, q^{\alpha+1}, q;q)_n}. \end{aligned} \tag{33}$$

By taking $(\gamma, t, x) = (1/\gamma, \gamma t, xy)$ and letting $\gamma \to 0$ in (33), we obtain (30). The proof of Lemma 9 is complete. □

Lemma 10. *For* $\min\{\alpha,\beta\} > -1$ *and* $m,n\in\mathbb{N}$, *one has*

$$\int_0^\infty \mathscr{L}_m^{(\alpha)}(x;q)\,\mathscr{L}_n^{(\beta)}(x;q)\,\frac{x^\beta}{(-x;q)_\infty}dx = (-1)^{m+n}q^{(\alpha-\beta)m+\binom{m}{2}+\binom{n}{2}-mn} \times(1-q)^{1+\beta}\begin{bmatrix}\beta-\alpha\\ m-n\end{bmatrix}\begin{bmatrix}n+\beta\\ n\end{bmatrix}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}, \quad (34)$$

$$\int_0^\infty \mathscr{M}_j^{(\alpha)}(x;q)\,\mathscr{M}_k^{(\beta)}(x;q)\,x^\beta(x;q)_\infty dx = (-1)^{k+j}q^{\binom{j+1}{2}+\binom{k+1}{2}-(\alpha+j)k+\binom{\beta+2}{2}} \times(1-q)^{1+\beta}\begin{bmatrix}\beta-\alpha\\ j-k\end{bmatrix}\begin{bmatrix}k+\beta\\ k\end{bmatrix}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}. \quad (35)$$

Proof. Interchanging the integral and summation by definition, the left hand side of (34) equals

$$\begin{aligned}
&\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\sum_{k=0}^m\sum_{j=0}^n\frac{(q^{-m};q)_k q^{\binom{k}{2}}q^{k(m+\alpha+1)}}{(q^{\alpha+1},q;q)_k}\times\frac{(q^{-n};q)_j q^{\binom{j}{2}}q^{j(n+\beta+1)}}{(q^{\beta+1},q;q)_j}\int_0^\infty\frac{x^{\beta+k+j}dx}{(-x;q)_\infty}\\
&=\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\sum_{k=0}^m\sum_{j=0}^n\frac{(q^{-m};q)_k q^{\binom{k}{2}}q^{k(m+\alpha+1)}}{(q^{\alpha+1},q;q)_k}\times\frac{(q^{-n};q)_j q^{\binom{j}{2}}q^{j(n+\beta+1)}}{(q^{\beta+1},q;q)_j}\\
&\quad\times\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}q^{-j\beta-\binom{j+1}{2}-k(\beta+j)-\binom{k+1}{2}}\times\left(q^{\beta+1};q\right)_{j+k}\\
&=\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\times\sum_{j=0}^n\frac{(q^{-n};q)_j q^{jn}}{(q;q)_j}\sum_{k=0}^m\frac{(q^{-m},q^{\beta+j+1};q)_k q^{k(m+\alpha-\beta-j)}}{(q^{\alpha+1},q;q)_k}\\
&=(-1)^m q^{(\alpha-\beta)m+\binom{m}{2}}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}\times\frac{(q^{\beta-\alpha-m+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\times\sum_{j=0}^n\frac{(q^{-n},q^{\beta-\alpha+1};q)_j}{(q,q^{\beta-\alpha-m+1};q)_j}q^{j(n-m)},
\end{aligned}\quad (36)$$

which is the right hand side of (34) by using the second formula of (23) and simplification. Similar to (34), the right hand side of (35) is equal to

$$\begin{aligned}
&q^{-m\alpha-n\beta}\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\times\sum_{k=0}^m\sum_{j=0}^n\frac{(q^{-m};q)_k(q^{-n};q)_j}{(q^{\alpha+1},q;q)_k(q^{\beta+1},q;q)_j}\times\int_0^\infty x^{\beta+k+j}(x;q)_\infty dx\\
&=q^{-m\alpha-n\beta}\frac{(q^{\alpha-\beta};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}(1-q)^{1+\beta}\times\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}q^{(\beta+1)m}q^{\binom{\beta+2}{2}}{}_2\phi_1\left[\begin{matrix}q^{-n},q^{\beta-\alpha+1}\\ q^{\beta-\alpha-m+1}\end{matrix};q\right],
\end{aligned}\quad (37)$$

which is equivalent to the right hand side of (35) by using the first formula of (23) and simplification. The proof of Lemma 10 is complete. □

Lemma 11. *If* $f(x)$ *is a polynomial of degree* m *about* x *and is defined in the infinite interval* $(0,\infty)$, *which can be expanded in a series of the form*

$$f(x)=\sum_{n=0}^m C_{mn}\mathscr{L}_n^{(\alpha)}(x;q) \quad \text{or} \quad f(x)=\sum_{n=0}^m D_{mn}\mathscr{M}_n^{(\alpha)}(x;q), \quad (38)$$

where C_{mn} *and* D_{mn} *are the nth Fourier-Laguerre coefficients, and both of them are independent of* x, *then one has*

$$C_{mn}=\frac{\Gamma_q(-\alpha)(q;q)_n q^n}{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n(1-q)^{\alpha+1}}\times\int_0^\infty\frac{x^\alpha}{(-x;q)_\infty}f(x)\,\mathscr{L}_n^{(\alpha)}(x;q)\,dx, \quad (39)$$

$$D_{mn}=\frac{\Gamma_q(-\alpha)(q;q)_n q^{\alpha n-n-\binom{\alpha+2}{2}}}{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n(1-q)^{\alpha+1}}\times\int_0^\infty x^\alpha(x;q)_\infty f(x)\,\mathscr{M}_n^{(\alpha)}(x;q)\,dx. \quad (40)$$

Proof. Multiplying (38) by $x^\alpha\mathscr{L}_n(x;q)/(-x;q)_\infty$ and integrating term by term over the interval $(0,\infty)$, using (6), we obtain the proof of (39). Similarly, taking $\beta=\alpha$ in (35), we deduce

$$\int_0^\infty \mathscr{M}_m^{(\alpha)}(x;q)\,\mathscr{M}_n^{(\alpha)}(x;q)\,x^\alpha(x;q)_\infty dx = -q^{\binom{\alpha+2}{2}+n-\alpha n}\frac{(q^{\alpha+1};q)_n(q^{-\alpha};q)_\infty}{(q;q)_n(q;q)_\infty}\frac{\pi\delta_{m,n}}{\sin\pi\alpha}, \quad (41)$$

so we also gain the proof of (40). The proof of Lemma 11 is complete. □

Lemma 12. *For* $\min\{\alpha,\beta\}>-1$, *one has*

$$\mathscr{L}_n^{(\alpha)}(xy;q)=\sum_{k=0}^{n}\begin{bmatrix}n+\alpha\\ n-k\end{bmatrix}y^k q^{(\alpha-\beta)k}\mathscr{L}_k^{(\beta)}(x;q)\times{}_2\phi_1\left[\begin{matrix}q^{k-n},q^{\beta+1+k}\\ q^{\alpha+k+1}\end{matrix};\ yq^{\alpha-\beta+n-k}\right],\tag{42}$$

$$\mathscr{M}_n^{(\alpha)}(xy;q)=\sum_{k=0}^{n}\begin{bmatrix}n+\alpha\\ n-k\end{bmatrix}y^k q^{(k-n+\beta)k-\alpha n}\mathscr{M}_k^{(\beta)}(x;q)\times{}_2\phi_1\left[\begin{matrix}q^{k-n},q^{\beta+1+k}\\ q^{\alpha+k+1}\end{matrix};\ yq\right].\tag{43}$$

Remark 13. Replacing x by $(1-q)x$ and letting $q\to 1$, we have [15, equation (11)]

$$L_n^{(\alpha)}(xy)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}y^k L_k^{(\beta)}(x)\,{}_2F_1\times\left[\begin{matrix}-n+k,\beta+k+1\\ \alpha+k+1\end{matrix};\ y\right].\tag{44}$$

Setting $\alpha=\beta$, (42) and (43) reduce to (29) and (30), respectively.

Proof. Let

$$A=\frac{\Gamma_q(-\beta)(q;q)_k q^k}{\pi\csc(-\beta\pi)(q^{\beta+1};q)_k(1-q)^{1+\beta}},\qquad B=(1-q)^{1+\beta}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}.\tag{45}$$

By Lemmas 10 and 11, the coefficient of $\mathscr{L}_n^{(\alpha)}(xy;q)$ expanded by $\mathscr{L}_n^{(\beta)}(x;q)$

$$\begin{aligned}C_{nk}&=A\cdot\sum_{j=0}^{n}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j}y^j}{(q;q)_{n-j}(q^{\alpha+1};q)_j}\times\int_0^\infty\mathscr{L}_j^{(\alpha)}(x;q)\mathscr{L}_k^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}dx\\
&=AB\cdot\sum_{j=0}^{n}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j}y^j}{(q;q)_{n-j}(q^{\alpha+1};q)_j}\times(-1)^{k+j}q^{(\alpha-\beta)j+\binom{j}{2}+\binom{k}{2}}\times\begin{bmatrix}\beta-\alpha\\ j-k\end{bmatrix}\begin{bmatrix}k+\beta\\ k\end{bmatrix}q^{-kj}\\
&=AB\cdot\sum_{j=0}^{n-k}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j-k}y^{j+k}}{(q;q)_{n-j-k}(q^{\alpha+1};q)_{j+k}}\times\begin{bmatrix}\beta-\alpha\\ j\end{bmatrix}\begin{bmatrix}k+\beta\\ k\end{bmatrix}(-1)^j q^{(\alpha-\beta)(j+k)-k+\binom{j}{2}}\\
&=AB\cdot\frac{(q^{\alpha+1};q)_n(y;q)_{n-k}y^k}{(q;q)_{n-k}(q^{\alpha+1};q)_k}q^{(\alpha-\beta-1)k}\times\begin{bmatrix}k+\beta\\ k\end{bmatrix}{}_3\phi_2\left[\begin{matrix}q^{k-n},q^{\alpha-\beta},0\\ \dfrac{q^{k-n+1}}{y},q^{\alpha+k+1}\end{matrix};\ q\right]\\
&=AB\cdot\frac{(q^{\alpha+1};q)_n y^k}{(q;q)_{n-k}(q^{\alpha+1};q)_k}q^{(\alpha-\beta-1)k}\times\begin{bmatrix}k+\beta\\ k\end{bmatrix}{}_2\phi_1\left[\begin{matrix}q^{k-n},q^{\beta+1+k}\\ q^{\alpha+k+1}\end{matrix};\ yq^{\alpha-\beta+n-k}\right]\end{aligned}\tag{46}$$

is equal to the right hand side of (42). Similarly, we have

$$\begin{aligned}D_{nk}&=A\cdot q^{-\binom{\beta+2}{2}-2k+\beta k}\times\int_0^\infty\mathscr{M}_n^{(\alpha)}(xy;q)\mathscr{M}_k^{(\beta)}(x;q)x^\beta(x;q)_\infty dx\\
&=AB\cdot q^{-\binom{\beta+2}{2}-2k+\beta k}\times\sum_{j=0}^{n}\frac{(-1)^{n-j}(q^{\alpha+1};q)_n(1/y;q)_{n-j}y^n}{(q;q)_{n-j}(q^{\alpha+1};q)_j}q^{\binom{j}{2}+\alpha j-\binom{n}{2}-\alpha n}\\
&\quad\times(-1)^{j+k}q^{\binom{j+1}{2}+\binom{k+1}{2}-(\alpha+j)k+\binom{\beta+2}{2}}\times\begin{bmatrix}\beta-\alpha\\ j-k\end{bmatrix}\begin{bmatrix}k+\beta\\ k\end{bmatrix}\\
&=AB\cdot q^{-2k+\beta k-\alpha n-\binom{n}{2}+\binom{k+1}{2}-\alpha k}\times\sum_{j=0}^{n-k}\frac{(-1)^{n+k}(q^{\alpha+1};q)_n(1/y;q)_{n-k}(q^{n-j-k+1};q)_j y^n}{(q^{n-j-k}/y;q)_j(q;q)_{n-k}(q^{\alpha+1};q)_{j+k}}\\
&\quad\times q^{(j+k)^2+\alpha(j+k)-(j+k)k}\times\frac{(q^{\beta-\alpha-j+1};q)_j}{(q;q)_j}\begin{bmatrix}k+\beta\\ k\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&= AB \cdot q^{-2k+\beta k-\alpha n-\binom{n}{2}+\binom{k+1}{2}} \\
&\times \frac{(-1)^{n+k}(q^{\alpha+1};q)_n(1/y;q)_{n-k}y^n}{(q;q)_{n-k}(q^{\alpha+1};q)_k}\begin{bmatrix} k+\beta \\ k \end{bmatrix} \\
&\times {}_2\phi_2\left[\begin{matrix} q^{k-n}, q^{\alpha-\beta} \\ yq^{k+1-n}, q^{\alpha+k+1} \end{matrix} ; yq^{\beta+k+2}\right],
\end{aligned} \tag{47}
$$

which is equivalent to the right hand side of (43) by (25) and simplification. The proof of Lemma 12 is complete. □

Proof of Theorems 2 and 3. By using formula (42), the left hand side of (7) is equal to

$$
\begin{aligned}
&\sum_{k=0}^{m}\begin{bmatrix} m+\alpha \\ m-k \end{bmatrix} y^k q^{(\alpha-\mu)k} \\
&\times {}_2\phi_1\left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1} \end{matrix} ; yq^{\alpha-\mu+m-k}\right] \\
&\times \sum_{j=0}^{n}\begin{bmatrix} n+\beta \\ n-j \end{bmatrix} z^j q^{(\beta-\mu)j} \\
&\times {}_2\phi_1\left[\begin{matrix} q^{j-n}, q^{\mu+1+j} \\ q^{\beta+j+1} \end{matrix} ; zq^{\beta-\mu+n-j}\right] \\
&\times \int_0^\infty \mathscr{L}_k^{(\mu)}(x;q)\,\mathscr{L}_j^{(\mu)}(x;q)\,\frac{x^\mu}{(-x;q)_\infty}dx.
\end{aligned} \tag{48}
$$

Similarly, with the help of formula (43), the left hand side of (8) equals

$$
\begin{aligned}
&\sum_{k=0}^{m}\begin{bmatrix} m+\alpha \\ m-k \end{bmatrix} y^k q^{(k-m+\mu)k-\alpha m} {}_2\phi_1 \\
&\times \left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1} \end{matrix} ; yq\right] \\
&\times \sum_{j=0}^{n}\begin{bmatrix} n+\beta \\ n-j \end{bmatrix} z^j q^{(j-n+\mu)j-\beta n} \\
&\times {}_2\phi_1\left[\begin{matrix} q^{j-n}, q^{\mu+1+j} \\ q^{\beta+j+1} \end{matrix} ; zq\right] \\
&\times \int_0^\infty \mathscr{M}_k^{(\mu)}(x;q)\,\mathscr{M}_j^{(\mu)}(x;q)\,x^\mu(x;q)_\infty dx.
\end{aligned} \tag{49}
$$

Using formulas (41) and (43) and noticing that the orthogonality of previous two types of q-Laguerre polynomials for the case of $k = j$, we can deduce (7) and (8). The proof of Theorems 2 and 3 is complete. □

3. Representations of q-Hermite Polynomials

Doha [34, page 5460] deduced the following result by third-order recurrence relation of the coefficients.

Proposition 14 (see [34, equation (49)]). *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$
\begin{aligned}
H_n(x) &= 2^n(1+\alpha)_n \\
&\times \sum_{k=0}^{n} {}_2F_2\left[\begin{matrix} \dfrac{-(n-k)}{2}, \dfrac{-(n-k-1)}{2} \\ \dfrac{-(\alpha+n)}{2}, \dfrac{-(\alpha+n-1)}{2} \end{matrix} ; -\frac{1}{4}\right] \\
&\times \frac{(-n)_k L_k^{(\alpha)}(x)}{(1+\alpha)_k}.
\end{aligned} \tag{50}
$$

In this section, we employ the technique of rearrangement of series

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathscr{A}(k,n) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathscr{A}(k,n-k) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\mathscr{A}(k,n-2k)
\end{aligned} \tag{51}
$$

to derive the following q-analogue of Proposition 14.

Theorem 15. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$
\mathscr{H}_n(x;q) = \left(q^{\alpha+1}, q; q\right)_n (1-q)^{-n/2}\sum_{k=0}^{n} C(n,k)\,\mathscr{L}_k^{(\alpha)}(x;q), \tag{52}
$$

where

$$
\begin{aligned}
C(n,k) &= \frac{(-1)^k q^{-(n^2+n+4\alpha n+4nk-2k^2)/4}}{(q^{\alpha+1};q)_k(q;q)_{n-k}} \\
&\times \sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s\left(q^{k-n};q\right)_{2s}(1-q)^s}{(q^{-(n+\alpha)};q)_{2s}(-q^{-n/2};q^{1/2})_{2s}(q;q)_s},
\end{aligned} \tag{53}
$$

and $[x]$ denotes the greatest integer not exceeding x.

Theorem 16. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$
\begin{aligned}
\mathscr{G}_n(x;q) &= \left(q^{\alpha+1}, q; q\right)_n (1-q)^{-n/2} q^{-n(n-1)/4} \\
&\times \sum_{k=0}^{n} D(n,k)\,\mathscr{M}_k^{(\alpha)}(x;q),
\end{aligned} \tag{54}
$$

where

$$
\begin{aligned}
&D(n,k) \\
&= \frac{(-1)^k q^{\binom{k}{2}+\alpha k}}{(q^{\alpha+1};q)_k(q;q)_{n-k}} \\
&\times \sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s q^{\binom{s}{2}}(1-q)^s\left(q^{k-n};q\right)_{2s} q^{(2s^2-s-4sk-4s\alpha)/2}}{(q;q)_s(q^{-(n+\alpha)};q)_{2s}(-q^{-n/2};q^{1/2})_{2s}}.
\end{aligned} \tag{55}
$$

Before the proof of Theorem 15 the following lemma is necessary.

Lemma 17. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$x^n = \left(q^{\alpha+1}, q; q\right)_n q^{-(\alpha n+n^2)} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{n-k}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-k}},$$
$$x^n = \left(q^{\alpha+1}, q; q\right)_n q^{n} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}+\alpha k} \mathscr{M}_k^{(\alpha)}(x;q)}{(q;q)_{n-k} (q^{\alpha+1};q)_k}. \quad (56)$$

Proof. Letting $f(x) = x^m$ in (38) and using the following fact [8, page 23]:

$$\int_0^\infty \mathscr{L}_n^{(\alpha)}(x;q) \frac{x^{\alpha+m}}{(-x;q)_\infty} dx = \frac{(q^{-m};q)_n \left(q^{\alpha+1};q\right)_m \Gamma(-\alpha)\Gamma(\alpha+1)}{(q;q)_n \Gamma_q(-\alpha) q^{\alpha m+\binom{m+1}{2}} (1-q)^{-1-\alpha}}, \quad (57)$$

similarly, we deduce the explicit representation of (39) and (40), respectively,

$$C_{mn} = \frac{\left(q^{\alpha+1};q\right)_m (q^{-m};q)_n q^n}{(q^{\alpha+1};q)_n q^{\alpha m+\binom{\alpha+1}{2}}},$$
$$D_{mn} = \frac{\left(q^{\alpha+1};q\right)_m (q^{-m};q)_n q^{m(n+1)}}{(q^{\alpha+1};q)_n q^{-\alpha n}}, \quad (58)$$

so we obtain the formula (56). The proof is complete. □

Lemma 18 (see [7, equations (3.29.5) and (3.28.5)]). *One has*

$$\sum_{n=0}^{\infty} \frac{t^n}{(q^{1/2};q^{1/2})_n} \mathscr{H}_n(x;q) = \frac{\left(-xt(1-q)^{-1/2};q^{1/2}\right)_\infty}{(-t^2;q)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/4}}{(q^{1/2};q^{1/2})_n} \mathscr{G}_n(x;q) = \frac{\left(t^2;q\right)_\infty}{\left(xt(1-q)^{-1/2};q^{1/2}\right)_\infty}. \quad (59)$$

Proof. By using [7, equations (3.29.5) and (3.28.5)]

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \widetilde{h}_n(x;q) t^n = \frac{(-xt;q)_\infty}{(-t^2;q^2)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{h_n(x;q)}{(q;q)_n} t^n = \frac{\left(t^2;q^2\right)_\infty}{(xt;q)_\infty}. \quad (60)$$

and replacing, respectively, by

$$\mathscr{H}_n(x;q) = q^{n(n-1)/4} \widetilde{h}_n\left(\frac{x}{\sqrt{1-q}};q^{1/2}\right),$$
$$\mathscr{G}_n(x;q) = q^{-n(n-1)/4} h_n\left(\frac{x}{\sqrt{1-q}};q^{1/2}\right), \quad (61)$$

we deduce the proof of Lemma 18. The proof is complete. □

Proof of Theorem 15. From the generating function of $\mathscr{H}_n(x;q)$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{(q^{1/2};q^{1/2})_n} \mathscr{H}_n(x;q)$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}(xt)^n}{(q^{1/2};q^{1/2})_n} (1-q)^{-n/2} \sum_{s=0}^{\infty} \frac{\left(-t^2\right)^s}{(q;q)_s}$$
$$= \sum_{n,s=0}^{\infty} \frac{q^{n(n-1)/4} t^{n+2s} (-1)^s x^n}{(q^{1/2};q^{1/2})_n (q;q)_s}$$
$$= \sum_{n,s=0}^{\infty} \frac{q^{n(n-1)/4} t^{n+2s} (-1)^s (1-q)^{-n/2}}{(q^{1/2};q^{1/2})_n (q;q)_s}$$
$$\times \left(q^{\alpha+1}, q; q\right)_n q^{-(\alpha n+n^2)}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^j q^{\binom{n-k}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-k}}$$
$$= \sum_{n,s,k=0}^{\infty} \Bigg[\Bigg(q^{(n+k)(n+k-1)/4} (-1)^s t^{n+2s+k} \left(q^{\alpha+1};q\right)_{n+k}$$
$$\times q^{-[\alpha(n+k)+(n+k)^2]} (1-q)^{-(n+k)/2} \Bigg)$$
$$\times \left(\left(q^{1/2};q^{1/2}\right)_{n+k} (q;q)_s \right)^{-1} \Bigg]$$
$$\times \frac{(-1)^k q^{\binom{n}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_n}$$
$$= \sum_{n,k=0}^{\infty} \sum_{s=0}^{[n/2]} \Bigg[\Bigg(q^{(n+k-2s)(n+k-2s-1)/4} (-1)^s t^{n+k}$$
$$\times \left(q^{\alpha+1};q\right)_{n-2s+k} q^{-[\alpha(n-2s+k)+(n-2s+k)^2]} \Bigg)$$
$$\times \left(\left(q^{1/2};q^{1/2}\right)_{n-2s+k} (q;q)_s \right)^{-1} \Bigg]$$
$$\times (1-q)^{-(n+k-2s)/2} \frac{(-1)^k q^{\binom{n-2s}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-2s}}$$
$$= \sum_{n,k=0}^{\infty} \frac{(-1)^k t^{n+k} \mathscr{L}_k^{(\alpha)}(x;q) (1-q)^{-(n+k)/2}}{(q^{\alpha+1};q)_k}$$
$$\times \sum_{s=0}^{[n/2]} \frac{(-1)^s \left(q^{\alpha+1}, q; q\right)_{n-2s+k} (1-q)^s}{(q^{1/2};q^{1/2})_{n+k-2s} (q;q)_{n-2s} (q;q)_s}$$
$$\times q^{(n+k-2s)(n+k-2s-1)/4}$$
$$\times q^{-[\alpha(n+k-2s)+(n+k-2s)^2]+\binom{n-2s}{2}}$$
$$= \sum_{n,k=0}^{\infty} \Bigg[\Bigg((-1)^k t^{n+k} \mathscr{L}_k^{(\alpha)}(x;q) (1-q)^{-(n+k)/2}$$

$$
\begin{aligned}
&\qquad \times\left(q^{\alpha+1};q\right)_{n+k}\left(-q^{1/2};q^{1/2}\right)_{n+k}\Big)\\
&\qquad \times\left(\left(q^{\alpha+1};q\right)_{k}(q;q)_n\right)^{-1}\Big]\\
&\quad\times\sum_{s=0}^{[n/2]}\frac{(-1)^s\left(q^{-n};q\right)_{2s}(1-q)^s}{\left(q^{-(n+k+\alpha)};q\right)_{2s}\left(-q^{-(n+k)/2};q^{1/2}\right)_{2s}(q;q)_s}\\
&\qquad\times q^{(n+k-2s)(n+k-2s-1)/4-[\alpha(n+k-2s)+(n+k-2s)^2]}\\
&\qquad\times q^{+\binom{n-2s}{2}-2s(\alpha+k)-s(n+k)+s(2s-1)/2}\\
&=\sum_{n=0}^{\infty}\left(q^{\alpha+1};q\right)_n\left(-q^{1/2};q^{1/2}\right)_n(1-q)^{-n/2}t^n\\
&\quad\times\sum_{k=0}^{n}\frac{(-1)^k\mathscr{L}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k(q;q)_{n-k}}\\
&\quad\times\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s\left(q^{k-n};q\right)_{2s}(1-q)^s}{\left(q^{-(n+\alpha)};q\right)_{2s}\left(-q^{-n/2};q^{1/2}\right)_{2s}(q;q)_s}\\
&\qquad\times q^{(n-2s)(n-2s-1)/4-[\alpha(n-2s)+(n-2s)^2]}\\
&\qquad\times q^{+\binom{n-k-2s}{2}-2s(\alpha+k)-sn+s(2s-1)/2}.
\end{aligned}\tag{62}
$$

Comparing the coefficients of t^n on both sides of (62), we obtain the results. □

Proof of Theorem 16. From the generating function of $\mathscr{G}_n(x;q)$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n q^{n(n-1)/4}}{\left(q^{1/2};q^{1/2}\right)_n}\mathscr{G}_n(x;q)\\
&\quad=\sum_{n,s=0}^{\infty}\frac{(-1)^s q^{\binom{s}{2}}x^n(1-q)^{-n/2}t^{n+2s}}{\left(q^{1/2};q^{1/2}\right)_n(q;q)_s}\\
&\quad=\sum_{n,s,k=0}^{\infty}\Big[\Big((-1)^s q^{\binom{s}{2}+n}(1-q)^{-(n+k)/2}t^{n+k+2s}\\
&\qquad\qquad\times\left(q^{\alpha+1};q\right)_{n+k}(-1)^k q^{\binom{k}{2}+\alpha k}\mathscr{M}_k^{(\alpha)}(x;q)\Big)\\
&\qquad\qquad\times\left(\left(q^{1/2};q^{1/2}\right)_{n+k}(q;q)_n\left(q^{\alpha+1};q\right)_k(q;q)_s\right)^{-1}\Big]\\
&\quad=\sum_{n,k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}+\alpha k+n}\mathscr{M}_k^{(\alpha)}(x;q)(1-q)^{-(n+k)/2}t^{n+k}}{\left(q^{\alpha+1};q\right)_k}\\
&\qquad\times\sum_{s=0}^{[n/2]}\frac{(-1)^s q^{\binom{s}{2}-2s}(1-q)^s\left(q^{\alpha+1};q\right)_{n-2s+k}}{\left(q^{1/2};q^{1/2}\right)_{n+k-2s}(q;q)_{n-2s}(q;q)_s}\\
&\quad=\sum_{n,k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}+\alpha k+n}\mathscr{M}_k^{(\alpha)}(x;q)(1-q)^{-(n+k)/2}t^{n+k}}{\left(q^{\alpha+1};q\right)_k}\\
&\qquad\times\frac{\left(q^{\alpha+1};q\right)_{n+k}\left(-q^{1/2};q^{1/2}\right)_{n+k}}{(q;q)_n}\\
&\qquad\times\sum_{s=0}^{[n/2]}\frac{(-1)^s q^{\binom{s}{2}-2s}(1-q)^s\left(q^{-n};q\right)_{2s}}{(q;q)_s\left(q^{-(n+k+\alpha)};q\right)_{2s}\left(-q^{-(n+k)/2};q^{1/2}\right)_{2s}}\\
&\qquad\times q^{-2s(k+\alpha)+s(2s-1)/2}\\
&\quad=\sum_{n=0}^{\infty}\left(q^{\alpha+1};q\right)_n\left(-q^{1/2};q^{1/2}\right)_n(1-q)^{-n/2}t^n\\
&\qquad\times\sum_{k=0}^{n}\frac{(-1)^k q^{\binom{k}{2}+\alpha k+n-k}\mathscr{M}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k(q;q)_{n-k}}\\
&\qquad\times\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s q^{\binom{s}{2}-2s}(1-q)^s\left(q^{k-n};q\right)_{2s}}{(q;q)_s\left(q^{-(n+\alpha)};q\right)_{2s}\left(-q^{-n/2};q^{1/2}\right)_{2s}}\\
&\qquad\times q^{-2s(k+\alpha)+s(2s-1)/2}.
\end{aligned}\tag{63}
$$

Equating the coefficients of t^n on both sides of (63), we obtain the results. □

Proof of Corollary 23. In view of the fact that

$$
\begin{aligned}
\left(q^{\beta};q\right)_{2n}&=\left(q^{\beta/2},-q^{\beta/2},q^{(\beta+1)/2},-q^{(\beta+1)/2};q\right)_n,\\
&\qquad |\beta|\in\mathbb{N},
\end{aligned}\tag{64}
$$

we have

$$
\begin{aligned}
&\lim_{q\to 1}\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s\left(q^{k-n};q\right)_{2s}(1-q)^s q^{\binom{s}{2}-2s}}{\left(q^{-(n+\alpha)};q\right)_{2s}\left(-q^{-n/2};q^{1/2}\right)_{2s}(q;q)_s}\\
&\quad=\lim_{q\to 1}\sum_{s=0}^{[(n-k)/2]}\Big[\Big((-1)^s\Big(q^{-(n-k)/2},-q^{-(n-k)/2},q^{-(n-k-1)/2},\\
&\qquad\qquad -q^{-(n-k-1)/2};q\Big)_S\Big)\\
&\qquad\qquad\times\Big(\Big(q^{-(n+\alpha)/2},-q^{-(n+\alpha)/2},\\
&\qquad\qquad q^{-(n+\alpha-1)/2},-q^{-(n+\alpha-1)/2};q\Big)_S\Big)^{-1}\Big]\\
&\qquad\times\frac{(1-q)^s q^{\binom{s}{2}-2s}}{\left(-q^{-n/2};q^{1/2}\right)_{2s}(q;q)_s}\\
&\quad={}_2F_2\left[\begin{matrix}\dfrac{-(n-k)}{2},\dfrac{-(n-k-1)}{2}\\[2mm] \dfrac{-(\alpha+n)}{2},\dfrac{-(\alpha+n-1)}{2}\end{matrix};\ -\frac{1}{4}\right],
\end{aligned}
$$

$$\begin{aligned}&\lim_{\substack{x\to(1-q)x,\\ q\to1}} \frac{\left(q^{\alpha+1};q\right)_n (q;q)_n}{(1-q)^n}\sum_{k=0}^{n} \frac{(-1)^k \mathscr{L}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k (q;q)_{n-k}}\\
&\quad\times q^{-n^2/4-n/4-\alpha n-nk+k^2/2}\\
&= \lim_{\substack{x\to(1-q)x,\\ q\to1}} \frac{\left(q^{\alpha+1},q;q\right)_n}{(1-q)^n} q^{-n(n-1)/4}\\
&\quad\times\sum_{k=0}^{n}\frac{(-1)^k q^{\binom{k}{2}+\alpha k}\mathscr{M}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k (q;q)_{n-k}}\\
&= (1+\alpha)_n\sum_{k=0}^{n}\frac{(-n)_k L_k^{(\alpha)}(x)}{(1+\alpha)_k}.
\end{aligned}\tag{65}$$

Combing (65) and (2), we deduce (50). The proof of Corollary 23 is complete. □

4. Integrals Related to Generalized q-Hermite Polynomials

The authors [23] deduced the following interesting result inspired by the relation of (16) and the orthogonality of q-Laguerre polynomials (6).

Proposition 19 (see [23, Theorem 1]). *The sequence of the q-polynomials $\{\mathscr{H}_n^{(\mu)}(x;q)\}$, which are defined by the relations* (16)*, satisfies the orthogonality relation*

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathscr{H}_m^{(\mu)}(x;q)\,\mathscr{H}_n^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty}dx\\
&\quad=\frac{\pi}{\cos\pi\mu}\frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty}q^{-n/2-\mu\theta_n}\\
&\qquad\times(q;q)_{[n/2]}\left(q^{\mu+1/2};q\right)_{[(n+1)/2]}\delta_{mn}
\end{aligned}\tag{66}$$

on the whole real line $\mathbb{R}$, where $\theta_n = n-2[n/2]$.

In this section, we will further consider multivariate q-Hermite polynomials by Theorems 2 and 3.

Theorem 20. *For $\theta_n = n-2[n/2]$, one has*

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathscr{H}_m^{(\alpha)}(xy;q)\,\mathscr{H}_n^{(\beta)}(xz;q)\,\frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty}dx\\
&\quad=(-1)^{[m/2]+[n/2]}(q;q)_{[m/2]}(q;q)_{[n/2]}\\
&\qquad\times(1-q)^{\mu+1/2+\theta_n}\frac{\pi\csc\left(-\mu+1/2-\theta_n\right)\pi}{\Gamma_q\left(-\mu+1/2-\theta_n\right)}\\
&\qquad\times\sum_{k=0}^{\min\{[m/2],[n/2]\}}q^{(\alpha+\beta-2\mu-1)k}(yz)^{2k+\theta_n}\\
&\qquad\times{}_2\phi_1\left[\begin{matrix}q^{k-[m/2]},q^{\mu+1/2+k+\theta_n}\\ q^{\alpha+k+1/2+\theta_n}\end{matrix};\ y^2q^{\alpha-\mu+[m/2]-k}\right]\\
&\qquad\times{}_2\phi_1\left[\begin{matrix}q^{k-[n/2]},q^{\mu+1/2+k+\theta_n}\\ q^{\beta+k+1/2+\theta_n}\end{matrix};\ z^2q^{\beta-\mu+[n/2]-k}\right]\\
&\qquad\times\left[\begin{matrix}\left[\dfrac{m}{2}\right]+\alpha-\dfrac{1}{2}+\theta_n\\ \left[\dfrac{m}{2}\right]-k\end{matrix}\right]\left[\begin{matrix}\left[\dfrac{n}{2}\right]+\beta-\dfrac{1}{2}+\theta_n\\ \left[\dfrac{n}{2}\right]-k\end{matrix}\right]\\
&\qquad\times\left[\begin{matrix}k+\mu-\dfrac{1}{2}+\theta_n\\ k\end{matrix}\right].
\end{aligned}\tag{67}$$

Theorem 21. *For $\theta_n = n-2[n/2]$, one has*

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathscr{G}_m^{(\alpha)}(xy;q)\,\mathscr{G}_n^{(\beta)}(xz;q)\,|x|^{2\mu}\left(x^2;q\right)_\infty dx\\
&\quad=(q;q)_{[m/2]}(q;q)_{[n/2]}(1-q)^{\mu+1/2+\theta_n}\\
&\qquad\times q^{\binom{\mu+3/2+\theta_n}{2}-(\alpha-1/2+\theta_n)[m/2]-(\beta-1/2+\theta_n)[n/2]}\\
&\qquad\times\frac{\pi\csc\left(-\mu+1/2-\theta_n\right)\pi}{\Gamma_q\left(-\mu+1/2-\theta_n\right)}\\
&\qquad\times\sum_{k=0}^{\min\{[m/2],[n/2]\}}q^{(2k+\mu+1/2-[m/2]-[n/2]+\theta_n)k}\\
&\qquad\times(yz)^{2k+\theta_n}{}_2\phi_1\left[\begin{matrix}q^{k-[m/2]},q^{\mu+1/2+k+\theta_n}\\ q^{\alpha+k+1/2+\theta_n}\end{matrix};y^2q\right]\\
&\qquad\times{}_2\phi_1\left[\begin{matrix}q^{k-[n/2]},q^{\mu+1/2+k+\theta_n}\\ q^{\beta+k+1/2+\theta_n}\end{matrix};z^2q\right]\\
&\qquad\times\left[\begin{matrix}\left[\dfrac{m}{2}\right]+\alpha-\dfrac{1}{2}+\theta_n\\ \left[\dfrac{m}{2}\right]-k\end{matrix}\right]\left[\begin{matrix}\left[\dfrac{n}{2}\right]+\beta-\dfrac{1}{2}+\theta_n\\ \left[\dfrac{n}{2}\right]-k\end{matrix}\right]\\
&\qquad\times\left[\begin{matrix}k+\mu-\dfrac{1}{2}+\theta_n\\ k\end{matrix}\right].
\end{aligned}\tag{68}$$

Remark 22. For $\alpha=\beta=\mu$ and $y=z=1$, Theorems 20 and 21 reduce to Proposition 19 and Corollary 23 respectively.

Corollary 23. *The sequence of the q-polynomials $\{\mathscr{G}_n^{(\mu)}(x;q)\}$, which are defined by the relations* (16)*, satisfies the orthogonality relation*

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathscr{G}_m^{(\mu)}(x;q)\,\mathscr{G}_n^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty}dx\\
&\quad=\frac{\pi}{\cos\pi\mu}\frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty}
\end{aligned}$$

$$\times q^{\binom{\mu+3/2}{2}-[n/2](\mu-3/2+\theta_n)+\theta_n} \times (q;q)_{[n/2]}\left(q^{\mu+1/2};q\right)_{[(n+1)/2]}\delta_{mn} \tag{69}$$

on the whole real line $\mathbb{R}$, *where* $\theta_n = n - 2[n/2]$.

Proof of Theorems 20 and 21. Let $\mathcal{I}(\alpha, y, m; \beta, z, n; \mu)$ and $\mathcal{J}(\alpha, y, m; \beta, z, n; \mu)$ represent the right hand side of (7) and (8) respectively. Since the weight function in (67) is an even function of the independent variable x by the definition (16), so the polynomials are evidently orthogonal to each other when the degrees m and n are either simultaneously even or odd. From (16) and Theorem 2 we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\alpha)}(xy;q)\,\mathcal{H}_{2n}^{(\beta)}(xz;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&= (-1)^{m+n}(q;q)_m(q;q)_n \times \int_{-\infty}^{\infty} \mathcal{L}_m^{(\alpha-1/2)}\left(x^2y^2;q\right)\mathcal{L}_n^{(\beta-1/2)} \times \left(x^2z^2;q\right)\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&= 2(-1)^{m+n}(q;q)_m(q;q)_n \times \int_{0}^{\infty} \mathcal{L}_m^{(\alpha-1/2)}\left(x^2y^2;q\right)\mathcal{L}_n^{(\beta-1/2)} \times \left(x^2z^2;q\right)\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&= (-1)^{m+n}(q;q)_m(q;q)_n \times \int_{0}^{\infty} \mathcal{L}_m^{(\alpha-1/2)}\left(y^2t;q\right)\mathcal{L}_n^{(\beta-1/2)}\left(z^2t;q\right)\frac{t^{\mu-1/2}}{(-t;q)_\infty}dt \\
&= (-1)^{m+n}(q;q)_m(q;q)_n \times \mathcal{I}\left(\alpha-\frac{1}{2}, y^2, m; \beta-\frac{1}{2}, z^2, n; \mu-\frac{1}{2}\right),
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{H}_{2m+1}^{(\alpha)}(xy;q)\,\mathcal{H}_{2n+1}^{(\beta)}(xz;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&= 2(-1)^{m+n}yz(q;q)_m(q;q)_n \times \int_{0}^{\infty} \mathcal{L}_m^{(\alpha+1/2)}\left(x^2y^2;q\right)\mathcal{L}_n^{(\beta+1/2)}\left(x^2z^2;q\right) \times \frac{|x|^{2\mu+2}}{(-x^2;q)_\infty}dx \\
&= (-1)^{m+n}yz(q;q)_m(q;q)_n \times \int_{0}^{\infty} \mathcal{L}_m^{(\alpha+1/2)}\left(y^2t;q\right)\mathcal{L}_n^{(\beta+1/2)}\left(z^2t;q\right)\frac{t^{\mu+1/2}}{(-t;q)_\infty}dt \\
&= (-1)^{m+n}yz(q;q)_m(q;q)_n \times \mathcal{I}\left(\alpha+\frac{1}{2}, y^2, m; \beta+\frac{1}{2}, z^2, n; \mu+\frac{1}{2}\right).
\end{aligned} \tag{71}$$

Putting (70) and (71) together and using Theorem 2, we complete the proof of Theorem 20 after some simplification. In the same way we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{G}_{2m}^{(\alpha)}(xy;q)\,\mathcal{G}_{2n}^{(\beta)}(xz;q)\,|x|^{2\mu}\left(-x^2;q\right)_\infty dx \\
&= q^{-\binom{m+1}{2}-\binom{n+1}{2}}(q;q)_m(q;q)_n\mathcal{J} \times \left(\alpha-\frac{1}{2}, y^2, m; \beta-\frac{1}{2}, z^2, n; \mu-\frac{1}{2}\right), \\
&\int_{-\infty}^{\infty} \mathcal{G}_{2m+1}^{(\alpha)}(xy;q)\,\mathcal{G}_{2n+1}^{(\beta)}(xz;q)\,|x|^{2\mu}\left(-x^2;q\right)_\infty dx \\
&= q^{-\binom{m+1}{2}-\binom{n+1}{2}}yz(q;q)_m(q;q)_n \times \mathcal{J}\left(\alpha+\frac{1}{2}, y^2, m; \beta+\frac{1}{2}, z^2, n; \mu+\frac{1}{2}\right),
\end{aligned} \tag{72}$$

which are two cases of Theorem 21; thus we obtain the proof. □

Proof of Proposition 19 and Corollary 23. Let us consider first that the case of both m and n is even, and just take $\alpha = \beta = \mu$ and $y = z = 1$ in Theorem 20. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\mu)}(x;q)\,\mathcal{H}_{2n}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&= (-1)^{m+n}(q;q)_m(q;q)_n(1-q)^{\mu+1/2} \times \frac{\pi\csc(-\mu+1/2)\pi}{\Gamma_q(-\mu+1/2)}\sum_{k=0}^{\min\{m,n\}} q^{-k} \\
&\quad\times {}_1\phi_0\left[\begin{matrix} q^{k-m} \\ - \end{matrix}; q^{m-k}\right] {}_1\phi_0\left[\begin{matrix} q^{k-n} \\ - \end{matrix}; q^{n-k}\right] \\
&\quad\times \begin{bmatrix} m+\alpha-\frac{1}{2} \\ m-k \end{bmatrix}\begin{bmatrix} n+\beta-\frac{1}{2} \\ n-k \end{bmatrix}\begin{bmatrix} k+\mu-\frac{1}{2} \\ k \end{bmatrix} \\
&= (q;q)_n(q;q)_n(1-q)^{\mu+1/2} \times \frac{\pi\csc(-\mu+1/2)\pi}{\Gamma_q(-\mu+1/2)}q^{-n}\begin{bmatrix} n+\mu-\frac{1}{2} \\ n \end{bmatrix}\delta_{mn} \\
&= \frac{\pi}{\cos\mu\pi}\frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty}q^{-n}(q;q)_n\left(q^{\mu+1/2};q\right)_n\delta_{mn},
\end{aligned} \tag{73}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{H}_{2m+1}^{(\mu)}(x;q)\,\mathscr{H}_{2n+1}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&\quad = (q;q)_n (q;q)_n (1-q)^{\mu+3/2} \\
&\qquad \times \frac{\pi\csc(-\mu-1/2)\pi}{\Gamma_q(-\mu-1/2)} q^{-n} \begin{bmatrix} n+\mu+\frac{1}{2} \\ n \end{bmatrix} \delta_{mn} \\
&\quad = -\frac{\pi}{\cos\mu\pi}\frac{\left(q^{-\mu-1/2};q\right)_\infty}{(q;q)_\infty} \\
&\qquad \times q^{-n}(q;q)_n \frac{(q;q)_{n+\mu+1/2}}{(q;q)_{\mu+1/2}}\delta_{mn} \\
&\quad = \frac{\pi}{\cos\mu\pi}\frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty} \\
&\qquad \times q^{-n-\mu-1/2}(q;q)_n \left(q^{\mu+1/2};q\right)_{n+1}\delta_{mn}.
\end{aligned} \tag{74}$$

Putting (73) and (74) together results in the orthogonality relation (66). The proof of Proposition 19 is complete. By the same way, we can deduce Corollary 23. This completes the proof. □

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