Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 956950, 35 pages
doi:10.1155/2012/956950

*Research Article*

# Strong Convergence Theorems for a Common Fixed Point of Two Countable Families of Relatively Quasi Nonexpansive Mappings and Applications

**Jingling Zhang, Yongfu Su, and Qingqing Cheng**

*Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*

Correspondence should be addressed to Yongfu Su, suyongfu@tjpu.edu.cn

Received 19 June 2012; Revised 3 August 2012; Accepted 7 August 2012

Academic Editor: Yeong-Cheng Liou



The purpose of this paper is to prove strong convergence theorems for common fixed points of two countable families of relatively quasi nonexpansive mappings in a uniformly convex and uniformly smooth real Banach space using the properties of generalized $f$-projection operator. In order to get the strong convergence theorems, a new iterative scheme by monotone hybrid method is presented and is used to approximate the common fixed points. Then, two examples of countable families of uniformly closed nonlinear mappings are given. The results of this paper modify and improve the results of Li et al. (2010), the results of Takahashi and Zembayashi (2008), and many others.

## 1. Introduction

Let $E$ be a real Banach space with the dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The duality mapping $J$ has the following properties:

(i) if $E$ is smooth, then $J$ is single-valued;

(ii) if $E$ is strictly convex, then $J$ is one-to-one;

(iii) if $E$ is reflexive, then $J$ is surjective;

(iv) if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$;

(v) if $E^*$ is uniformly convex, then $J$ is uniformly continuous on bounded subsets of $E$ and $J$ is single valued and also one-to-one (see [1–4]).

Let $E$ be a smooth Banach space with the dual $E^*$. The functional $\phi : E \times E \rightarrow R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{1.2}$$

for all $x, y \in E$.

Let $C$ be a closed convex subset of $E$, and let $T$ be a mapping from $C$ into itself. We denote by $F(T)$ the set of fixed points of $T$. A point $p$ in $C$ is said to be an asymptotic fixed point of $T$ [5] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\widehat{F}(T)$. A mapping $T$ from $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [1, 6–8] if $F(T) = \widehat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [1, 6–8].

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [9] which is well known as Mann's iteration process and is defined as follows:

$$\begin{aligned} &x_0 \quad \text{chosen arbitrarily}, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0, 1]$. Fourteen years later, Halpern [10] proposed the new innovation iteration process which resembles Manns iteration (1.3), it is defined by

$$\begin{aligned} &x_0 \quad \text{chosen arbitrarily}, \\ &x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

where the element $u \in C$ is fixed. Seven years later, Ishikawa [2] enlarged and improved Mann's iteration (1.3) to the new iteration method, it is often cited as Ishikawa's iteration process which is defined recursively by

$$\begin{aligned} &x_0 \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

In both Hilbert space [10–12] and uniformly smooth Banach space [13–15] the iteration process (1.4) has been proved to be strongly convergent if the sequence $\{\alpha_n\}$ satisfies the following conditions:

(i) $\alpha_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

By the restriction of condition (ii), it is widely believed that Halpern's iteration process (1.4) has slow convergence though the rate of convergence has not been determined. Halpern [10] proved that conditions (i) and (ii) are necessary in the strong convergence of (1.4) for a nonexpansive mapping $T$ on a closed convex subset $C$ of a Hilbert space $H$. Moreover, Wittmann [12] showed that (1.4) converges strongly to $P_{F(T)}u$ when $\{\alpha_n\}$ satisfies (i), (ii), and (iii), where $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

Both iterations processes (1.3) and (1.5) have only weak convergence, in general Banach space (see [16] for more details). As a matter of fact, process (1.3) may fail to converge while process (1.5) can still converge for a Lipschitz pseudo contractive mapping in a Hilbert space [17]. For example, Reich [18] proved that if $E$ is a uniformly convex Banach space with Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of $T$. However, we note that Manns iteration process (1.3) has only weak convergence even in a Hilbert space [16].

Some attempts to modify the Mann's iteration method so that strong convergence guaranteed has recently been made. Nakajo and Takahashi [19] proposed the following modification of the Mann iteration method for a single nonexpansive mapping $T$ in a Hilbert space $H$:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily},\\
&y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n\cap Q_n}(x_0),
\end{aligned} \tag{1.6}$$

where $C$ is a closed convex subset of $H$, $P_K$ denotes the metric projection from $H$ onto a closed convex subset $K$ of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denotes the fixed points set of $T$.

The ideas to generalize the process (1.6) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [8] presented their ideas as the following method for a single relatively nonexpansive mapping $T$ in a Banach space $E$:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily}, \\
&y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n), \\
&C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\
&Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
&x_{n+1} = \Pi_{C_n \cap Q_n}(x_0).
\end{aligned} \tag{1.7}$$

They proved the following convergence theorem.

**Theorem MT.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by* (1.7)*, where $J$ is the duality mapping on $E$. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from $C$ onto $F(T)$.*

In 2007, Plubtieng and Ungchittrakool [20] proposed the following hybrid algorithms for two relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

**Theorem SK 1.** *Let $E$ be a uniformly convex and uniformly smooth real Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T, S$ be two relatively nonexpansive mappings from $C$ into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{aligned}
&x_0 \in C \quad \textit{chosen arbitrarily}, \\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\
&z_n = J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right), \\
&H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\
&W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
&x_{n+1} = \Pi_{H_n \cap W_n}(x_0),
\end{aligned} \tag{1.8}$$

*with the following restrictions:*

(i) $0 \le \alpha_n < 1$, $\limsup_{n\to\infty}\alpha_n < 1$;

(ii) $0 \le \beta_n^{(1)}, \beta_n^{(1)}, \beta_n^{(3)} \le 1$, $\lim_{n\to\infty}\beta_n^{(1)} = 0$, $\liminf_{n\to\infty}\beta_n^{(2)}\beta_n^{(3)} > 0$.

*Then the $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\Pi_F$ is the generalized projection from $C$ onto $F$.*

**Theorem SK 2.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T, S$ be two relatively nonexpansive mappings from $C$ into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily},\\
&y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n),\\
&z_n = J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right),\\
&H_n = \left\{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n\left(\|x_0\|^2 + 2\langle z, J x_n - J x_0\rangle\right)\right\},\\
&W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\},\\
&x_{n+1} = \Pi_{H_n \cap W_n}(x_0),
\end{aligned} \tag{1.9}$$

*with the following restrictions:*

(i) $0 < \alpha_n < 1$, $\limsup_{n\to\infty}\alpha_n < 1$;

(ii) $0 \le \beta_n^{(1)}, \beta_n^{(1)}, \beta_n^{(3)} \le 1$, $\lim_{n\to\infty}\beta_n^{(1)} = 0$, $\liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$.

*Then the $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\Pi_F$ is the generalized projection from $C$ onto $F$.*

In 2010, Su et al. [21] proposed the following hybrid algorithms for two countable families of weak relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

**Theorem SXZ 1.** *Let $E$ be a uniformly convex and uniformly smooth real Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from $C$ into itself such that $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily},\\
&z_n = J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right),\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
&C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\right\},\\
&C_0 = \{z \in C : \phi(z, y_0) \le \phi(z, x_0)\},\\
&Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\right\},\\
&Q_0 = C,\\
&x_{n+1} = \Pi_{C_n \cap Q_n}(x_0)
\end{aligned} \tag{1.10}$$

*with the conditions:*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(2)} > 0$;

(iii) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(3)} > 0$;

(iv) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

**Theorem SXZ 2.** *Let* $E$ *be a uniformly convex and uniformly smooth real Banach space, let* $C$ *be a nonempty closed convex subset of* $E$*, and let* $\{T_n\}, \{S_n\}$ *be two countable families of weak relatively nonexpansive mappings from* $C$ *into itself such that* $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$*. Define a sequence* $\{x_n\}$ *in* $C$ *by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily},\\
&z_n = J^{-1}\left(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right),\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
&C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \le \left(1 - \alpha_n \beta_n^{(1)}\right)\phi(z, x_n) + \alpha_n \beta_n^{(1)} \phi(z, x_0)\right\},\\
&C_0 = \{z \in C : \phi(z, y_0) \le \phi(z, x_0)\},\\
&Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\right\},\\
&Q_0 = C,\\
&x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{aligned} \tag{1.11}$$

*with the conditions:*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\lim_{n\to\infty}\beta_n^{(1)} = 0$;

(iii) $\limsup_{n\to\infty}\beta_n^{(2)}\beta_n^{(3)} > 0$;

(iv) $0 \le \alpha_n \le \alpha < 1 \quad for \quad some \quad \alpha \in (0,1)$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Recently, Li et al. [22] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized $f$-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{aligned}
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\\
&C_n = \{w \in C_n : G(w, J y_n) \le G(w, J x_n)\},\\
&x_{n+1} = \Pi_{C_{n+1}}^{f}(x_0), \quad n \ge 1.
\end{aligned} \tag{1.12}$$

They proved a strong convergence theorem for finding an element in the fixed points set of $T$. We remark here that the results of Li et al. [22] extended and improved on the results of Matsushita and Takahashi [8].

Motivated by the above-mentioned results and the ongoing research, it is our purpose in this paper to prove a strong convergence theorem for two countable families of relatively quasi nonexpansive mappings in a uniformly convex and uniformly smooth real Banach space using the properties of generalized $f$-projection operator. Our results extend the results of Li et al. [22], Takahashi and Zembayashi [23], and many other recent known results in the literature.

## 2. Preliminaries

Let $E$ be a smooth Banach space with the dual $E^*$. The functional $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{2.1}$$

for all $x, y \in E$. Observe that, in a Hilbert space $H$, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

Recall that if $C$ is a nonempty, closed, and convex subset of a Hilbert space $H$ and $P_C : H \to C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive. This is true only when $H$ is a real Hilbert space. In this connection, Alber [24] has recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.2}$$

existence and uniqueness of the operator $\Pi_C$ follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping $J$. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the functional $\phi$ that

$$(\|x\| - \|y\|)^2 \le \phi(y, x) \le \left(\|y\|^2 + \|x\|^2\right), \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) - 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y \in E$. See [25] for more details.

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

*Remark 2.1.* If $E$ is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of $J$, we have $Jx = Jy$. Since $J$ is one-to-one, then we have $x = y$; see [12, 15, 26] for more details.

Let $C$ be a closed convex subset of $E$, and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from $C$ into itself. We denote by $F$ the set of common fixed points of $\{T_n\}_{n=0}^{\infty}$, that is,

$F = \bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$, where $F(T_n)$ denote the set of fixed points of $T_n$, for all $n \geq 0$. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$. The set of asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. A point $p$ in $C$ is said to be a *strong asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if $C$ contains a sequence $\{x_n\}$ which converges strongly to $p$ such that $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$. The set of strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$ [21].

*Definition 2.2.* Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of relatively nonexpansive mappings if the following conditions are satisfied:

(i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;

(ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$;

(iii) $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

*Definition 2.3.* Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of weak relatively nonexpansive mappings if the following conditions are satisfied:

(i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;

(ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$;

(iii) $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

*Definition 2.4.* Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of relatively quasi nonexpansive mappings if the following conditions are satisfied:

(i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;

(ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$.

*Definition 2.5.* A mapping $T$ is said to be relatively nonexpansive mappings if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$;

(iii) $\widehat{F}(T) = F(T)$.

*Definition 2.6.* A mapping $T$ is said to be weak relatively nonexpansive mappings if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$;

(iii) $\widetilde{F}(T) = F(T)$.

*Definition 2.7.* A mapping $T$ is said to be relatively quasi nonexpansive mappings if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$.

The Definition 2.5 (Definitions 2.6 and 2.7) is a special form of the Definition 2.2 (Definitions 2.3 and 2.4) as $T_n \equiv T$ for all $n \geq 0$. The following conclusions are obvious: (1) relatively nonexpansive mapping must be weak relatively nonexpansive mapping; (2) weak relatively nonexpansive mapping must be relatively quasi nonexpansive mapping.

The hybrid algorithms for fixed point of relatively nonexpansive mappings and applications have been studied by many authors, for example, [1, 6, 7, 17, 27, 28]. In recent years, the definition of relatively quasi nonexpansive mapping has been presented and studied by many authors [7, 17, 26, 28]. Now we give an example which is a countable family of relatively quasi nonexpansive mappings but not a countable family of relatively nonexpansive mappings.

*Example 2.8.* Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$
$$\|\xi\| = \left( \sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \tag{2.5}$$
$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots),\ \eta = (\eta_1, \eta_2, \eta_3, \dots, \eta_n \dots) \in l^2.$$

It is well known that $l^2$ is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots), \\ x_1 &= (1, 1, 0, 0, \dots), \\ x_2 &= (1, 0, 1, 0, 0, \dots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \dots), \\ &\vdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \dots, \xi_{n,k}, \dots), \\ &\vdots \end{aligned} \tag{2.6}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1,\ n+1, \\ 0 & \text{if } k \neq 1,\ k \neq n+1, \end{cases} \tag{2.7}$$

for all $n \geq 1$.

Define a countable family of mappings $T_n : E \longrightarrow E$ as follows:

$$T_n(x) = \begin{cases} \dfrac{n}{n+1} x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n, \end{cases} \tag{2.8}$$

for all $n \geq 0$.

*Conclusion 2.9.* $\{x_n\}$ converges weakly to $x_0$.

*Proof.* For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \longrightarrow 0, \tag{2.9}$$

as $n \longrightarrow \infty$. That is, $\{x_n\}$ converges weakly to $x_0$ □

*Conclusion 2.10.* $\{x_n\}$ is not a Cauchy sequence, so that, it does not converge strongly to any element of $l^2$.

*Proof.* In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

*Conclusion 2.11.* $T_n$ has a unique fixed point 0, that is, $F(T_n) = \{0\}$, for all $n \geq 0$.

*Proof.* The conclusion is obvious. □

*Conclusion 2.12.* $x_0$ is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$.

*Proof.* Since $\{x_n\}$ converges weakly to $x_0$ and

$$\|T_n x_n - x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \longrightarrow 0 \tag{2.10}$$

as $n \longrightarrow \infty$, so that, $x_0$ is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$. □

*Conclusion 2.13.* $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi nonexpansive mappings.

*Proof.* Since $E = L^2$ is a Hilbert space, for any $n \geq 0$ we have

$$\phi(0, T_n x) = \|0 - T_n x\|^2 = \|T_n x\|^2 \leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E. \tag{2.11}$$

then $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi nonexpansive mappings □

*Conclusion 2.14.* $\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mappings.

*Proof.* From Conclusions 2.11 and 2.12, we have $\bigcap_{n=0}^{\infty} F(T_n) \neq \widehat{F}(\{T_n\}_{n=0}^{\infty})$, so that, $\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mapping. □

Next, we recall the concept of generalized $f$-projector operator, together with its properties. Let $G : C \times E^* \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.12}$$

where $\xi \in C$, $\varphi \in E$, $\rho$ is a positive number, and $f : C \to R \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From the definitions of $G$ and $f$, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to $\varphi$ when $\xi$ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to $\xi$ when $\varphi$ is fixed.

*Definition 2.15* (see [29]). Let $E$ be a real Banach space with its dual $E^*$. Let $C$ be a nonempty, closed and convex subset of $E$. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized $f$-projection operator if

$$\Pi_C^f \varphi = \left\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \right\}, \quad \forall \varphi \in E^*. \tag{2.13}$$

For the generalized $f$-projection operator, Wu and Huang [42] proved the following theorem basic properties

**Lemma 2.16** (see [29]). *Let $E$ be a real reflexive Banach space with its dual $E^*$. Let $C$ be a nonempty, closed, and convex subset of $E$. Then the following statements hold:*

(i) *$\Pi_C^f$ is a nonempty closed convex subset of $C$ for all $\varphi \in E^*$;*

(ii) *if $E$ is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f$ if and only if*

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.14}$$

(iii) *if $E$ is strictly convex and $f : C \to R \cup \{+\infty\}$ is positive homogeneous ($i.e., f(tx) = tf(x)$ $for\ all$ $t > 0$ such that $tx \in C, where$ $x \in C$), then $\Pi_C^f$ is a single valued mapping.*

Fan et al. [30] showed that the condition $f$ is a positive homogeneous which appeared in Lemma 2.13 can be removed.

**Lemma 2.17** (see [30]). *Let $E$ be a real reflexive Banach space with its dual $E^*$ and $C$ a nonempty, closed, and convex subset of $E$. Then if $E$ is strictly convex, then $\Pi_C^f$ is a single-valued mapping.*

Recall that $J$ is a single-valued mapping when $E$ is a smooth Banach space. There exists a unique element $\varphi \in E$. such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.15) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx \rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.15}$$

Now, we consider the second generalized $f$-projection operator in a Banach space.

*Definition 2.18.* Let $E$ be a real Banach space and $C$ a nonempty, closed, and convex subset of $E$. We say that $\Pi_C^f : E \to 2^C$ is a generalized $f$-projection operator if

$$\Pi_C^f(x) = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \right\}, \quad \forall \in E. \tag{2.16}$$

Obviously, the definition of relatively quasi nonexpansive mapping $T$ is equivalent to

(R1) $F(T) \neq \emptyset$;

(R2) $G(p, JTx) \leq G(p, Jx)$, for all $x \in C$, $p \in F(T)$.

**Lemma 2.19** (see [31]). *Let $E$ be a Banach space and let $f : E \to R \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exists $x^* \in E$ and $\alpha \in R$ such that*

$$f(x) \geq (x, x^*) + \alpha, \quad \forall x \in E. \tag{2.17}$$

We know that the following lemmas hold for operator $\Pi_C^f$.

**Lemma 2.20** (see [22]). *Let $C$ be a nonempty, closed, and convex subset of a smooth and reflexive Banach space $E$. Then the following statements hold:*

(i) *$\Pi_C^f x$ is a nonempty closed and convex subset of $C$ for all $x \in E$;*

(ii) *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.18}$$

(iii) *if $E$ is strictly convex, then $\Pi_C^f x$ is a single valued mapping.*

**Lemma 2.21** (see [22]). *Let $C$ be a nonempty, closed, and convex subset of a smooth and reflexive Banach space $E$. let $x \in E$ and $\hat{x} \in \Pi_C^f$, then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \tag{2.19}$$

The fixed points set $F(T)$ of a relatively quasi nonexpansive mapping is closed convex as given in the following lemma.

**Lemma 2.22** (see [32, 33]). *Let $C$ be a nonempty, closed, and convex subset of a smooth and reflexive Banach space $E$. let $T$ be a closed relatively quasi nonexpansive mapping of $C$ into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

**Lemma 2.23** (see [25]). *Let $E$ be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of $E$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

**Lemma 2.24** (see [24, 25, 27]). *Let $C$ be a nonempty closed convex subset of a smooth real Banach space $E$ and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.20}$$

**Lemma 2.25** (see [28]). *Let $E$ be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of $E$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|) \tag{2.21}$$

*for all $x, y$, and $z \in B_r(0)$ and $\lambda, \mu$, and $\gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.*

It is easy to prove the following result.

**Lemma 2.26.** *Let $E$ be a strictly convex and smooth real Banach space, let $C$ be a closed convex subset of $E$, and let $T$ be a relatively quasi nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.*

**Lemma 2.27.** *Let $E$ be a $p$-uniformly convex Banach space with $p \geq 2$. Then, for all $x, y \in E$, $j(x) \in J_p(x)$ and $j(y) \in J_p(y)$,*

$$\langle x - y, j(x) - j(y) \rangle \geq \frac{c^p}{c^{p-2} p} \|x - y\|^p, \tag{2.22}$$

*where $J_p$ is the generalized duality mapping from $E$ into $E^*$ and $1/c$ is the $p$-uniformly convexity constant of $E$.*

Observe that an infinite family of operators $\{T_n\}_{n=1}^{\infty}$ in a Banach space is said to be uniformly closed, if $x_n \to x$, $x_n - T_n x_n \to 0$ then $T_n x = x$ (i.e., $x \in \cap_{n=1}^{\infty} F(T_n)$). Obviously, a countable family of uniformly closed of relatively quasi nonexpansive mappings is a countable family of weak relatively nonexpansive mappings.

## 3. Main Results

**Theorem 3.1.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $\{T_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ be two countable families of uniformly closed of relatively quasi nonexpansive mappings of $C$ into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$.*

*Let $f : E \to R$ be a convex and lower semicontinuous mapping with $C \subset int\,(D(f))$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$
\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right), \\
y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n) Jz_n), \\
C_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : G(z, Jy_n) \le G(z, Jx_n)\right\}, \\
C_0 &= \{z \in C : G(z, Jy_0) \le G(z, Jx_0)\}, \\
Q_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\right\}, \\
Q_0 &= C, \\
x_{n+1} &= \Pi^f_{C_n \bigcap Q_n}(x_0),
\end{aligned}
\tag{3.1}
$$

*with the conditions*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(iii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iv) $0 \le \alpha_n \le \alpha < 1 \quad for\ some\ \alpha \in (0,1)$.

*Then $\{x_n\}$ converges strongly to $x^*$, where $x^* = \Pi^f_F x_0$.*

*Proof. Step 1*. We show that $C_n$ and $Q_n$ are closed and convex for each $n \ge 0$.

From the definitions of $C_n$ and $Q_n$, it is obvious that $Q_n$ is closed and convex and $C_n$ is closed for each $n \ge 0$. Moreover, since $G(z, Jy_n) \le G(z, Jx_n)$ is equivalent to

$$\phi(z, y_n) \le \phi(z, x_n), \tag{3.2}$$

and is equivalent to

$$2\langle z, Jx_n - Jy_n\rangle \le \|x_n\|^2 + \|y_n\|^2, \tag{3.3}$$

it follows that $C_n$ is convex for each $n \ge 0$. So, $C_n \bigcap Q_n$ is a closed convex subset of $E$ for all $n \in N \cup \{0\}$.

*Step 2*. We show that $F \subset C_n \bigcap Q_n$ for all $n \ge 0$. Observe that

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right). \tag{3.4}$$

Hence from the definition of $G(x, Jy)$ and the convexity of $\|\cdot\|^2$we have, for all $p \in F$, that

$$\begin{aligned}
G(p, Jz_n) &= G\Big(p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\Big) \\
&= \|p\|^2 - 2\Big\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\Big\rangle \\
&\quad + \Big\|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\Big\|^2 + 2\rho f(p) \\
&\leq \|p\|^2 - 2\Big\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\Big\rangle \\
&\quad + \beta_n^{(1)} \|Jx_n\|^2 + \beta_n^{(2)} \|JT_n x_n\|^2 + \beta_n^{(3)} \|JS_n x_n\|^2 + 2\rho f(p) \\
&= \beta_n^{(1)} G(p, Jx_n) + \beta_n^{(2)} G(p, JT_n x_n) + \beta_n^{(3)} G(p, JS_n x_n) \\
&\leq \beta_n^{(1)} G(p, Jx_n) + \beta_n^{(2)} G(p, Jx_n) + \beta_n^{(3)} G(p, Jx_n) \\
&= G(p, Jx_n).
\end{aligned} \tag{3.5}$$

By the similar reason we have, for all $p \in F$, that

$$y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) Jz_n), \tag{3.6}$$

$$\begin{aligned}
G(p, Jy_n) &= G(p, \alpha_n Jx_n + (1-\alpha_n) Jz_n) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n) Jz_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n) Jz_n\|^2 + 2\rho f(p) \\
&\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n) Jz_n\rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + (1-\alpha_n) \|Jz_n\|^2 + 2\rho f(p) \\
&= \alpha_n G(p, Jx_n) + (1-\alpha_n) G(p, Jz_n) \\
&\leq \alpha_n G(p, Jx_n) + (1-\alpha_n) G(p, Jx_n) \\
&= G(p, Jx_n).
\end{aligned} \tag{3.7}$$

That is, $p \in C_n$ for all $n \geq 0$.

Next, we show that $F \subset Q_n$ for all $n \geq 0$, we prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0)$, by Definition 2.15 we have

$$G(x_{n+1}, Jx_0) \leq G(z, Jx_0), \quad \forall z \in C_n \bigcap Q_n. \tag{3.8}$$

It is equivalent to

$$\phi(x_{n+1}, Jx_0) \leq \phi(z, Jx_0), \tag{3.9}$$

and is equivalent to

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad \forall z \in C_n \bigcap Q_n. \tag{3.10}$$

As $F \subset C_n \bigcap Q_n$, by the induction assumptions, the last inequality holds, in particular, for all $z \in F$. This together with the definition of $Q_{n+1}$ implies that $F \subset Q_{n+1}$.

*Step 3*. We show that $x_n \to x^*$ as $n \to \infty$, and $x^* \in F$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to R$ is a convex and lower semi-continuous, applying Lemma 2.19, we see that there exists $u^* \in E^*$ and $\alpha \in R$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E. \tag{3.11}$$

It follows that

$$\begin{aligned}
G(x_{n+1}, Jx_0) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho f(x_{n+1}) \\
&\geq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho\langle x_{n+1}, u^* \rangle + 2\rho\alpha \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 - \rho u^* \rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_{n+1}\|^2 - 2\|x_{n+1}\| \left\| Jx_0 - \rho u^* \right\| + \|x_0\|^2 + 2\rho\alpha \\
&= \left(\|x_{n+1}\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 - \left\|Jx_0 - \rho u^*\right\|^2 + \|x_0\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.12}$$

Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0)$, for each $p \in F$, it follows from (3.12) that

$$\begin{aligned}
G(p, Jx_0) &\geq G(x_{n+1}, Jx_0) \\
&\geq \left(\|x_{n+1}\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 - \left\|Jx_0 - \rho u^*\right\|^2 + \|x_0\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.13}$$

This implies that $\{x_n\}_{n=1}^{\infty}$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$.

Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0) \in C_n \bigcap Q_n \subset C_{n-1} \bigcap Q_{n-1}$, by Lemma 2.21 we have

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \tag{3.14}$$

It is obvious that

$$G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0), \tag{3.15}$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ exists.

By the fact that $C_{n+m} \bigcap Q_{n+m} \subset C_{n-1} \bigcap Q_{n-1}$ and $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0) \in C_{n-1} \bigcap Q_{n-1}$, by Lemma 2.21 we obtain

$$\phi(x_{n+m}, x_n) + G(x_n, Jx_0) \leq G(x_{n+m}, Jx_0). \tag{3.16}$$

Taking the limit as $m, n \to \infty$ in (3.16), we obtain

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \tag{3.17}$$

which holds uniformly for all $m$. By using Lemma 2.21, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0, \tag{3.18}$$

which holds uniformly for all $m$. Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $x^* \in C$ such that $x_n \to x^*$. In particular, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.19}$$

Since $x_{n+1} = \Pi^f_{C_n \cap Q_n}(x_0) \in C_n$, from the definition of $C_n$, we know that

$$G(x_{n+1}, Jy_n) \le G(x_{n+1}, Jx_n) \tag{3.20}$$

is equivalent to

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n). \tag{3.21}$$

So, $\phi(x_{n+1}, y_n) \to 0$.

By using Lemma 2.23, we get that

$$\|x_{n+1} - y_n\| \longrightarrow 0, \tag{3.22}$$

as $n \to \infty$.

Hence $y_n \to x^*$ as $n \to \infty$. Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} (1 - \alpha_n)\|Jz_n - Jx_n\| = \|Jy_n - Jx_n\| = 0. \tag{3.23}$$

Since $0 \le \alpha_n \le \alpha < 1$, then

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0. \tag{3.24}$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0 \tag{3.25}$$

so that $z_n \to x^*$ as $n \to \infty$.

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_nx_n\}$, and $\{JS_nx_n\}$. From the definition of $\phi(x,y)$ and

$$z_n = J^{-1}\Big(\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big), \tag{3.26}$$

we have, for all $p \in F$ that

$$\begin{aligned}\phi(p,z_n) &= \phi\Big(p, J^{-1}\Big(\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big)\Big)\\ &= \|p\|^2 - 2\Big\langle p, \beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big\rangle\\ &\quad + \Big\|\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big\|^2.\end{aligned} \tag{3.27}$$

Therefore by using Lemma 2.25 (inequality (2.21)), for all $p \in F$, we have

$$\begin{aligned}\phi(p,z_n) &\le \|p\|^2 - 2\Big\langle p, \beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big\rangle\\ &\quad + \beta_n^{(1)}\|Jx_n\|^2 + \beta_n^{(2)}\|JT_nx_n\|^2 + \beta_n^{(3)}\|JS_nx_n\|^2\\ &\quad - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\\ &= \beta_n^{(1)}\phi(p,x_n) + \beta_n^{(2)}\phi(p,T_nx_n) + \beta_n^{(3)}\phi(p,S_nx_n)\\ &\quad - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\\ &\le \beta_n^{(1)}\phi(p,x_n) + \beta_n^{(2)}\phi(p,x_n) + \beta_n^{(3)}\phi(p,x_n)\\ &\quad - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\\ &= \phi(p,x_n) - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|),\end{aligned} \tag{3.28}$$

and hence

$$\beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|) \le \phi(p,x_n) - \phi(p,z_n) \longrightarrow 0, \tag{3.29}$$

as $n \to \infty$. By using the same way, we can prove that

$$\beta_n^{(1)}\beta_n^{(3)}g(\|Jx_n - JS_nx_n\|) \le \phi(p,x_n) - \phi(p,z_n) \longrightarrow 0, \tag{3.30}$$

as $n \to \infty$. From the properties of the mapping $g$, we have

$$\|Jx_n - JT_nx_n\| \longrightarrow 0, \tag{3.31}$$

as $n \to \infty$, and

$$\|Jx_n - JS_nx_n\| \longrightarrow 0, \tag{3.32}$$

as $n \to \infty$. Since $J^{-1}$ is also uniformly norm-to-norm continuous on any bounded sets, we have

$$\|x_n - T_n x_n\| \longrightarrow 0, \tag{3.33}$$

as $n \to \infty$, and

$$\|Jx_n - JS_n x_n\| \longrightarrow 0, \tag{3.34}$$

as $n \to \infty$. Since $x_n \to x^*$ and $T_n$, $S_n$ are uniformly closed, $x^* = T_n x^*$, and $x^* = S_n x^*$.

*Step 4*. we show that $x^* = \Pi_F^f x_0$.

Since $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$ is a closed and convex set, from Lemma 2.19, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_{n-1}\bigcap Q_{n-1}}^f (x_0)$ and $w \in F \subset C_{n-1} \bigcap Q_{n-1}$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 0. \tag{3.35}$$

We know that $G(\xi, J\varphi)$ is convex and lower semicontinuous with respect to $\xi$ when $\varphi$ is fixed. This implies that

$$G(x^*, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0). \tag{3.36}$$

From the definition of $\Pi_F^f x_0$ and $x^* \in F$, we see that $x^* = w$. This completes the proof. $\square$

Based on Theorem 3.1, we have the following.

**Corollary 3.2.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $\{T_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ be two countable families of weak relatively nonexpansive mappings of $C$ into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$\begin{aligned}
&z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right),\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) Jz_n),\\
&C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\},\\
&C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\},\\
&Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\right\},\\
&Q_0 = C,\\
&x_{n+1} = \Pi_{C_n \bigcap Q_n}(x_0),
\end{aligned} \tag{3.37}$$

*with the conditions:*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(iii) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(3)} > 0$;

(iv) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

*Then* $\{x_n\}$ *converges strongly to* $x^*$, *where* $x^* = \Pi_F x_0$.

*Proof.* Putting $f(x) \equiv 0$, we can conclude from Theorem 3.1 the desired conclusion immediately. □

**Theorem 3.3.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $\{T_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}$ be two countable families of uniformly closed of relatively quasi nonexpansive mappings of $C$ into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. Let $f : E \to R$ be a convex and lower semicontinuous mapping with $C \subset int(D(f))$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$
\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right),\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
C_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : G(z, J y_n) \le \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)}\right) G(z, J x_0)\right.\\
&\qquad \left. + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right) G(z, J x_n)\right\},\\
C_0 &= \{z \in C : G(z, J y_0) \le G(z, J x_0)\},\\
Q_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\right\},\\
Q_0 &= C,\\
x_{n+1} &= \Pi^f_{C_n \bigcap Q_n}(x_0),
\end{aligned}
\tag{3.38}
$$

*with the conditions:*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\lim_{n\to\infty}\beta_n^{(1)} = 0$;

(iii) $\limsup_{n\to\infty}\beta_n^{(2)}\beta_n^{(3)} > 0$;

(iv) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

*Then* $\{x_n\}$ *converges strongly to* $x^*$, *where* $x^* = \Pi^f_F x_0$.

*Proof.* *Step 1*. We show that $C_n$ and $Q_n$ are closed and convex for each $n \ge 0$.

From the definitions of $C_n$ and $Q_n$, it is obvious that $Q_n$ is closed and convex and $C_n$ is closed for each $n \geq 0$. Moreover, since $G(z, Jy_n) \leq (\beta_n^{(1)} - \alpha_n\beta_n^{(1)})G(z, Jx_0) + (1 - \beta_n^{(1)} + \alpha_n\beta_n^{(1)})G(z, Jx_n)$ is equivalent to

$$\begin{aligned} &2\left\langle z, \left(\beta_n^{(1)} - \alpha_n\beta_n^{(1)}\right)Jx_0 + \left(1 - \beta_n^{(1)} + \alpha_n\beta_n^{(1)}\right)Jx_n - Jy_n\right\rangle \\ &\quad \leq \left(\beta_n^{(1)} - \alpha_n\beta_n^{(1)}\right)\|x_0\|^2 + \left(1 - \beta_n^{(1)} + \alpha_n\beta_n^{(1)}\right)\|x_n\|^2 - \left\|y_n\right\|^2. \end{aligned} \tag{3.39}$$

it follows that $C_n$ is convex for each $n \geq 0$. So, $C_n \bigcap Q_n$ is a closed convex subset of $E$ for all $n \in N \cup \{0\}$.

*Step 2.* We show that $F \subset C_n \bigcap Q_n$ for all $n \geq 0$. Observe that

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right). \tag{3.40}$$

Hence from the definition of $G(x, Jy)$ and the convexity of $\|\cdot\|^2$we have, for all $p \in F$, that

$$\begin{aligned} G(p, Jz_n) &= G\left(p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right) \\ &= \left\|p\right\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right\rangle \\ &\quad + \left\|\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right\|^2 + 2\rho f(p) \\ &\leq \left\|p\right\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right\rangle \\ &\quad + \beta_n^{(1)}\|Jx_0\|^2 + \beta_n^{(2)}\|JT_nx_n\|^2 + \beta_n^{(3)}\|JS_nx_n\|^2 + 2\rho f(p) \\ &= \beta_n^{(1)} G(p, Jx_0) + \beta_n^{(2)} G(p, JT_nx_n) + \beta_n^{(3)} G(p, JS_nx_n) \\ &\leq \beta_n^{(1)} G(p, Jx_0) + \beta_n^{(2)} G(p, Jx_\text{n}) + \beta_n^{(3)} G(p, Jx_n) \\ &= \beta_n^{(1)} G(p, Jx_0) + \left(1 - \beta_n^{(1)}\right)G(p, Jx_n). \end{aligned} \tag{3.41}$$

By the similar reason we have, for all $p \in F$, that

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n),\\
G(p, Jy_n) &= G(p, \alpha_n Jx_n + (1-\alpha_n)Jz_n)\\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle\\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)Jz_n\|^2 + 2\rho f(p)\\
&\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle\\
&\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|Jz_n\|^2 + 2\rho f(p)\\
&= \alpha_n G(p, Jx_n) + (1-\alpha_n)G(p, Jz_n)\\
&\leq \alpha_n G(p, Jx_n) + (1-\alpha_n)\left(\beta_n^{(1)} G(p, Jx_0) + \left(1-\beta_n^{(1)}\right)G(p, Jx_n)\right)\\
&= (1-\alpha_n)\beta_n^{(1)} G(p, Jx_0) + \left(1-\beta_n^{(1)} + \alpha_n\beta_n^{(1)}\right)G(p, Jx_n).
\end{aligned} \tag{3.42}$$

That is, $p \in C_n$ for all $n \geq 0$.

Next, we show that $F \subset Q_n$ for all $n \geq 0$, we prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0)$, by Definition 2.15 we have

$$G(x_{n+1}, Jx_0) \leq G(z, Jx_0), \quad \forall z \in C_n \bigcap Q_n. \tag{3.43}$$

It is equivalent to

$$\phi(x_{n+1}, Jx_0) \leq \phi(z, Jx_0), \tag{3.44}$$

and is equivalent to

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \geq 0, \quad \forall z \in C_n \bigcap Q_n. \tag{3.45}$$

As $F \subset C_n \bigcap Q_n$, by the induction assumptions, the last inequality holds, in particular, for all $z \in F$. This together with the definition of $Q_{n+1}$ implies that $F \subset Q_{n+1}$.

*Step 3*. We show that $x_n \to x^*$ as $n \to \infty$, and $x^* \in F$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to R$ is a convex and lower semi-continuous, applying Lemma 2.19, we see that there exists $u^* \in E^*$ and $\alpha \in R$ such that

$$f(y) \geq \langle y, u^*\rangle + \alpha, \quad \forall y \in E. \tag{3.46}$$

It follows that

$$\begin{aligned}
G(x_{n+1}, Jx_0) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_{n+1}) \\
&\geq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_{n+1}, u^*\rangle + 2\rho\alpha \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 - \rho u^*\rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_{n+1}\|^2 - 2\|x_{n+1}\|\left\|Jx_0 - \rho u^*\right\| + \|x_0\|^2 + 2\rho\alpha \\
&= \left(\|x_{n+1}\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 - \left\|Jx_0 - \rho u^*\right\|^2 + \|x_0\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.47}$$

Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0)$, for each $p \in F$, it follows from (3.47) that

$$\begin{aligned}
G(p, Jx_0) &\geq G(x_{n+1}, Jx_0) \\
&\geq \left(\|x_{n+1}\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 - \left\|Jx_0 - \rho u^*\right\|^2 + \|x_0\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.48}$$

This implies that $\{x_n\}_{n=1}^{\infty}$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$.

Since $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0) \in C_n \bigcap Q_n \subset C_{n-1} \bigcap Q_{n-1}$, by Lemma 2.21, we have

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \tag{3.49}$$

It is obvious that

$$G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0), \tag{3.50}$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ exists.

By the fact that $C_{n+m} \bigcap Q_{n+m} \subset C_{n-1} \bigcap Q_{n-1}$ and $x_{n+1} = \Pi^f_{C_n \bigcap Q_n}(x_0) \in C_{n-1} \bigcap Q_{n-1}$, by Lemma 2.21 we obtain

$$\phi(x_{n+m}, x_n) + G(x_n, Jx_0) \leq G(x_{n+m}, Jx_0). \tag{3.51}$$

Taking the limit as $m, n \to \infty$ in (3.51), we obtain

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \tag{3.52}$$

which holds uniformly for all $m$. By using Lemma 2.23, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0, \tag{3.53}$$

which holds uniformly for all $m$. Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $x^* \in C$ such that $x_n \to x^*$. In particular, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.54}$$

Since $x_{n+1} = \Pi^f_{C_n \cap Q_n}(x_0) \in C_n$, from the definition of $C_n$, we know that

$$\begin{aligned} G(x_{n+1}, Jy_n) &\leq G(z, Jy_n) \\ &\leq \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)}\right) G(x_{n+1}, Jx_0) + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right) G(x_{n+1}, Jx_n), \end{aligned} \tag{3.55}$$

and is equivalent to

$$\begin{aligned} \phi(x_{n+1}, y_n) + 2\rho f(x_{n+1}) \leq{}& \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)}\right) (\phi(x_{n+1}, x_0) + 2\rho f(x_{n+1})) \\ &+ \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right) (\phi(x_{n+1}, x_n) + 2\rho f(x_{n+1})), \end{aligned} \tag{3.56}$$

and is equivalent to

$$\phi(x_{n+1}, y_n) \leq \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)}\right) \phi(x_{n+1}, x_0) + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right) \phi(x_{n+1}, x_n), \tag{3.57}$$

So, $\phi(x_{n+1}, y_n) \to 0$.

By using Lemma 2.23, we get that

$$\left\| x_{n+1} - y_n \right\| \longrightarrow 0, \tag{3.58}$$

as $n \to \infty$,

and hence $y_n \to x^*$ as $n \to \infty$. Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \to \infty} (1 - \alpha_n) \| Jz_n - Jx_n \| = \left\| Jy_n - Jx_n \right\| = 0. \tag{3.59}$$

Since $0 \leq \alpha_n \leq \alpha < 1$, then

$$\lim_{n \to \infty} \| Jz_n - Jx_n \| = 0. \tag{3.60}$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \to \infty} \| z_n - x_n \| = 0, \tag{3.61}$$

so that $z_n \to x^*$ as $n \to \infty$.

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_n x_n\}$, and $\{JS_n x_n\}$. From the definition of $\phi(x, y)$ and

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right), \tag{3.62}$$

we have, for all $p \in F$ that

$$\begin{aligned}\phi(p, z_n) &= \phi\left(p, J^{-1}\left(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right)\right)\\ &= \|p\|^2 - 2\left\langle p, \beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n \right\rangle \\ &\quad + \left\|\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right\|^2.\end{aligned} \tag{3.63}$$

Therefore by using Lemma 2.25 (inequality (2.21)), for all $p \in F$, we have

$$\begin{aligned}\phi(p, z_n) &\leq \|p\|^2 - 2\left\langle p, \beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n \right\rangle \\ &\quad + \beta_n^{(1)} \|J x_0\|^2 + \beta_n^{(2)} \|J T_n x_n\|^2 + \beta_n^{(3)} \|J S_n x_n\|^2 \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|J T_n x_n - J S_n x_n\|) \\ &= \beta_n^{(1)} \phi(p, x_0) + \beta_n^{(2)} \phi(p, T_n x_n) + \beta_n^{(3)} \phi(p, S_n x_n) \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|J T_n x_n - J S_n x_n\|) \\ &\leq \beta_n^{(1)} \phi(p, x_0) + \beta_n^{(2)} \phi(p, x_n) + \beta_n^{(3)} \phi(p, x_n) \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|J T_n x_n - J S_n x_n\|) \\ &= \beta_n^{(1)} \phi(p, x_0) + \left(1 - \beta_n^{(1)}\right) \phi(p, x_n) - \beta_n^{(2)} \beta_n^{(3)} g(\|J T_n x_n - J S_n x_n\|),\end{aligned} \tag{3.64}$$

and hence

$$\beta_n^{(2)} \beta_n^{(3)} g(\|J x_n - J T_n x_n\|) \leq \beta_n^{(1)} \phi(p, x_0) + \left(1 - \beta_n^{(1)}\right) \phi(p, x_n) - \phi(p, z_n). \tag{3.65}$$

From $\lim_{n\to\infty} \beta_n^{(1)} = 0$ and $x_n \to x^*$, $z_n \to x^*$,we have

$$\beta_n^{(2)} \beta_n^{(3)} g(\|J T_n x_n - J S_n x_n\|) \longrightarrow 0, \tag{3.66}$$

as $n \to \infty$. From the properties of the mapping $g$, we have

$$\|J T_n x_n - J S_n x_n\| \longrightarrow 0, \tag{3.67}$$

as $n \to \infty$.

Since

$$z_n = J^{-1}\left(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right), \tag{3.68}$$

then we have

$$Jz_n = \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n. \tag{3.69}$$

Therefore

$$\begin{aligned}
\|Jx_n - Jz_n\| &= \left\| Jx_n - \left( \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right) \right\| \\
&= \Big\| \beta_n^{(1)} (Jx_n - Jx_0) + \beta_n^{(2)} (Jx_n - JT_n x_n) \\
&\qquad + \beta_n^{(3)} (Jx_n - JS_n x_n) \Big\| \\
&\geq \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\
&\quad - \left\| \beta_n^{(1)} (Jx_n - Jx_0) \right\|,
\end{aligned} \tag{3.70}$$

which leads to

$$\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \leq \|Jx_n - Jz_n\| + \left\| \beta_n^{(1)} (Jx_n - Jx_0) \right\|. \tag{3.71}$$

Since $x_n \rightarrow x^*$, $z_n \rightarrow x^*$ and $\lim_{n\to\infty} \beta_n^{(1)} = 0$. Then from above inequality we obtain

$$\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \longrightarrow 0. \tag{3.72}$$

On the other hand, by using the property of norm $\|.\|$, we have

$$\begin{aligned}
&\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\
&\quad = \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) + \beta_n^{(3)} (Jx_n - JT_n x_n) - \beta_n^{(3)} (Jx_n - JT_n x_n) \right\| \\
&\quad = \left\| \left( \beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) + \beta_n^{(3)} (JTx_n - JS_n x_n) \right\| \\
&\quad \geq \left\| \left( \beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) \right\| - \left\| \beta_n^{(3)} (JTx_n - JS_n x_n) \right\|,
\end{aligned} \tag{3.73}$$

which leads to the following inequality

$$\begin{aligned}
\left\| \left( \beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) \right\| &\leq \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\
&\quad + \left\| \beta_n^{(3)} (JTx_n - JS_n x_n) \right\|.
\end{aligned} \tag{3.74}$$

Therefore, by using (3.67) and (3.72) we have

$$\left\| \left(\beta_n^{(2)} + \beta_n^{(3)}\right)(Jx_n - JT_nx_n) \right\| \longrightarrow 0. \tag{3.75}$$

This together with condition (ii) of Theorem 3.3 implies that

$$\|Jx_n - JT_nx_n\| \longrightarrow 0. \tag{3.76}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets, then we have

$$\|x_n - T_nx_n\| \longrightarrow 0, \tag{3.77}$$

as $n \to \infty$. By using the same way, we can prove that

$$\|x_n - S_nx_n\| \longrightarrow 0 \tag{3.78}$$

as $n \to \infty$. Since $x_n \to x^*$ and $T_n$, $S_n$ are uniformly closed, $x^* = T_nx^*$, and $x^* = S_nx^*$, so that $x^* \in F$.

*Step 4.* we show that $x^* = \Pi_F^f x_0$.

Since $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$ is a closed and convex set, from Lemma 2.19, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_{n-1}\bigcap Q_{n-1}}^f (x_0)$ and $w \in F \subset C_{n-1} \bigcap Q_{n-1}$ we have

$$G(x_n, Jx_0) \le G(w, Jx_0), \quad \forall n \ge 0. \tag{3.79}$$

We know that $G(\xi, J\varphi)$ is convex and lower semicontinuous with respect to $\xi$ when $\varphi$ is fixed. This implies that

$$G(x^*, Jx_0) \le \liminf_{n\to\infty} G(x_n, Jx_0) \le \limsup_{n\to\infty} G(x_n, Jx_0) \le G(w, Jx_0). \tag{3.80}$$

From the definition of $\Pi_F^f x_0$ and $x^* \in F$, we see that $x^* = w$. This completes the proof. □

Based on Theorem 3.3, we have the following.

**Corollary 3.4.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $\{T_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ be two countable families of weak relatively* *nonexpansive mappings of C into itself such that* $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. *For any given gauss* $x_0 \in C$, *define a sequence* $\{x_n\}$ *in C by the following algorithm:*

$$
\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right), \\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\
C_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)}\right)\phi(z, x_0) \right. \\
&\qquad \left. + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right)\phi(z, x_n)\right\}, \\
C_0 &= \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
Q_n &= \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\right\}, \\
Q_0 &= C, \\
x_{n+1} &= \Pi_{C_n \bigcap Q_n}(x_0),
\end{aligned}
\tag{3.81}
$$

*with the conditions:*

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;

(iii) $\limsup_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0$;

(iv) $0 \leq \alpha_n \leq \alpha < 1 \quad for \quad some \quad \alpha \in (0,1)$.

*Then* $\{x_n\}$ *converges strongly to* $x^*$, *where* $x^* = \Pi_F x_0$.

*Proof.* Putting $f(x) \equiv 0$, we can conclude from Theorem 3.3 the desired conclusion immediately. □

## 4. Applications

Let $E$ be a real Banach space and let $E^*$ be the dual space of $E$. Let $C$ be a closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$. The equilibrium problem is to find $x \in C$ such that

$$
f(x, y) \geq 0, \quad \forall y \in C. \tag{4.1}
$$

The set of solutions of (4.1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$ let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $p \in EP(f)$ if and only if $\langle Tp, y - p \rangle \geq 0$ for all $y \in C$, that is, $p$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.1). Some methods have been proposed to solve the equilibrium problem in Hilbert spaces.

For solving the equilibrium problem, let us assume that a bifunction $f$ satisfies the following conditions:

(A1) $f(x,x) = 0$, for all $x \in E$;

(A2) $f$ is monotone, that is, $f(x,y) + f(y,x) \leq 0$, for all $x, y \in E$;

(A3) for all $x, y, z \in E$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semi-continuous.

**Lemma 4.1** (see [34]). *Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$ satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{4.2}$$

**Lemma 4.2** (see [34]). *Let $C$ be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space $E$, and let $f$ be a bifunction from $C\times C$ to $R = (-\infty, +\infty)$ satisfying* (A1)–(A4). *For $r > 0$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C \right\}, \tag{4.3}$$

*for all $x \in E$. Then, the following hold*

(1) *$T_r$ is single valued;*

(2) *$T_r$ is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{4.4}$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex;*

(5) *$T_r$ is also a relatively nonexpansive mapping.*

**Lemma 4.3** (see [34]). *Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$ satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$, $q \in F(T_r)$, then the following holds*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{4.5}$$

**Lemma 4.4.** *Let $E$ be a $p$-uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$ satisfying* (A1)–(A4). *Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = r > 0$. Then the sequence of mappings $\{T_{r_n}\}$ is uniformly closed.*

*Proof.* (1) Let $\{x_n\}$ be a convergent sequence in $C$. Let $z_n = T_{r_n}x_n$ for all $n$, then

$$f(z_n, y) + \frac{1}{r_n}\langle y - z_n, Jz_{\mathrm{n}} - Jx_n\rangle \geq 0, \quad \forall y \in C, \tag{4.6}$$

$$f(z_{n+m}, y) + \frac{1}{r_{n+m}}\langle y - z_{n+m}, Jz_{n+m} - Jx_{n+m}\rangle \geq 0, \quad \forall y \in C. \tag{4.7}$$

Putting $y = z_{n+m}$ in (4.6) and $y = z_n$ in (4.7), we have

$$\begin{aligned} f(z_n, z_{n+m}) + \frac{1}{r_n}\langle z_{n+m} - z_n, Jz_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ f(z_{n+m}, z_n) + \frac{1}{r_{n+m}}\langle z_n - z_{n+m}, Jz_{n+m} - Jx_{n+m}\rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{4.8}$$

So, from (A2) we have

$$\left\langle z_{n+m} - z_n, \frac{Jz_n - Jx_n}{r_n} - \frac{Jz_{n+m} - Jx_{n+m}}{r_{n+m}}\right\rangle \geq 0, \tag{4.9}$$

and hence

$$\left\langle z_{n+m} - z_n, Jz_n - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0. \tag{4.10}$$

Thus, we have

$$\left\langle z_{n+m} - z_n, Jz_n - Jz_{n+m} + Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0, \tag{4.11}$$

which implies that

$$\langle z_{n+m} - z_n, Jz_{n+m} - Jz_n\rangle \leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0. \tag{4.12}$$

By using Lemma 2.25, we obtain

$$\begin{aligned} \frac{c^p}{c^{p-2}p}\|z_{n+m} - z_n\|^p &\leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \\ &= \left\langle z_{n+m} - z_n, \left(1 - \frac{r_n}{r_{n+m}}\right)Jz_{n+m} + \frac{r_n}{r_{n+m}}Jx_{n+m} - Jx_n\right\rangle. \end{aligned} \tag{4.13}$$

Therefore, we get

$$\frac{c^p}{c^{p-2}p}\|z_{n+m} - z_n\|^{p-1} \leq \left|1 - \frac{r_n}{r_{n+m}}\right|\|Jz_{n+m}\| + \left\|\frac{r_n}{r_{n+m}}Jx_{n+m} - Jx_n\right\|. \tag{4.14}$$

On the other hand, for any $p \in EP(f)$, from $z_n = T_{r_n} x_n$, we have

$$\|z_n - p\| = \|T_{r_n} x_n - p\| \leq \|x_n - p\|, \tag{4.15}$$

so that $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} r_n = r > 0$, this together with (4.14) implies that $\{z_n\}$ is a Cauchy sequence. Hence $T_{r_n} x_n = z_n$ is convergent.

(2) By using the Lemma 4.2, we know that

$$\bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f) \neq \emptyset. \tag{4.16}$$

(3) From (1) we know that, $\lim_{n\to\infty} T_{r_n} x$ exists for all $x \in C$. So, we can define a mapping $T$ from $C$ into itself by

$$Tx = \lim_{n\to\infty} T_{r_n} x, \quad \forall x \in C. \tag{4.17}$$

It is obvious that, $T$ is nonexpansive. It is easy to see that

$$EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T). \tag{4.18}$$

On the other hand, let $w \in F(T)$, $w_n = T_{r_n} w$, we have

$$f(w_n, y) + \frac{1}{r_n}\langle y - w_n, Jw_n - Jw\rangle \geq 0, \quad \forall y \in C. \tag{4.19}$$

By (A2) we know

$$\frac{1}{r_n}\langle y - w_n, Jw_n - Jw\rangle \geq f(y, w_n), \quad \forall y \in C. \tag{4.20}$$

Since $w_n \to Tw = w$ and from (A4), we have $f(y, w) \leq 0$, for all $y \in C$. Then, for $t \in (0,1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y). \end{aligned} \tag{4.21}$$

Therefore, we have

$$f(ty + (1-t)w, y) \geq 0. \tag{4.22}$$

Letting $t \downarrow 0$ and using (A3), we get

$$f(w, y) \geq 0, \quad \forall y \in C \tag{4.23}$$

and hence $w \in EP(f)$. From the above two respects, we know that $F(T) = \bigcap_{n=0}^{\infty} F(T_{r_n})$.

Next we show $\{T_{r_n}\}$ is uniformly closed. Assume $x_n \to x$ and $\|x_n - T_{r_n} x_n\| \to 0$, from the above results we know that $Tx = \lim_{n\to\infty} T_{r_n} x$. On the other hand, from $\|x_n - T_{r_n} x_n\| \to 0$, we also get $\lim_{n\to\infty} T_{r_n} x = x$, so that $x \in F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. That is, the sequence of mappings $\{T_{r_n}\}$ is uniformly closed. This completes the proof. □

Let $A$ be a multi-valued operator from $E$ to $E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator $A$ is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \tag{4.24}$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if it is graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well known.

**Theorem 4.5.** *Let $E$ be a reflexive, strictly convex, and smooth Banach space and let $A$ be a monotone operator from $E$ to $E^*$. Then $A$ is maximal if and only if $R(J + rA) = E^*$, for all $r > 0$.*

Let $E$ be a reflexive, strictly convex, and smooth Banach space, and let $A$ be a maximal monotone operator from $E$ to $E^*$. Using Theorem 4.5 and strict convexity of $E$, one obtains that for every $r > 0$ and $x \in E$, there exists a unique $x_r$ such that

$$Jx \in Jx_r + rAx_r. \tag{4.25}$$

Then one can defines a single-valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1} J$ and such a $J_r$ is called the resolvent of $A$, one knows that $A^{-1}0 = F(J_r)$ for all $r > 0$.

**Theorem 4.6.** *Let $E$ be a uniformly convex and a uniformly smooth Banach space, and let $A$ be a maximal monotone operator from $E$ to $E^*$, let $J_r$ be a resolvent of $A$ for $r > 0$. Then for any sequence $\{r_n\}_{n=1}^{\infty}$ such that $\liminf_{n\to\infty} r_n > 0$, $\{J_{r_n}\}_{n=1}^{\infty}$ is a uniformly closed sequence of relatively quasi nonexpansive mappings.*

*Proof.* Firstly, we show that $\{J_{r_n}\}_{n=1}^{\infty}$ is uniformly closed. Let $\{z_n\} \subset E$ be a sequence such that $z_n \to p$ and $\lim_{n\to\infty} \|z_n - J_{r_n} z_n\| = 0$. Since $J$ is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}(Jz_n - JJ_{r_n} z_n) \longrightarrow 0. \tag{4.26}$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n} z_n) \in AJ_{r_n} z_n \tag{4.27}$$

and the monotonicity of $A$ that

$$\left\langle w - J_{r_n} z_n, w^* - \frac{1}{r_n}(Jz_n - JJ_{r_n} z_n) \right\rangle \geq 0 \tag{4.28}$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have $\langle w - p, w^* \rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore from the maximality of $A$, we obtain $p \in A^{-1}0 = F(J_{r_n})$ for all $n \geq 1$, that is, $p \in \bigcap_{n=1}^{\infty} F(J_{r_n})$.

Next we show $J_{r_n}$ is a relatively quasi nonexpansive mapping for all $n \geq 1$. For any $w \in E$ and $p \in F(J_{r_n}) = A^{-1}0$, from the monotonicity of $A$, we have

$$\begin{aligned}
\phi(p, J_{r_n} w) &= \|p\|^2 - 2\langle p, JJ_{r_n} w\rangle + \|J_{r_n} w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w - Jw\rangle + \|J_{r_n} w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p - J_{r_n} w, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\
&= \|p\|^2 - 2\langle J_{r_n} w - p, Jw - JJ_{r_n} w\rangle \\
&\quad + 2\langle J_{r_n} w, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\
&\leq \|p\|^2 + 2\langle J_{r_n} w, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_{r_n} w\|^2 + 2\langle J_{r_n} w, Jw\rangle - \|w\|^2 \\
&= \phi(p, w) - \phi(J_{r_n} w, w) \\
&\leq \phi(p, w).
\end{aligned} \tag{4.29}$$

This implies that $J_{r_n}$ is a relatively quasi nonexpansive mapping for all $n \geq 1$. This completes the proof. □

## References


[1] D. Butnariu, S. Reich, and A. J. Zaslavski, "Weak convergence of orbits of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.

[2] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[3] S. Reich, "Book Review: geometry of Banach spaces, duality mappings and nonlinear problems," *American Mathematical Society. Bulletin. New Series*, vol. 26, no. 2, pp. 367–370, 1992.

[4] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama, Yokohama, Japan, 2000.

[5] S. Reich, "A weak convergence theorem for the alternating method with Bregman distances," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 313–318, Dekker, New York, NY, USA, 1996.

[6] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banach spaces," *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.

[7] Y. Censor and S. Reich, "Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization," *Optimization. A Journal of Mathematical Programming and Operations Research*, vol. 37, no. 4, pp. 323–339, 1996.

[8] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[9] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[10] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[11] P.-L. Lions, "Approximation de points fixes de contractions," vol. 284, no. 21, pp. A1357–A1359, 1977.

[12] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[13] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.

[14] W. Takahashi, *Totsu Kaiseki to Fudōten Kinji*, vol. 2 of *Sūrikaiseki Shiriizu*, Yokohama, Yokohama, Japan, 2000.

[15] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.

[16] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[17] C. E. Chidume and S. A. Mutangadura, "An example of the Mann iteration method for Lipschitz pseudocontractions," *Proceedings of the American Mathematical Society*, vol. 129, no. 8, pp. 2359–2363, 2001.

[18] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[19] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[20] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 149, no. 2, pp. 103–115, 2007.

[21] Y. Su, H.-k. Xu, and X. Zhang, "Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications," *Nonlinear Analysis*, vol. 73, no. 12, pp. 3890–3906, 2010.

[22] X. Li, N.-j. Huang, and D. O'Regan, "Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1322–1331, 2010.

[23] W. Takahashi and K. Zembayashi, "Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 528476, 11 pages, 2008.

[24] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Dekker, New York, NY, USA, 1996.

[25] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[26] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1990.

[27] YA. I. Alber and S. Reich, "An iterative method for solving a class of nonlinear operator equations in Banach spaces," *Panamerican Mathematical Journal*, vol. 4, no. 2, pp. 39–54, 1994.

[28] Y. J. Cho, H. Zhou, and G. Guo, "Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 47, no. 4-5, pp. 707–717, 2004.

[29] K.-q. Wu and N.-j. Huang, "The generalised $f$-projection operator with an application," *Bulletin of the Australian Mathematical Society*, vol. 73, no. 2, pp. 307–317, 2006.

[30] J. Fan, X. Liu, and J. Li, "Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces," *Nonlinear Analysis*, vol. 70, no. 11, pp. 3997–4007, 2009.

[31] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[32] X. Qin, Y. J. Cho, and S. M. Kang, “Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces,” *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[33] W. Nilsrakoo and S. Saejung, “Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 312454, 19 pages, 2008.

[34] S. Ai and Y. Su, “Hybrid algorithm for common fixed points of uniformly closed countable families of hemirelatively nonexpansive mappings and applications,” *Journal of Applied Mathematics*, vol. 2012, Article ID 401960, 20 pages, 2012.