

Research Article

The Existence of a Nontrivial Solution for a p -Kirchhoff Type Elliptic Equation in \mathbb{R}^N

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Using Mountain Pass lemma, under some appropriate assumptions, we establish the existence of one nontrivial solution for a class of p -Kirchhoff-type elliptic equations in \mathbb{R}^N .

1. Introduction and Main Result

In this paper, we consider the existence of solution for the following elliptic problem:

$$\begin{aligned} & - \left(a + b \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \right) \operatorname{div} (g(x) |\nabla u|^{p-2} \nabla u) \\ & + h(x) |u|^{p-2} u = f(x, u), \quad x \in \mathbb{R}^N, \end{aligned} \quad (1)$$

$$u(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty,$$

where $a > 1$, $b > 0$, $N \geq 3$, $g(x)$, and $h(x)$ are positive functions, $f(x, t) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function, and further assumptions will be listed later. Problems like (1) originally came from the stationary problem of a model introduced by Kirchhoff [1]. Due to the existence of the integration over the whole space, problems like (1) are also called nonlocal problems.

In recent years, the Kirchhoff-type equations with p -Laplacian operator has been considered by many authors; see [2–6]. When $h(x) \equiv 0$, the authors in [6] considered the following similar Kirchhoff-type elliptic problem on the bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} & - \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \operatorname{div} (|\nabla u|^{p-2} \nabla u) \\ & = f(x, u), \quad x \in \Omega, \end{aligned} \quad (2)$$

$$u(x) = 0 \quad \text{on } \partial\Omega.$$

The function $M(t)$ in (2) is required to meet the condition of $m_0 \leq M(t) \leq m_1$ for some constants $m_0, m_1 > 0$. The authors proved that problem (2) has at least one positive solution under some other additional conditions. Problem (2) was also considered in [3], where the function $f(x, u)$ is odd about u . For $p = 2$, we refer to [7–9]. The authors in [8] studied the following Kirchhoff type problem:

$$\begin{aligned} & - \left(\int_{\Omega} a + b |\nabla u|^2 dx \right) \Delta u + u = f(x, u), \quad x \in \mathbb{R}^N, \\ & u(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow +\infty. \end{aligned} \quad (3)$$

By the Fountain theorem, the author proved the existence of infinitely many solutions. Note that one of the assumptions made on the function $f(x, u)$ in (3) is that $f(x, u) = -f(x, -u)$ for any $(x, u) \in (\mathbb{R}^N, \mathbb{R})$. In the present paper, however, the function $f(x, u)$ is not required to be odd about u as that of [3, 8]. When $a = b = 0$ and $g(x) \equiv 1$, problem (1) becomes the p -Laplacian elliptic equations without nonlocal term, and this kind of problem is also studied by many authors. For these works, we refer to [10–14] and the references therein. In [15], Liu discussed the following elliptic problem:

$$\begin{aligned} & - \operatorname{div} (|\nabla u|^{p-2} \nabla u) + V(x) |u|^{p-2} u = f(x, u), \\ & u(x) \in W^{1,p}(\mathbb{R}^N). \end{aligned} \quad (4)$$

The author proved that problem (4) has at least one ground state. The weight function $V(x)$ is required to be bounded; more precisely, there exist constants $\alpha, \beta > 0$ such that

$$V(x) \in C(\mathbb{R}^N), \quad 0 < \alpha \leq V(x) \leq \beta < +\infty. \quad (5)$$

We point out that the weight function $h(x)$ in problem (1) is permitted to tends to infinity.

In this paper, inspired by [4, 8, 15], we consider the existence of solution of problem (1). By the variational method, we will prove that problem (1) has at least one nontrivial weak solution. Since problem (1) is considered in the whole space \mathbb{R}^N , the loss of compactness of the Sobolev embedding renders the variational technique more delicate.

In this paper, we make the following hypothesis:

(A₁) $g(x) \geq 1$, $h(x) \geq h_0 > 0$ and $h(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$,

(A₂) there exists $\mu \geq 2p$ such that $0 \leq \mu F(x, t) \leq tf(x, t)$, where $F(x, t) = \int_0^t f(x, s)ds$,

(A₃) there exist constants $c > 0$ and $p < q < p^*$ such that $|f(x, t)| \leq c|t|^{q-1}$.

Remark 1. Throughout this paper, we denote by c the constant which may vary from line to line but remains independent of the relevant quantities. Note that there exist many functions $g(x)$, $h(x)$, and $f(x, t)$ such that satisfy the assumptions of Theorem 3, for example, $g(x) = 1 + x^4$, $h(x) = e^{x^2}$, $f(x, t) = t^{5p-1}/(1 + x^4)$.

Let $W_0^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space with the norm of

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p + |u|^p dx \right)^{1/p}. \quad (6)$$

Denote

$$E = \left\{ u \in W_0^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^p + h(x)|u|^p) dx < \infty \right\}. \quad (7)$$

Then, E is a Sobolev space with the norm of

$$\|u\|_E = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + h(x)|u|^p) dx \right)^{1/p}. \quad (8)$$

We give another space

$$X = \left\{ u \in E : \int_{\mathbb{R}^N} (g(x)|\nabla u|^p + h(x)|u|^p) dx < \infty \right\} \quad (9)$$

endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} (g(x)|\nabla u|^p + h(x)|u|^p) dx \right)^{1/p}. \quad (10)$$

It is not difficult to check that X is a Banach space. The Euler functional of problem (1) is

$$\begin{aligned} J(u) = & \frac{a}{p} \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx + \frac{b}{2p} \left(\int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \right)^2 \\ & + \frac{1}{p} \int_{\mathbb{R}^N} h(x) |u|^p dx - \int_{\mathbb{R}^N} F(x) dx, \end{aligned} \quad (11)$$

where $F(u) = \int_0^u f(x, t)dt$. Then, the assumptions (A₁)–(A₃) imply that $J(u) \in C^1(X, \mathbb{R}^1)$, and for any $\varphi \in X$, there holds

$$\begin{aligned} \langle J'(u), \varphi \rangle = & a \int_{\mathbb{R}^N} g(x) |\nabla u|^{p-2} \nabla u \nabla \varphi dx \\ & + b \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \int_{\mathbb{R}^N} g(x) |\nabla u|^{p-2} \nabla u \nabla \varphi dx \\ & + \int_{\mathbb{R}^N} h(x) |u|^{p-2} u \varphi dx - \int_{\mathbb{R}^N} f(x, u) \varphi dx. \end{aligned} \quad (12)$$

Particularly,

$$\begin{aligned} \langle J'(u), u \rangle = & a \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \\ & + b \left(\int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \right)^2 \\ & + \int_{\mathbb{R}^N} h(x) |u|^p dx - \int_{\mathbb{R}^N} f(x, u) u dx. \end{aligned} \quad (13)$$

Definition 2. A function $u \in X$ is said to be a weak solution of (1) if and only if (12) holds for any $\varphi \in X$.

Our main result is listed later.

Theorem 3. Assume (A₁)–(A₃). Then, problem (1) has at least one nontrivial weak solution in X .

This paper is organized as follows. In Section 2, we introduce some definitions and prove several lemmas which will be used later. In Section 3, we give the proof of Theorem 3 by making use of the Mountain Pass lemma.

2. Preliminary Results

In this section, we give some important lemmas, which will be needed in the proof of our main result. Particularly, one result of compact embedding on unbounded domain will be proved.

Lemma 4. Assume (A₁) and $p \leq q < p^*$. Then, the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact.

Proof. We split the proof into two cases.

Case 1 ($q = p$). Let

$$B_R = \{u \in X : \|u\|_X < R\}, \quad B_R^c = \mathbb{R}^N \setminus B_R \quad (14)$$

and $\eta = \inf_{B_R^c} h(x)$; then (A_1) shows that $\eta \rightarrow +\infty$ as $R \rightarrow \infty$. Furthermore, one gets that

$$\int_{B_R^c} |u|^p dx \leq \eta^{-1} \int_{B_R^c} h(x) |u|^p dx, \quad (15)$$

which implies that

$$\begin{aligned} \left(\int_{B_R^c} |u|^p dx \right)^{1/p} &\leq \eta^{-1/p} \left(\int_{B_R^c} h(x) |u|^p dx \right)^{1/p} \\ &\leq \eta^{-1/p} \left(\int_{\mathbb{R}^N} h(x) |u|^p dx \right)^{1/p} \\ &\leq \eta^{-1/p} \|u\|_X. \end{aligned} \quad (16)$$

It follows from (16) that

$$\lim_{R \rightarrow \infty} \sup_{u \in X \setminus \{0\}} \frac{\|u\|_{L^p(B_R^c)}}{\|u\|_X} = 0. \quad (17)$$

Let $\{u_n\}$ be a bounded sequence of X such that $\|u_n\|_X \leq c$. Then, there exists $u \in X$ such that $u_n \rightharpoonup u$. For any $\varepsilon > 0$, it follows from (17) that there exists R_ε large enough such that

$$\|u_n\|_{L^p(B_{R_\varepsilon}^c)} \leq \varepsilon c^{-1} \|u_n\|_X \leq \varepsilon \quad (n = 1, 2, \dots). \quad (18)$$

Note that the embedding $X \hookrightarrow L^p(B_{R_\varepsilon})$ is compact. Therefore, for these ε and R_ε , there exists $N_\varepsilon > 0$ large enough such that

$$\|u_n - u\|_{L^p(B_{R_\varepsilon})} \leq \varepsilon \quad (19)$$

for all $n > N_\varepsilon$. Thus, one can get from (18) and (19) that

$$\begin{aligned} \|u_n - u\|_{L^p(\mathbb{R}^N)} &\leq \|u_n - u\|_{L^p(B_{R_\varepsilon})} \\ &\quad + \|u_n\|_{L^p(B_{R_\varepsilon}^c)} + \|u\|_{L^p(B_{R_\varepsilon}^c)} < 3\varepsilon, \end{aligned} \quad (20)$$

which implies that $u_n \rightarrow u$ strongly in $L^q(\mathbb{R}^N)$.

Case 2 ($p < q < p^*$). By the Hölder and Young inequalities, one can get from (A_1) that

$$\begin{aligned} \int_{B_R^c} |u|^q dx &\leq \left(\int_{B_R^c} |u|^p dx \right)^{(p^*-q)/(p^*-p)} \\ &\quad \times \left(\int_{B_R^c} |u|^{p^*} dx \right)^{(q-p)/(p^*-p)} \\ &\leq \eta^{-(p^*-q)/(p^*-p)} \left(\int_{B_R^c} h(x) |u|^p dx \right)^{(p^*-q)/(p^*-p)} \\ &\quad \times \left(\int_{B_R^c} |u|^{p^*} dx \right)^{(q-p)/(p^*-p)} \end{aligned}$$

$$\begin{aligned} &\leq c\eta^{-(p^*-q)/(p^*-p)} \left[\left(\int_{B_R^c} h(x) |u|^p dx \right)^{q/p} \right. \\ &\quad \left. + \left(\int_{B_R^c} |u|^{p^*} dx \right)^{q/p^*} \right] \\ &\leq c\eta^{-(p^*-q)/(p^*-p)} \left[\left(\int_{B_R^c} h(x) |u|^p dx \right)^{q/p} \right. \\ &\quad \left. + \left(\int_{B_R^c} g(x) |\nabla u|^p dx \right)^{q/p} \right] \\ &\leq c\eta^{-(p^*-q)/(p^*-p)} \|u\|_X^q. \end{aligned} \quad (21)$$

Since $p < q < p^*$ and $\eta \rightarrow +\infty$ as $R \rightarrow \infty$, then

$$\lim_{R \rightarrow \infty} \sup_{u \in X \setminus \{0\}} \frac{\|u\|_{L^q(B_R^c)}}{\|u\|_X} = 0. \quad (22)$$

Therefore, similar to the proof of Case 1, the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact. \square

Now, we give the definitions of Palais-Smale (simply $(PS)_c$ sequence and $(PS)_c$ condition.

Definition 5. Let $c \in \mathbb{R}^1$, $J(u) \in C^1(X, \mathbb{R})$, and X be a Banach space. The sequence $\{u_n\} \subset X$ is said to be a $(PS)_c$ sequence if there holds

$$J(u_n) \rightarrow c, \quad J'(u_n) \rightarrow 0 \quad \text{in } X^* \text{ as } n \rightarrow \infty. \quad (23)$$

A functional $J(u)$ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence in X contains a convergent subsequence.

Next, we will prove that the functional $J(u)$ satisfies the $(PS)_c$ condition.

Lemma 6. Assume (A_1) – (A_3) . Then, $J(u)$ satisfies the $(PS)_c$ condition on X for any $c \in \mathbb{R}^1$.

Proof. Let $\{u_n\}$ be an any $(PS)_c$ sequence in X . We divide the proof into two steps.

Step 1 ($\{u_n\}$ is bounded in X). Note that $a > 1$; then, it follows from (23) and (A_2) that

$$\begin{aligned} c + \|u_n\|_X &\geq J(u_n) - \frac{1}{2p} \langle J'(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{2p} \right) a \int_{\mathbb{R}^N} g(x) |\nabla u_n|^p dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} h(x) |u_n|^p dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{2p} f(x, u_n) u_n - F \right) dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2p} \int_{\mathbb{R}^N} g(x) |\nabla u_n|^p dx \\
&\quad + \frac{1}{2p} \int_{\mathbb{R}^N} h(x) |u_n|^p dx = \frac{1}{2p} \|u_n\|_X^p,
\end{aligned} \tag{24}$$

which implies that $\{u_n\}$ is bounded in X .

Step 2 ($\{u_n\}$ converges strongly in X). Since $\{u_n\}$ is bounded in the separable X , there exist $u_0 \in X$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in X . Now, we want to prove that

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x, u_n) u_n dx \\
&\quad - \int_{\mathbb{R}^N} f(x, u_n) u_0 dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{25}$$

In fact, it follows from (A_3) that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} f(x, u_n) u_n dx - \int_{\mathbb{R}^N} f(x, u_n) u_0 dx \right| \\
&\leq \int_{\mathbb{R}^N} |u_n|^{q-1} |u_n - u_0| dx.
\end{aligned} \tag{26}$$

Thus, in order to prove (25), we need only to prove that

$$\int_{\mathbb{R}^N} |u_n|^{q-1} |u_n - u_0| dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Here, we divide the proof (27) into two cases:

(i)

$$p \leq q < \frac{(N-1)p}{N-p}. \tag{28}$$

By the Hölder inequality, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{q-1} |u_n - u_0| dx \\
&\leq \left(\int_{\mathbb{R}^N} |u_n|^{p(q-1)/(p-1)} dx \right)^{(p-1)/p} \\
&\quad \times \left(\int_{\mathbb{R}^N} |u_n - u_0|^p dx \right)^{1/p}.
\end{aligned} \tag{29}$$

It is easy to check that $p \leq p(q-1)/(p-1) < p^*$. Then, the compact embedding in Lemma 4 shows that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |u_n|^{p(q-1)/(p-1)} dx \right)^{(p-1)/p} < \infty, \\
&\left(\int_{\mathbb{R}^N} |u_n - u_0|^p dx \right)^{1/p} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{30}$$

which gives (27).

(ii)

$$\frac{(N-2)p}{N-p} \leq q < p^*. \tag{31}$$

Let $\lambda_1 = p^*/(q-2)$. Since $1 < p < N$ and $p(N-2)/(N-p) \leq q < p^*$, there exist $p \leq \lambda_2 = \lambda_3 < p^*$ such that

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 1. \tag{32}$$

The Hölder inequality shows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{q-1} |u_n - u_0| dx \leq \left(\int_{\mathbb{R}^N} |u_n|^{(q-2)\lambda_1} dx \right)^{1/\lambda_1} \\
&\quad \times \left(\int_{\mathbb{R}^N} |u_n|^{\lambda_2} dx \right)^{1/\lambda_2} \\
&\quad \times \left(\int_{\mathbb{R}^N} |u_n - u_0|^{\lambda_3} dx \right)^{1/\lambda_3}.
\end{aligned} \tag{33}$$

Therefore, the embedding in Lemma 4 implies that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |u_n|^{(q-2)\lambda_1} dx \right)^{1/\lambda_1} < \infty, \\
&\left(\int_{\mathbb{R}^N} |u_n|^{\lambda_2} dx \right)^{1/\lambda_2} < \infty, \\
&\left(\int_{\mathbb{R}^N} |u_n - u_0|^{\lambda_3} dx \right)^{1/\lambda_3} \longrightarrow 0,
\end{aligned} \tag{34}$$

which also gives (27).

Note that $(N-1)p/(N-p) < p^*$ and

$$\begin{aligned}
&\frac{(N-2)p}{N-p} \geq p, \quad p \geq 2, \\
&\frac{(N-2)p}{N-p} < p, \quad 1 < p < 2.
\end{aligned} \tag{35}$$

Therefore, the previous cases (i) and (ii) imply that (25) holds for all $p \leq q < p^*$.

Denote

$$\begin{aligned}
D_n &= \langle J'(u_n), u_n \rangle - \langle J'(u_n), u_0 \rangle \\
&\quad + \int_{\mathbb{R}^N} f(x, u_n) u_n dx - \int_{\mathbb{R}^N} f(x, u_n) u_0 dx.
\end{aligned} \tag{36}$$

Then, it follows from (23) and (25) that $D_n = o(1)$ and

$$\begin{aligned}
o(1) = D_n &= a \int_{\mathbb{R}^N} g(x) |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u_0) dx \\
&\quad + b \left(\int_{\mathbb{R}^N} g(x) |\nabla u_n|^p dx \right) \\
&\quad \times \int_{\mathbb{R}^N} g(x) |\nabla u_n|^{p-2} \nabla u \nabla (u_n - u_0) dx \\
&\quad + \int_{\mathbb{R}^N} h(x) |u_n|^{p-2} u_n (u_n - u_0) dx \\
&\quad + \int_{\mathbb{R}^N} f(x, u_n) u_n dx - \int_{\mathbb{R}^N} f(x, u_n) u_0 dx.
\end{aligned} \tag{37}$$

Since $u_n \rightharpoonup u_0$ in X , one obtains from the Hölder inequality that

$$\begin{aligned} o(1) = & -a \int_{\mathbb{R}^N} g(x) |\nabla u_0|^{p-2} \nabla u_0 \nabla (u_n - u_0) dx \\ & -b \left(\int_{\mathbb{R}^N} g(x) |\nabla u_n|^p dx \right) \\ & \times \int_{\mathbb{R}^N} g(x) |\nabla u_0|^{p-2} \nabla u_0 \nabla (u_n - u_0) dx \\ & - \int_{\mathbb{R}^N} h(x) |u_0|^{p-2} u_0 (u_n - u_0) dx. \end{aligned} \quad (38)$$

Thus, it follows from (37) and (38) that

$$\begin{aligned} o(1) = & a \int_{\mathbb{R}^N} g(x) (|\nabla u_n|^{p-2} \nabla u_n \\ & - |\nabla u_0|^{p-2} \nabla u_0) \nabla (u_n - u_0) dx \\ & + b \left(\int_{\mathbb{R}^N} g(x) |\nabla u_n|^p dx \right) \\ & \times \int_{\mathbb{R}^N} g(x) (|\nabla u_n|^{p-2} \nabla u_n \\ & - |\nabla u_0|^{p-2} \nabla u_0) \nabla (u_n - u_0) dx \\ & + \int_{\mathbb{R}^N} h(x) (|u_n|^{p-2} u_n - |u_0|^{p-2} u_0) (u_n - u_0) dx \\ & + \int_{\mathbb{R}^N} f(x, u_n) u_n dx - \int_{\mathbb{R}^N} f(x, u_n) u_0 dx. \end{aligned} \quad (39)$$

Since $a \geq 1$, $b > 0$, and $g(x)$, $h(x)$ are positive functions, we get from (25) and (39) that

$$\begin{aligned} & \int_{\mathbb{R}^N} g(x) (|\nabla u_n|^{p-2} \nabla u_n \\ & - |\nabla u_0|^{p-2} \nabla u_0) \nabla (u_n - u_0) dx \longrightarrow 0, \quad (40) \\ & \int_{\mathbb{R}^N} h(x) (|u_n|^{p-2} u_n - |u_0|^{p-2} u_0) (u_n - u_0) dx \longrightarrow 0. \end{aligned}$$

Therefore, we can deduce from (40) and the following standard inequalities in [16]:

$$|\xi - \zeta|^p \leq \begin{cases} c \langle |\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta, \xi - \zeta \rangle & \text{for } p \geq 2, \\ c \langle |\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta, \xi - \zeta \rangle^{p/2} \\ \quad \times (|\xi|^p + |\zeta|^p)^{(2-p)/2} & \text{for } 1 < p < 2 \end{cases} \quad (41)$$

that

$$\|u_n - u_0\|_X \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (42)$$

Then, we complete the proof. \square

3. Existence of Solution

In this section, the proof of Theorem 3 is mainly based on the following Mountain Pass lemma [17] (also see [18]).

Lemma 7 (Mountain Pass lemma). *Let X be a Banach space and $J \in C^1(X, \mathbb{R})$ satisfies $(PS)_c$ condition. Suppose that $J(0) = 0$ and*

- (I₁) *there are constant $\rho, \alpha > 0$ such that $J_{\partial B_\rho} \geq \alpha$,*
- (I₂) *there is an $e \in X \setminus B_\rho$ such that $J(e) \leq 0$.*

Then, J possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g[0,1]} J(u) \geq \alpha > 0, \quad (43)$$

where

$$\Gamma = \{g \in C([0, 1], X) : g(0) = 0, g(1) = e\}. \quad (44)$$

In view of Lemma 6, the functional $J(u)$ satisfies the $(PS)_c$ condition. It is obvious that $J(0) = 0$; then, in order to apply Lemma 7, we need only to prove that $J(u)$ satisfies the geometric conditions of the Mountain Pass lemma.

Lemma 8. *Assume (A_1) – (A_2) . Then,*

- (I₁) *there exist $\rho, \alpha > 0$ such that $J(u) \geq \alpha > 0$ with $\|u\|_X = \rho$,*
- (I₂) *there is an $e \in X$ such that $\|e\|_X > \rho$ and $J(e) \leq 0$.*

Proof. It follows from (A_2) , (A_3) , and the embedding in Lemma 4 that

$$\begin{aligned} J(u) \geq & \frac{a}{p} \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx + \frac{b}{2p} \left(\int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \right)^2 \\ & + \frac{1}{p} \int_{\mathbb{R}^N} h(x) |u|^p dx - \frac{1}{\mu} \int_{\mathbb{R}^N} |u|^q dx \\ \geq & \frac{1}{p} \int_{\mathbb{R}^N} (g(x) |\nabla u|^p + h(x) |u|^p) dx - c \|u\|_X^q \\ = & \frac{1}{p} \|u\|_X^p - c \|u\|_X^q. \end{aligned} \quad (45)$$

Since $p < q$, there exists a sufficient small $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha > 0$ with $\|u\|_X = \rho$. On the other hand, for fixed $u_0 \in X$ and $t > 0$, it follows from (A_2) that

$$\begin{aligned} J(tu_0) \leq & \frac{at^p}{p} \int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \\ & + \frac{bt^{2p}}{2p} \left(\int_{\mathbb{R}^N} g(x) |\nabla u|^p dx \right)^2 + \frac{t^p}{p} \int_{\mathbb{R}^N} h(x) |u|^p dx \\ & - t^\mu \int_{\mathbb{R}^N} |u_0|^\mu dx \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \end{aligned} \quad (46)$$

Then, there is a large $t_0 > 0$ such that $\|t_0 u_0\|_X > \rho$ and $J(t_0 u_0) < 0$. One may choose $e = t_0 u_0$; then, $\|e\|_X > \rho$ and $J(e) < 0$. Thus, the proof of Lemma 8 is complete. \square

Proof of Theorem 3. It follows from Lemmas 7 and 8 that the functional $J(u)$ has a critical point u such that $J(u) = c > 0$; that is, problem (1) has at least one weak solution. On the other hand, $J(0) = 0$; then, problem (1) has at least one nontrivial weak solution. \square

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