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*Research Article*

# Upper and Lower Bounds for Ranks of the Matrix Expression $X - XAX$

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We consider the question of how to take $X$ such that the nonlinear matrix expression $X - XAX$ attains its maximal and minimal possible ranks.

## 1. Introduction

Throughout this paper $C^{m\times n}$ denotes the set of all $m \times n$ matrices over the complex field $C$. $I_k$ denotes the identity matrix of order $k$ and $O_{m\times n}$ is the $m \times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For a matrix $A \in C^{m\times n}$, $A^*$ and $r(A)$ denote the conjugate transpose and the rank of the matrix $A$, respectively. $(A, B)$ denotes a row block matrix consisting of $A \in C^{m\times n}$ and $B \in C^{m\times k}$.

Let $A \in C^{m\times n}$; a generalized inverse $X$ of $A$ is a matrix which satisfies some of the following four Penrose equations [1]:

$$\begin{aligned} AXA &= A, \qquad & XAX &= X, \\ (AX)^* &= AX, \qquad & (XA)^* &= XA. \end{aligned} \tag{1}$$

For a subset $\{i, j, \ldots, k\}$ of the set $\{1, 2, 3, 4\}$, the set of $n \times m$ matrices satisfying the equations $(i), (j), \ldots, (k)$ from (1) is denoted by $A\{i, j, \ldots, k\}$. A matrix in $A\{i, j, \ldots, k\}$ is called an $\{i, j, \ldots, k\}$-inverse of $A$ and is denoted by $A^{(i,j,\ldots,k)}$. For example, an $n \times m$ matrix $X$ of the set $A\{2\}$ is called a $\{2\}$-inverse of $A$ and is denoted by $X = A^{(2)}$. The unique $\{1, 2, 3, 4\}$-inverse of $A$ is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of $A$. For convenience, the symbols $E_A$ and $F_A$ stand for the two orthogonal projectors $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. We refer the reader to [2–4] for basic results on generalized inverses.

In matrix theory and applications, there exists a nonlinear matrix expression that involves variable entries:

$$P(X) = X - XAX, \tag{2}$$

where $A \in C^{m\times m}$ is a given complex matrix and $X \in C^{m\times m}$ is a variable matrix. These nonlinear matrix expressions vary with respect to the choice of $X$. One of the fundamental problems for (2) is to determine the maximal and minimal possible ranks of the matrix expression $P(X)$ when $X$ is running over $C^{m\times m}$. Since the rank of matrix is an integer between 0 and the minimum of row and column numbers of the matrix [5], then the maximal and minimal ranks of $P(X)$ can be attained for some $X$.

The investigation of extremal ranks of matrix expressions has many direct motivations in matrix analysis. For example, a matrix expression $D - AXB$ of order $n$ is nonsingular if and only if the maximal rank of $D - AXB$ with respect to $X$ is $n$; two consistent matrix equations $X_1 = X_1AX_1$ and $X_2 = X_2BX_2$ have a common solution if and only if the minimal rank of the difference $X_1 - X_2$ of their solutions is zero; a nonlinear matrix equation $X = XAX$ is consistent if and only if the minimal rank of $X - XAX$ with respect to $X$ is zero. From the definition of the $\{2\}$-inverse of a matrix, we know that the solution $X$ of the nonlinear matrix equation $X = XAX$ is a $\{2\}$-inverse of matrix $A$; that is, using the minimal rank of $X - XAX$, we can find out the general expression of the $\{2\}$-inverses of a matrix $A$, which is a matrix such that the nonlinear matrix expression $X - XAX$ attains its minimal rank. In general, for any two matrix expressions $P(X_1, X_2, \ldots, X_m)$ and $Q(Y_1, Y_2, \ldots, Y_n)$ of the same size, there are $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ such that $P(X_1, X_2, \ldots, X_m) = Q(Y_1, Y_2, \ldots, Y_n)$ if and only if

$$\min_{X_1,X_2,\ldots,X_m;\,Y_1,Y_2,\ldots,Y_n} r\left(P\left(X_1, X_2, \ldots, X_m\right) - Q\left(Y_1, Y_2, \ldots, Y_n\right)\right) = 0. \tag{3}$$

These examples imply that the extremal ranks of matrix expressions have close links with many topics in matrix analysis and applications. Various statements on maximal and minimal ranks of matrix expressions are quite easy to understand for the people who know linear algebra. But the question now is how to give simple or closed forms for the extremal ranks of a matrix expression with respect to its variant matrices. The study on maximal and minimal ranks of matrix expression started in late 1980s. If want to know more about this question the reader can see [6–18].

The work in this paper includes two parts. First, in Section 2, we will consider how to choose a matrix $X \in C^{m\times m}$, such that $X - XAX$ has the maximal possible rank. Second, in Section 3, we will determine the minimal rank of $X - XAX$ and present a general expression of the {2}-inverses of matrix $A \in C^{m\times m}$.

In order to find the extremal ranks of the nonlinear matrix expression $X - XAX$, we need the following lemmas, which will be used in this paper.

**Lemma 1** (see [19]). *Let $P(X_1, X_2) = A - B_1X_1C_1 - B_2X_2C_2$, where $A \in C^{m\times n}$, $B_1 \in C^{m\times p_1}$, $B_2 \in C^{m\times p_2}$, $C_1 \in C^{q_1\times n}$, and $C_2 \in C^{q_2\times n}$ are given matrices. Then for any variable matrices $X_1 \in C^{p_1\times q_1}$ and $X_2 \in C^{p_2\times q_2}$, one has*

$$\max_{X_1,X_2} r\left(P\left(X_1, X_2\right)\right) = \min\left\{r\left(A, B_1, B_2\right), r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix}, r\begin{pmatrix} A & B_1 \\ C_2 & O \end{pmatrix}, r\begin{pmatrix} A & B_2 \\ C_1 & O \end{pmatrix}\right\}, \tag{4}$$

$$\min_{X_1,X_2} r\left(P\left(X_1, X_2\right)\right) = r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix} + r\left(A, B_1, B_2\right) + \max\left\{S_1, S_2\right\}, \tag{5}$$

*where*

$$\begin{aligned} S_1 &= r\begin{pmatrix} A & B_1 \\ C_2 & O \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_2 & O & O \end{pmatrix} - r\begin{pmatrix} A & B_1 \\ C_1 & O \\ C_2 & O \end{pmatrix}, \\ S_2 &= r\begin{pmatrix} A & B_2 \\ C_1 & O \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_1 & O & O \end{pmatrix} - r\begin{pmatrix} A & B_2 \\ C_1 & O \\ C_2 & O \end{pmatrix}. \end{aligned} \tag{6}$$

**Lemma 2** (see [20]). *Let $A$–$BXC$ be a linear matrix expression over the complex field $C$, where $A \in C^{m\times n}$, $B \in C^{m\times k}$, and $C \in C^{l\times n}$ are given; $X \in C^{k\times l}$ is a variant matrix. Then the maximal rank of $A - BXC$ with respect to $X$ is*

$$\max_X r\left(A - BXC\right) = \min\left\{r\left(A, B\right), r\begin{pmatrix} A \\ C \end{pmatrix}\right\}, \tag{7}$$

*the general expression of $X$ satisfying* (7) *is*

$$X = \left(E_{A_2}B\right)^{\dagger} E_{A_2} A F_{A_1} \left(CF_{A_1}\right)^{\dagger} + U, \tag{8}$$

*where $A_1 = E_BA = (I_m - BB^{\dagger})A$, $A_2 = AF_C = A(I_n - C^{\dagger}C)A$, $E_{A_2} = I_m - A_2A_2^{\dagger}$, and $F_{A_1} = I_n - A_1^{\dagger}A_1$, and the matrix $U \in C^{k\times l}$ is chosen such that*

$$r\left(E_{A_2}BUCF_{A_1}\right) = \min\left\{r\left(E_{A_2}B\right), r\left(CF_{A_1}\right)\right\}. \tag{9}$$

**Lemma 3** (see [21]). *Let $A \in C^{m\times n}$ and $B \in C^{m\times k}$. Then*

$$(A, B)^{\dagger} = \begin{pmatrix} \left(I_n + TT^*\right)^{-1}\left(A^{\dagger} - A^{\dagger}BC^{\dagger}\right) \\ T^*\left(I_n + TT^*\right)^{-1}\left(A^{\dagger} - A^{\dagger}BC^{\dagger}\right) + C^{\dagger} \end{pmatrix}, \tag{10}$$

*where $C = (I_m - AA^{\dagger})B$, $T = A^{\dagger}B(I_k - C^{\dagger}C)$, and $(I_n + TT^*)^{-1} = I_n - T(I_k + T^*T)^{-1}T^*$.*

**Lemma 4** (see [22]). *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$, and $C \in C^{l\times n}$. Then*

$$\begin{gathered} r\left(A, B\right) = r\left(A\right) + r\left(E_AB\right) = r\left(B\right) + r\left(E_BA\right), \\ r\begin{pmatrix} A \\ C \end{pmatrix} = r\left(A\right) + r\left(CF_A\right) = r\left(C\right) + r\left(AF_C\right), \\ r\left(A + B\right) \le r\left(A\right) + r\left(B\right), \end{gathered} \tag{11}$$

*where $E_A = I_m - AA^{\dagger}$, $E_B = I_m - BB^{\dagger}$, $F_A = I_n - A^{\dagger}A$ and $F_C = I_n - C^{\dagger}C$.*

## 2. The Maximal Rank of $X$−$XAX$ with respect to $X$

Let $A \in C^{m\times m}$ be a given matrix; in this section, we will present the maximal rank of the nonlinear matrix expression $X - XAX$, with respect to the variable matrix $X \in C^{m\times m}$. The relative results are included in the following three lemmas.

**Lemma 5.** *Let $A \in C^{m\times m}$, and $I = I_m$ denotes the identity matrix of order $m$. Then*

$$\begin{pmatrix} I & I \\ O & A \end{pmatrix}^{\dagger} = \begin{pmatrix} \left(I + F_A\right)^{-1} & -A^{\dagger} \\ F_A\left(I + F_A\right)^{-1} & A^{\dagger} \end{pmatrix}. \tag{12}$$

*Proof.* By Lemma 3 with $M = \begin{pmatrix} I \\ O \end{pmatrix}$ and $N = \begin{pmatrix} I \\ A \end{pmatrix}$, we have

$$\begin{aligned} \begin{pmatrix} I & I \\ O & A \end{pmatrix}^{\dagger} &= (M, N)^{\dagger} \\ &= \begin{pmatrix} \left(I + TT^*\right)^{-1}\left(M^{\dagger} - M^{\dagger}NK^{\dagger}\right) \\ T^*\left(I + TT^*\right)^{-1}\left(M^{\dagger} - M^{\dagger}NK^{\dagger} + K^{\dagger}\right) \end{pmatrix}, \end{aligned} \tag{13}$$

where

$$M^{\dagger} = (I, O), \qquad K = \left(I - MM^{\dagger}\right) N = \begin{pmatrix} O \\ A \end{pmatrix}, \qquad K^{\dagger} = \left(O, A^{\dagger}\right), \tag{14}$$

$$M^{\dagger} - M^{\dagger} N K^{\dagger} = \left(I, A^{\dagger}\right), \qquad T = M^{\dagger} N \left(I - K^{\dagger} K\right) = F_A = I - A^{\dagger} A. \tag{15}$$

Combining (13), (14) with (15), we have

$$\begin{pmatrix} I & I \\ O & A \end{pmatrix}^{\dagger} = \begin{pmatrix} (I+F_A)^{-1} & -(I+F_A)^{-1} A^{\dagger} \\ F_A (I+F_A)^{-1} & -F_A (I+F_A)^{-1} A^{\dagger} + A^{\dagger} \end{pmatrix} = \begin{pmatrix} (I+F_A)^{-1} & -A^{\dagger} \\ F_A (I+F_A)^{-1} & A^{\dagger} \end{pmatrix}. \tag{16}$$

The second equality holds as

$$(I+F_A)^{-1} = I - F_A (I+F_A)^{-1} F_A, \qquad F_A A^{\dagger} = \left(I - A^{\dagger} A\right) A^{\dagger} = O. \tag{17}$$

□

**Lemma 6.** *Let $A \in C^{m\times m}$, and $I = I_m$ denotes the identity matrix of order $m$. Then*

$$\max_U r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix} - \begin{pmatrix} I & I \\ O & A \end{pmatrix} U \begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) = m + r(A). \tag{18}$$

*The general expression of $U$ satisfying* (18) *is*

$$U = \begin{pmatrix} A^{\dagger} + V_1 & V_2 \\ -A^{\dagger} + V_3 & V_4 \end{pmatrix}, \tag{19}$$

*where $V_1, V_2, V_3, V_4 \in C^{m\times m}$ are chosen such that*

$$r\left(\begin{pmatrix} I & I \\ O & A \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}\begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) = m + r(A). \tag{20}$$

*Proof.* By Lemma 2 with $P = \begin{pmatrix} O & O \\ O & A \end{pmatrix}$, $Q = \begin{pmatrix} I & I \\ O & A \end{pmatrix}$, and $W = \begin{pmatrix} O & -A \\ -I & O \end{pmatrix}$, we have

$$\begin{aligned} &\max_U r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix} - \begin{pmatrix} I & I \\ O & A \end{pmatrix} U \begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) \\ &= \max_U r(P - QUW) \\ &= \min\left\{ r(P, Q), r\begin{pmatrix} P \\ W \end{pmatrix} \right\} \\ &= \min\left\{ r\begin{pmatrix} O & O & I & I \\ O & A & O & A \end{pmatrix}, r\begin{pmatrix} O & O \\ O & A \\ O & -A \\ -I & O \end{pmatrix} \right\} \\ &= m + r(A). \end{aligned} \tag{21}$$

From Lemmas 2 and 5, we have the general expression of $U$ satisfying (21) as

$$U = \left(E_{A_2} Q\right)^{\dagger} E_{A_2} P F_{A_1} \left(W F_{A_1}\right)^{\dagger} + V, \tag{22}$$

where

$$V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}, \qquad Q^{\dagger} = \begin{pmatrix} (I+F_A)^{-1} & -A^{\dagger} \\ F_A (I+F_A)^{-1} & A^{\dagger} \end{pmatrix}, \qquad W^{\dagger} = \begin{pmatrix} O & -I \\ -A^{\dagger} & O \end{pmatrix},$$

$$\begin{aligned} A_1 = E_Q P &= \left(I - QQ^{\dagger}\right) P \\ &= \left(I - \begin{pmatrix} I & I \\ O & A \end{pmatrix}\begin{pmatrix} (I+F_A)^{-1} & -A^{\dagger} \\ F_A (I+F_A)^{-1} & A^{\dagger} \end{pmatrix}\right)\begin{pmatrix} O & O \\ O & A \end{pmatrix} \\ &= \begin{pmatrix} O & O \\ O & E_A \end{pmatrix}\begin{pmatrix} O & O \\ O & A \end{pmatrix} = O, \end{aligned} \tag{23}$$

$$\begin{aligned} A_2 = P F_W &= P\left(I - W^{\dagger} W\right) \\ &= \begin{pmatrix} O & O \\ O & A \end{pmatrix}\left(I - \begin{pmatrix} O & -I \\ -A^{\dagger} & O \end{pmatrix}\begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) \\ &= \begin{pmatrix} O & O \\ O & A \end{pmatrix}\begin{pmatrix} O & O \\ O & F_A \end{pmatrix} = O. \end{aligned} \tag{24}$$

Combining (22), (23) with (24), we have $E_{A_2} = I$, $F_{A_1} = I$ and

$$\begin{aligned} U &= Q^{\dagger} P W^{\dagger} + V \\ &= \begin{pmatrix} (I+F_A)^{-1} & -A^{\dagger} \\ F_A (I+F_A)^{-1} & A^{\dagger} \end{pmatrix}\begin{pmatrix} O & O \\ O & A \end{pmatrix}\begin{pmatrix} O & -I \\ -A^{\dagger} & O \end{pmatrix} \\ &\quad + \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \\ &= \begin{pmatrix} A^{\dagger} + V_1 & V_2 \\ -A^{\dagger} + V_3 & V_4 \end{pmatrix}, \end{aligned} \tag{25}$$

where $V_1, V_2, V_3, V_4 \in C^{m\times m}$ are chosen such that

$$\begin{aligned} &r\left(E_{A_2} Q \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} W F_{A_1}\right) \\ &= r\left(\begin{pmatrix} I & I \\ O & A \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}\begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) \\ &= \min\left\{ r\left(E_{A_2} Q\right), r\left(W F_{A_1}\right) \right\} \\ &= \min\left\{ r(Q), r(W) \right\} \\ &= m + r(A). \end{aligned} \tag{26}$$

□

**Lemma 7.** *Let $A \in C^{m\times m}$. Then there exist some matrices $V \in C^{m\times m}$, such that*

$$\max_V r\left[\left(A^{\dagger} + V\right) - \left(A^{\dagger} + V\right) A \left(A^{\dagger} + V\right)\right] = m. \tag{27}$$

*Proof.* Applying Lemma 4, we have

$$\begin{aligned} & r\left[\left(A^{\dagger}+V\right)-\left(A^{\dagger}+V\right)A\left(A^{\dagger}+V\right)\right] \\ & = r\left(A^{\dagger}-A^{\dagger}AA^{\dagger}+V-VA^{\dagger}A-A^{\dagger}AV-VAV\right) \\ & \le r\left(A^{\dagger}-A^{\dagger}AA^{\dagger}\right)+r\left(V-VA^{\dagger}A-A^{\dagger}AV-VAV\right) \\ & = 0+r\left(V-VA^{\dagger}A-A^{\dagger}AV-VAV\right) \\ & = r\begin{pmatrix} V-VAA^{\dagger}-A^{\dagger}AV & VA \\ AV & A \end{pmatrix}-r(A) \\ & = r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} -I \\ O \end{pmatrix}V\left(I-AA^{\dagger},A\right)\right. \\ & \qquad \left. -\begin{pmatrix} A^{\dagger}A \\ -A \end{pmatrix}V(I,O)\right)-r(A). \end{aligned} \tag{28}$$

From (27) and (28), we have

$$\begin{aligned} & \max_{V} r\left[\left(A^{\dagger}+V\right)-\left(A^{\dagger}+V\right)A\left(A^{\dagger}+V\right)\right] \\ & = \max_{V} r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} -I \\ O \end{pmatrix}V\left(I-AA^{\dagger},A\right)\right. \\ & \qquad \left. -\begin{pmatrix} A^{\dagger}A \\ -A \end{pmatrix}V(I,O)\right)-r(A). \end{aligned} \tag{29}$$

By Lemma 1, we have

$$\begin{aligned} & \max_{V} r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} -I \\ O \end{pmatrix}V\left(I-AA^{\dagger},A\right)\right. \\ & \qquad \left. -\begin{pmatrix} A^{\dagger}A \\ -A \end{pmatrix}V(I,O)\right) \\ & = \min\left\{ r\begin{pmatrix} O & O & -I & A^{\dagger}A \\ O & A & O & A \end{pmatrix}, r\begin{pmatrix} O & O \\ O & A \\ I-AA^{\dagger} & A \\ I & O \end{pmatrix},\right. \\ & \qquad \left. r\begin{pmatrix} O & O & -I \\ O & A & O \\ I & O & O \end{pmatrix}, r\begin{pmatrix} O & O & A^{\dagger}A \\ O & A & -A \\ I-AA^{\dagger} & A & O \end{pmatrix}\right\} \\ & = \min\left\{m+r(A), m+r(A), 2m+r(A), m+r(A)\right\} \\ & = m+r(A). \end{aligned} \tag{30}$$

Combining (29) with (30), we get the result (27). □

According to Lemmas 5, 6, and 7, we immediately obtain the following theorem.

**Theorem 8.** *Let $A \in C^{m\times m}$ be a given matrix, and $X \in C^{m\times m}$ is a variant matrix. Then*

$$\max_{X} r(X-XAX) = m. \tag{31}$$

*In consequence,*

(1) *there always exists $X \in C^{m\times m}$, such that $X - XAX$ is nonsingular;*

(2) *the matrices $X$ satisfying* (31) *are given by $X = A^{\dagger} + V$ and $V \in C^{m\times m}$ is the same as in* (27).

*Proof.* First describe a special congruence transformation for a block matrix, which reduces the calculation of the maximal rank of $X - XAX$:

$$\begin{aligned} & \max_{X} r(X-XAX) \\ & = \max_{X}\begin{pmatrix} X & XA \\ AX & A \end{pmatrix}-r(A) \\ & = \max_{X} r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} -I \\ -A \end{pmatrix}X(I,O)-\begin{pmatrix} I \\ O \end{pmatrix}X\begin{pmatrix} O \\ -A \end{pmatrix}\right) \\ & \quad -r(A). \end{aligned} \tag{32}$$

By Lemma 1, we have

$$\begin{aligned} & \max_{X} r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} -I \\ -A \end{pmatrix}X(I,O)-\begin{pmatrix} I \\ O \end{pmatrix}X\begin{pmatrix} O \\ -A \end{pmatrix}\right) \\ & = \min\left\{ r\begin{pmatrix} O & O & -I & I \\ O & A & -A & O \end{pmatrix}, r\begin{pmatrix} O & O \\ O & A \\ I & O \\ O & -A \end{pmatrix},\right. \\ & \qquad \left. r\begin{pmatrix} O & O & -I \\ O & A & -A \\ O & -A & O \end{pmatrix}, r\begin{pmatrix} O & O & I \\ O & A & O \\ I & O & O \end{pmatrix}\right\} \\ & = \min\left\{m+r(A), m+r(A), m+r(A), 2m+r(A)\right\} \\ & = m+r(A). \end{aligned} \tag{33}$$

Combining (32) with (33), we have

$$\max_{X} r(X-XAX) = m. \tag{34}$$

On the other hand, from another special congruence transformation for a block matrix, we have

$$\begin{aligned} & \max_{X} r(X-XAX) \\ & = \max_{X}\begin{pmatrix} X & XA \\ AX & A \end{pmatrix}-r(A) \\ & = \max_{X} r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} I & I \\ O & A \end{pmatrix}\begin{pmatrix} X & O \\ O & X \end{pmatrix}\begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) \\ & \quad -r(A). \end{aligned} \tag{35}$$

Combining formula (35) with Lemma 6, we have

$$\begin{aligned}
&\max_X r(X - XAX) \\
&= \max_X r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix} - \begin{pmatrix} I & I \\ O & A \end{pmatrix}\begin{pmatrix} X & O \\ O & X \end{pmatrix}\begin{pmatrix} O & -A \\ -I & O \end{pmatrix}\right) - r(A) \\
&= \min\left\{ r\begin{pmatrix} O & O & I & I \\ O & A & O & A \end{pmatrix}, r\begin{pmatrix} O & O \\ O & A \\ O & -A \\ -I & O \end{pmatrix} \right\} - r(A) \\
&= m + r(A) - r(A) \\
&= m.
\end{aligned} \tag{36}$$

That is, theres always exist $X \in C^{m\times m}$, such that $X - XAX$ is nonsingular.

From the results in Lemmas 6 and 7, we obtain that there always exist $X = A^{\dagger} + V$, such that

$$\max_X r(X - XAX) = \max_V r\left[\left(A^{\dagger} + V\right) - \left(A^{\dagger} + V\right)A\left(A^{\dagger} + V\right)\right] = m. \tag{37}$$

□

## 3. The Minimal Rank of $X - XAX$ with respect to $X$

In this section, we will present the minimal rank of the nonlinear matrix expression $X - XAX$. Moreover, we will consider how to choose a matrix $X$, such that $X - XAX$ has the minimal possible rank.

**Theorem 9.** *Let $A \in C^{m\times m}$ be a given matrix, and $X \in C^{m\times m}$ is a variant matrix. Then*

$$\min_X r(X - XAX) = 0. \tag{38}$$

*In consequence, there exists $X \in C^{m\times m}$, such that the nonlinear matrix equation $X = XAX$ is consistent.*

*Proof.* By formula (5) in Lemma 1, we have

$$\begin{aligned}
&\min_X r(X - XAX) \\
&= \min_X \begin{pmatrix} X & XA \\ AX & A \end{pmatrix} - r(A) \\
&= \max_X r\left(\begin{pmatrix} O & O \\ O & A \end{pmatrix} - \begin{pmatrix} -I \\ -A \end{pmatrix} X (I, O) - \begin{pmatrix} I \\ O \end{pmatrix} X \begin{pmatrix} O \\ -A \end{pmatrix}\right) \\
&\quad - r(A) \\
&= r\begin{pmatrix} O & O \\ O & A \\ O & -A \\ -I & O \end{pmatrix} + r\begin{pmatrix} O & O & I & I \\ O & A & O & A \end{pmatrix} \\
&\quad + \max\{S_1, S_2\} - r(A) \\
&= 2m + r(A) + \max\{S_1, S_2\},
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
S_1 &= r\begin{pmatrix} O & O & I \\ O & A & O \\ -I & O & O \end{pmatrix} - r\begin{pmatrix} O & O & I & I \\ O & A & O & A \\ -I & O & O & O \end{pmatrix} \\
&\quad - r\begin{pmatrix} O & O & I \\ O & A & O \\ O & -A & O \\ -I & O & O \end{pmatrix} = -2m - r(A),
\end{aligned} \tag{40}$$

$$\begin{aligned}
S_2 &= r\begin{pmatrix} O & O & I \\ O & A & A \\ O & -A & O \end{pmatrix} - r\begin{pmatrix} O & O & I & I \\ O & A & O & A \\ O & -A & O & O \end{pmatrix} \\
&\quad - r\begin{pmatrix} O & O & I \\ O & A & A \\ O & -A & O \\ -I & O & O \end{pmatrix} = -2m - r(A).
\end{aligned} \tag{41}$$

Combining (39), (40) with (41), we have

$$\min_X r(X - XAX) = 0. \tag{42}$$

□

**Corollary 10.** *Let $A \in C^{m\times m}$ be a given matrix. Then the matrix $X \in C^{m\times m}$ satisfying the matrix equation $X = XAX$ is given by*

$$X = \left(A^{\dagger} + F_A V\right) A \left(A^{\dagger} + W E_A\right), \tag{43}$$

*where $V, W \in C^{m\times m}$ are two variant matrices.*

*Proof.* Putting $X = (A^{\dagger} + F_A V)A(A^{\dagger} + WE_A)$ into $X - XAX$ yields

$$\begin{aligned}
&X - XAX \\
&= \left(A^{\dagger} + F_A V\right) A \left(A^{\dagger} + W E_A\right) \\
&\quad - \left(A^{\dagger} + F_A V\right) A A^{\dagger} A \left(A^{\dagger} + F_A V\right) A \left(A^{\dagger} + W E_A\right) \\
&= \left(A^{\dagger} + F_A V\right) A \left(A^{\dagger} + W E_A\right) \\
&\quad - \left(A^{\dagger} + F_A V\right) A A^{\dagger} A \left(A^{\dagger} + W E_A\right) \\
&= O.
\end{aligned} \tag{44}$$

□

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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