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Research Article

Blow-Up Criteria of Smooth Solutions for the Cahn-Hilliard-Boussinesq System with Zero Viscosity in a Bounded Domain

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We prove that a smooth solution of the 3D Cahn-Hilliard-Boussinesq system with zero viscosity in a bounded domain breaks down if a certain norm of vorticity blows up at the same time. Here, this norm is weaker than bmo-norm.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$, and n is the unit outward normal vector to $\partial\Omega$. We consider the following Cahn-Hilliard-Boussinesq system with zero viscosity in $\Omega \times (0, \infty)$ [1]:

$$\partial_t u + (u \cdot \nabla) u + \nabla \pi = \mu \nabla \phi + \theta e_3, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t \theta + u \cdot \nabla \theta = \Delta \theta, \tag{1.3}$$

$$\partial_t \phi + u \cdot \nabla \phi = \Delta \mu, \tag{1.4}$$

$$-\Delta \phi + f'(\phi) = \mu, \tag{1.5}$$

$$u \cdot n = 0, \qquad \frac{\partial \theta}{\partial n} = \frac{\partial \phi}{\partial n} = \frac{\partial \mu}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.6}$$

$$(u, \theta, \phi)(x, 0) = (u_0, \theta_0, \phi_0)(x), \quad x \in \Omega, \tag{1.7}$$

where u, the fluid velocity field, θ, the temperature, ϕ, the order parameter, π, and the pressure are the unknowns. $e_3 := (0,0,1)^t$. μ is a chemical potential. $f(\phi) := (1/4)(\phi^2 - 1)^2$ is the double well potential.

When $\theta = \phi \equiv 0$, (1.1) and (1.2) are Euler equations. Ogawa-Taniuchi [3] proved that a smooth solution breaks down if a certain norm of vorticity blows up at the same time. Here this norm is weaker than bmo-norm.

Before presenting our results, we introduce some function spaces and some notations.

Let $\eta, \phi_j, j = 0, \pm 1, \pm 2, \pm 3, \ldots$ be the Littlewood-Paley dyadic decomposition of unity that satisfies

$$\eta \in C_0^\infty(B(0,1)), \qquad \phi \in C_0^\infty\left(B(0,2) \setminus B\left(0, \frac{1}{2}\right)\right), \qquad \phi_j(\xi) = \phi\left(2^{-j}\xi\right),$$
$$\eta(\xi) + \sum_{j=0}^{\infty} \phi_j(\xi) = 1, \tag{1.8}$$

for any $\xi \in \mathbb{R}^3$, where $B(x,r)$ denotes the ball centered at x of radius r.

We first recall the space of Besov type introduced by Vishik [2].

Definition 1.1 (see [2]). Let $\Theta(\alpha)(\geq 1)$ be a nondecreasing function on $[1,\infty)$. $V_\Theta := \{f \in \mathcal{S}'; \|f\|_{V_\Theta} < \infty\}$ is introduced by the norm

$$\|f\|_{V_\Theta} := \sup_{N=1,2,\ldots} \frac{\left\|\left(\eta\widehat{f}\right)^\vee\right\|_{L^\infty} + \sum_{j=0}^N \left\|\left(\phi_j\widehat{f}\right)^\vee\right\|_{L^\infty}}{\Theta(N)}, \tag{1.9}$$

where $\widehat{f}$ and $\check{f}$ denote the Fourier and inverse Fourier transforms.

We note that

$$\|f\|_{V_\Theta} \leq C\|f\|_{\dot{B}^0_{\infty,\infty}} \leq C\|f\|_{\mathrm{bmo}} \leq C\|f\|_{L^\infty}, \quad \text{if } \Theta(N) \geq N. \tag{1.10}$$

Now let us introduce the space of bmo type in [3].

Definition 1.2. Let $\beta(r)$ be a positive function on $(0,1]$, and $\Omega \subset \mathbb{R}^3$ is a domain with $\partial\Omega \in C^\infty$.

(1) $\mathrm{bmo}_\beta(\mathbb{R}^3)$ is the space defined as a set for an $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ function f such that

$$\|f\|_{\mathrm{bmo}_\beta(\mathbb{R}^3)} := \sup_{0<r<1, x\in\mathbb{R}^3} \frac{1}{|B(x,r)|\beta(r)} \int_{B(x,r)} \left|f(y) - \overline{f}_{B(x,r)}\right| dy$$
$$+ \sup_{x\in\mathbb{R}^3} \frac{1}{|B(x,1)|} \int_{B(x,1)} |f(y)| dy < \infty, \tag{1.11}$$

where $\overline{f}_B := (1/|B|) \int_B f(y) dy$.

(2) On $\Omega \subset \mathbb{R}^3$ we define bmo_β as restrictions of the above space $\mathrm{bmo}_\beta(\mathbb{R}^3)$:

$$\mathrm{bmo}_\beta(\Omega) := \left\{ f|_\Omega ; f \in \mathrm{bmo}_\beta\left(\mathbb{R}^3\right) \right\}, \tag{1.12}$$

where $f|_\Omega$ is the restriction of f on Ω. The norm of this space is defined by

$$\|f\|_{\mathrm{bmo}_\beta(\Omega)} := \inf\left\{ \left\|\tilde{f}\right\|_{\mathrm{bmo}_\beta(\mathbb{R}^3)} ; \tilde{f} \in \mathrm{bmo}_\beta\left(\mathbb{R}^3\right) \text{ with } \tilde{f} = f \text{ in } \Omega \right\}. \tag{1.13}$$

In particular if $\beta(r) = 1$, we write $\mathrm{bmo}_\beta(\mathbb{R}^3) = \mathrm{bmo}(\mathbb{R}^3)$ and $\mathrm{bmo}_\beta(\Omega) = \mathrm{bmo}(\Omega)$. Obviously, $\mathrm{bmo} \subset \mathrm{bmo}_\beta$ if $\beta \geq 1$.

Definition 1.3. Let $\Theta(\alpha)(\geq 1)$ be a nondecreasing function on $[1, \infty)$

$$Y_\Theta(\Omega) := \left\{ f \in L^1(\Omega); \|f\|_{Y_\Theta(\Omega)} < \infty \right\}, \tag{1.14}$$

where

$$\begin{gathered} \|f\|_{Y_\Theta(\Omega)} := \sup_{p \geq 1} \frac{\|f\|_{L^p}}{\Theta(p)}, \\ M_\Theta(\Omega) := \left\{ f \in L^1(\Omega); \|f\|_{M_\Theta(\Omega)} < \infty \right\}, \end{gathered} \tag{1.15}$$

where

$$\|f\|_{M_\Theta(\Omega)} := \sup_{p \geq 1} \frac{1}{\Theta(p)} \sup_{0<r<1, x \in \mathbb{R}^3} \left(r^{-3+3/p} \int_{B(x,r)\cap\Omega} |f(y)| dy \right). \tag{1.16}$$

We note that these spaces have the following relations:

$$\|f\|_{M_\Theta(\Omega)} \leq C \|f\|_{Y_\Theta(\Omega)} \leq C \|f\|_{\mathrm{bmo}(\Omega)}. \tag{1.17}$$

From now on we impose the following assumptions.

Assumption 1.4. Let $\beta(r) := \Theta(\log(e + 1/r)) / \log(e + 1/r)$.

(H1) $\Theta(\alpha)$ is a positive and nondecreasing function on $[0, \infty)$ satisfying

$$\int^{+\infty} \frac{d\alpha}{\Theta(\alpha)} = \infty, \quad \Theta(\alpha) \geq \alpha. \tag{1.18}$$

(H2) For all $s \geq 1$ there exists $C(s)$ such that

$$\Theta(s\alpha) \leq C(s)\Theta(\alpha), \quad \forall \alpha \geq 1. \tag{1.19}$$

(H3) $\beta(r)$ is a nonincreasing function on $(0,1]$.

Then Ogawa-Taniuchi [3] proved the following blowup criterion:

$$\int_0^T \|\omega(t)\|_{\mathrm{bmo}_\beta(\Omega)} + \|\omega(t)\|_{M_\Theta(\Omega_\epsilon)} dt = \infty, \tag{1.20}$$

where $\omega := \operatorname{curl} u$ and for all $\epsilon > 0$ and $\Omega_\epsilon := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < \epsilon\}$ or

$$\int_0^T \|\omega(t)\|_{\mathrm{bmo}_\beta(\Omega_{3\epsilon})} + \|\omega(t)\|_{M_\Theta(\Omega_{3\epsilon})} + \left\|\rho\omega(t)\right\|_{V_\Theta} dt = \infty, \tag{1.21}$$

for all $0 < \epsilon < \epsilon_0$ and all $\rho \in C^\infty(\mathbb{R}^3)$ with $\rho \equiv 1$ in $\Omega \setminus \Omega_\epsilon$ and $\rho \equiv 0$ in $\mathbb{R}^3 \setminus \Omega$. ϵ_0 is a small positive constant depending only on Ω.

Since $\beta(r) \geq 1$, we see

$$\|f\|_{\mathrm{bmo}_\beta(\Omega)} \leq \|f\|_{\mathrm{bmo}(\Omega)}. \tag{1.22}$$

By this inequality and (1.17), (1.20) implies

$$\int_0^T \|\omega(t)\|_{\mathrm{bmo}(\Omega)} dt = \infty. \tag{1.23}$$

The aim of this paper is to prove a similar result for the problem (1.1)–(1.7). It is easy to show that the problem (1.1)–(1.7) has a unique local smooth solution. Thus, we omit the details here. However, the global regularity is still open, which this paper aims to study. We will prove that.

Theorem 1.5. *Let* $u_0 \in H^3$, $\theta_0 \in H^2$, $\phi_0 \in H^4$, $\operatorname{div} u_0 = 0$ *in* Ω, $u_0 \cdot n = \partial\theta_0/\partial n = \partial\phi_0/\partial n = \partial\mu_0/\partial n = 0$ *on* $\partial\Omega$. *Suppose that* (u, θ, ϕ) *is a local smooth solution to the problem* (1.1)–(1.7) *on* $[0, T)$. *If T is maximal, then* (1.20) *and* (1.21) *hold true.*

In Section 2, we will give some preliminaries. Section 3 is devoted to the proof of Theorem 1.5.

2. Preliminaries

Lemma 2.1 (see [4]). *For any* $u \in W^{s,p}$ *with* $\operatorname{div}\, u = 0$ *in* Ω *and* $u \cdot n = 0$ *on* $\partial\Omega$, *there holds*

$$\|u\|_{W^{s,p}} \leq C(\|u\|_{L^p} + \|\operatorname{curl}\, u\|_{W^{s-1,p}}), \tag{2.1}$$

for any $s \geq 1$ *and* $p \in (1, \infty)$.

Lemma 2.2 (see [5]). *Let $s \geq 1$.*

(1) *If $f, g \in H^s(\Omega) \cap C(\Omega)$, then*

$$\|fg\|_{H^s(\Omega)} \leq C\Big(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)}\Big). \tag{2.2}$$

(2) *If $f \in H^s(\Omega) \cap C^1(\Omega)$ and $g \in H^{s-1}(\Omega) \cap C(\Omega)$, then for $|\alpha| \leq s$,*

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)} \leq C\Big(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)}\|g\|_{H^{s-1}(\Omega)}\Big). \tag{2.3}$$

Lemma 2.3 (see [3]). *For all $\epsilon > 0$, there holds*

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega)} \leq C\Big(1 + \|u\|_{L^2(\Omega)} + \|\mathrm{curl}\ u\|_{bmo_\beta(\Omega)} + \|\mathrm{curl}\ u\|_{M_\Theta(\Omega_\epsilon)}\Big) \\ \times \Theta\Big(\log\Big(e + \|u\|_{H^3(\Omega)}\Big)\Big), \end{aligned} \tag{2.4}$$

for all $u \in H^3(\Omega)$ with $\mathrm{div}\ u = 0$ *in Ω and $u \cdot n = 0$ on $\partial\Omega$.*

Lemma 2.4 (see [3]). *There exists a constant ϵ_0 depending only on Ω such that the following holds.*

For all $0 < \epsilon < \epsilon_0$, and for all $\rho \in C^\infty(\mathbb{R}^3)$ with $\rho \equiv 1$ in $\Omega \setminus \Omega_\epsilon$ and $\rho \equiv 0$ in $\mathbb{R}^3 \setminus \Omega$, there exists constant C depending only on ϵ, ρ, Ω and Θ such that

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega)} \leq C\Big(1 + \|u\|_{L^2(\Omega)} + \|\mathrm{curl}\ u\|_{bmo_\beta(\Omega_{3\epsilon})} + \|\mathrm{curl}\ u\|_{M_\Theta(\Omega_{3\epsilon})} \\ + \|\rho\, \mathrm{curl}\ u\|_{V_\Theta}\Big)\Theta\Big(\log\Big(e + \|u\|_{H^3(\Omega)}\Big)\Big), \end{aligned} \tag{2.5}$$

for all $u \in H^3(\Omega)$ with $\mathrm{div}\ u = 0$ *in Ω and $u \cdot n = 0$ on $\partial\Omega$.*

Lemma 2.5 (see [6]). *Let φ be nonnegative function on $(0, T)$ with $\int_0^T \varphi(t)dt < \infty$, let $\Theta(\alpha)$ be a positive and nondecreasing for $\alpha \geq 1$ and $\int^{+\infty}(d\alpha/\Theta(\alpha)) = \infty$. Assume that $v \in C([0, T))$ and*

$$0 \leq v(t) \leq v(0) + \int_0^t \varphi(s)\Theta(v(s))ds \quad \forall 0 \leq t < T. \tag{2.6}$$

Then, $\sup_{0\leq t\leq T} v(t) < \infty$.

3. Proof of Theorem 1.5

In this section, all the integrations with respect to spacial variable are on the domain ω (we omit it for simplicity).

Since the proof of (1.21) is similar to that of (1.20), we only need to prove (1.20). By the standard argument of continuation of local solutions, it suffices to prove that if

$$\int_0^T \|\omega(t)\|_{\mathrm{bmo}_\beta(\Omega)} + \|\omega(t)\|_{M_\Theta(\Omega_\epsilon)} dt < \infty \quad \text{for some } \epsilon > 0, \tag{3.1}$$

then

$$u \in L^\infty\left(0,T;H^3\right), \qquad \theta \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^3\right), \qquad \phi \in L^\infty\left(0,T;H^4\right). \tag{3.2}$$

First, by the maximum principle, it follows from (1.2) and (1.3) that

$$\|\theta\|_{L^\infty(0,T;L^\infty)} \le C. \tag{3.3}$$

Testing (1.3) by θ, using (1.2), we see that

$$\frac{1}{2}\frac{d}{dt}\int \theta^2 dx + \int |\nabla\theta|^2 dx = 0, \tag{3.4}$$

whence

$$\|\theta\|_{L^2(0,T;H^1)} \le C. \tag{3.5}$$

Testing (1.4) by ϕ, using (1.2) and (1.6), we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \phi^2 dx + \int |\Delta\phi|^2 dx &= \int f'(\phi)\Delta\phi\, dx = \int \left(\phi^3 - \phi\right)\Delta\phi\, dx \\
&= -3\int \phi^2 |\nabla\phi|^2 dx - \int \phi\Delta\phi\, dx \\
&\le -\int \phi\Delta\phi\, dx \le \|\phi\|_{L^2}\|\Delta\phi\|_{L^2} \\
&\le \frac{1}{2}\|\Delta\phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^2}^2,
\end{aligned} \tag{3.6}$$

which gives

$$\|\phi\|_{L^\infty(0,T;L^2)} + \|\phi\|_{L^2(0,T;H^2)} \le C. \tag{3.7}$$

Testing (1.1) and (1.4) by u and μ, respectively, using (1.2), (1.5), (1.6), and (3.3), we infer that

$$\frac{d}{dt}\int \frac{1}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}u^2 dx + \int |\nabla\mu|^2 dx = \int \theta e_3 u\, dx \le \|\theta\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2}, \tag{3.8}$$

which yields

$$\|\phi\|_{L^\infty(0,T;H^1)} \le C, \tag{3.9}$$

$$\|u\|_{L^\infty(0,T;L^2)} \le C, \tag{3.10}$$

$$\|\nabla\mu\|_{L^2(0,T;L^2)} \le C. \tag{3.11}$$

In the following calculations, we will use the following Gagliardo-Nirenberg inequality:

$$\|\phi\|_{L^\infty}^2 \le C\|\phi\|_{H^1}\|\phi\|_{H^2}. \tag{3.12}$$

It follows from (1.5), (3.11), (3.7), (3.9) and (3.12) that

$$\begin{aligned}
&\int_0^T\int |\nabla\Delta\phi|^2 dxdt \\
&= \int_0^T\int |\nabla(f'(\phi)-\mu)|^2 dxdt \\
&\le C\int_0^T\int |\nabla\mu|^2 dxdt + C\int_0^T\int |\nabla f'(\phi)|^2 dxdt \\
&\le C + C\int_0^T\int \left|\nabla\left(\phi^3-\phi\right)\right|^2 dxdt \\
&\le C + C\int_0^T\int \phi^2|\nabla\phi|^2 dxdt \\
&\le C + C\|\nabla\phi\|_{L^\infty(0,T;L^2)}^2\int_0^T \|\phi\|_{L^\infty}^4 dt \\
&\le C + C\int_0^T \|\phi\|_{L^\infty}^4 dt \\
&\le C + C\int_0^T \|\phi\|_{H^1}^2\|\phi\|_{H^2}^2 dt \\
&\le C + C\|\phi\|_{L^\infty(0,T;H^1)}^2\int_0^T \|\phi\|_{H^2}^2 dt \\
&\le C,
\end{aligned} \tag{3.13}$$

which implies

$$\begin{aligned} &\|\nabla\phi\|_{L^2(0,T;L^\infty)} \le C, \\ &\|\phi\|_{L^4(0,T;L^\infty)} \le C. \end{aligned} \tag{3.14}$$

Testing (1.4) by $\Delta^2\phi$, using (3.9), (3.10), and (3.14), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |\Delta\phi|^2 dx + \int \left|\Delta^2\phi\right|^2 dx \\ &\quad = \int \Delta f'(\phi)\cdot\Delta^2\phi\, dx - \int (u\cdot\nabla)\phi\cdot\Delta^2\phi\, dx \\ &\quad \le \left(\|\Delta f'(\phi)\|_{L^2} + \|u\|_{L^2}\|\nabla\phi\|_{L^\infty}\right)\left\|\Delta^2\phi\right\|_{L^2} \\ &\quad \le C\Big(\|\phi\|_{L^\infty}^2\|\Delta\phi\|_{L^2} + \|\phi\|_{L^\infty}\|\nabla\phi\|_{L^\infty}\|\nabla\phi\|_{L^2} + \|\nabla\phi\|_{L^\infty}\Big)\left\|\Delta^2\phi\right\|_{L^2} \\ &\quad \le \frac{1}{2}\left\|\Delta^2\phi\right\|_{L^2}^2 + C\|\phi\|_{L^\infty}^4\|\Delta\phi\|_{L^2}^2 + C\|\phi\|_{H^2}^2\|\nabla\phi\|_{L^\infty}^2 + C\|\nabla\phi\|_{L^\infty}^2, \end{aligned} \tag{3.15}$$

which leads to

$$\|\phi\|_{L^\infty(0,T;H^2)} + \|\phi\|_{L^2(0,T;H^4)} \le C. \tag{3.16}$$

Testing (1.3) by $-\Delta\theta$, using (1.2) and (1.6), we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |\nabla\theta|^2 dx + \int (\Delta\theta)^2 dx \\ &\quad = \int (u\cdot\nabla)\theta\cdot\Delta\theta\, dx \\ &\quad = \sum_i \int \partial_i((u\cdot\nabla\theta)\partial_i\theta)dx - \sum_i\int \partial_i(u\cdot\nabla\theta)\cdot\partial_i\theta\, dx \\ &\quad = \int \operatorname{div}((u\cdot\nabla\theta)\nabla\theta)dx - \sum_i\int \partial_i u\cdot\nabla\theta\cdot\partial_i\theta\, dx - \frac{1}{2}\sum_i\int u\cdot\nabla(\partial_i\theta)^2 dx \\ &\quad = -\sum_i\int \partial_i u\cdot\nabla\theta\cdot\partial_i\theta\, dx \\ &\quad \le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Equations (1.3) and (1.6) can be rewritten as

$$\begin{aligned} &\Delta\theta = g := \partial_t\theta + u\cdot\nabla\theta \quad \text{as } \Omega\times(0,\infty), \\ &\frac{\partial\theta}{\partial n} = 0, \quad \text{on } \partial\Omega\times(0,\infty). \end{aligned} \tag{3.18}$$

By the classical regularity theory of elliptic equation, using (3.10), we get

$$\begin{aligned}\|\theta\|_{H^3} &\le C\|g\|_{H^1}\\ &\le C\|\partial_t\theta\|_{H^1} + C\|u\cdot\nabla\theta\|_{H^1}\\ &\le C\|\partial_t\theta\|_{H^1} + C\|u\|_{L^2}\|\nabla\theta\|_{L^\infty} + C\|u\|_{L^6}\|\Delta\theta\|_{L^3} + C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}\\ &\le C\|\partial_t\theta\|_{H^1} + C\|\nabla\theta\|_{L^\infty} + C\|u\|_{L^6}\|\Delta\theta\|_{L^3} + C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}.\end{aligned} \tag{3.19}$$

Now using the following Gagliardo-Nirenberg inequalities:

$$\begin{gathered}\|\nabla\theta\|_{L^\infty} \le C\|\theta\|_{L^\infty}^{1/3}\|\theta\|_{H^3}^{2/3},\\ \|\Delta\theta\|_{L^3} \le C\|\theta\|_{L^\infty}^{1/3}\|\theta\|_{H^3}^{2/3},\\ \|u\|_{L^6}^3 \le C\|u\|_{L^2}^2\|u\|_{H^3},\end{gathered} \tag{3.20}$$

we obtain

$$\|\theta\|_{H^3} \le C\|\partial_t\theta\|_{H^1} + C + C\|u\|_{H^3} + C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}. \tag{3.21}$$

Taking curl to (1.1), using (1.2) and (1.5), we have

$$\partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u - \operatorname{curl}\left(\Delta\phi\nabla\phi\right) + \operatorname{curl}\left(\theta e_3\right). \tag{3.22}$$

Taking Δ to (3.22), testing by $\Delta\omega$, using (1.2) and (1.6), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\Delta\omega|^2dx &= -\int(\Delta(u\cdot\nabla\omega) - u\nabla\Delta\omega)\Delta\omega\,dx + \int\Delta(\omega\cdot\nabla u)\cdot\Delta\omega\,dx\\ &\quad - \int\Delta\operatorname{curl}\left(\Delta\phi\cdot\nabla\phi\right)\cdot\Delta\omega\,dx + \int\Delta\operatorname{curl}\left(\theta e_3\right)\cdot\Delta\omega\,dx\\ &\le (\|\Delta(u\cdot\nabla\omega) - u\nabla\Delta\omega\|_{L^2} + \|\Delta(\omega\cdot\nabla u)\|_{L^2}\\ &\quad + \left\|\Delta\operatorname{curl}\left(\Delta\phi\nabla\phi\right)\right\|_{L^2} + \|\Delta\operatorname{curl}\left(\theta e_3\right)\|_{L^2})\|\Delta\omega\|_{L^2}\\ &:= (I_1 + I_2 + I_3 + I_4)\|\Delta\omega\|_{L^2}.\end{aligned} \tag{3.23}$$

Using (1.2), (1.6), Lemma 2.2, I_1 and I_2 can be bounded as follows:

$$\begin{aligned}
I_1 &= \sum_i \|\Delta\partial_i(u_i\omega) - u_i\partial_i\Delta\omega\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\Delta\omega\|_{L^2} + C\|\omega\|_{L^\infty}\left\|\nabla^3 u\right\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}, \\
I_2 &\le C\|\omega\|_{L^\infty}\|u\|_{H^3} + C\|\nabla u\|_{L^\infty}\|\omega\|_{H^2} \\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}.
\end{aligned} \tag{3.24}$$

Using Lemma 2.2, I_3 can be bonded as follows

$$\begin{aligned}
I_3 &= \left\|\sum_j \Delta \operatorname{curl} \partial_j(\partial_j\phi\nabla\phi)\right\|_{L^2} \\
&\le C\|\nabla\phi\|_{L^\infty}\|\nabla\phi\|_{H^4} \\
&\le C\|\nabla\phi\|_{L^\infty}\|\phi\|_{H^5}.
\end{aligned} \tag{3.25}$$

Inserting the above estimates into (3.23), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int|\Delta\omega|^2dx &\le C\left(\|\nabla u\|_{L^\infty}\|u\|_{H^3} + \|\nabla\phi\|_{L^\infty}\|\phi\|_{H^5} + \|\theta\|_{H^3}\right)\|\Delta\omega\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}^3 + C\|\nabla\phi\|_{L^\infty}^2\|\Delta\omega\|_{L^2}^2 + C\|\Delta\omega\|_{L^2}^2 \\
&\quad + \delta\|\phi\|_{H^5}^2 + \delta\|\theta\|_{H^3}^2,
\end{aligned} \tag{3.26}$$

for any $0 < \delta < 1$.

Testing (1.1) by $\partial_t u$, using (1.2), (1.6), (3.3), (3.10), and (3.16), and noting that

$$\int(\mu\cdot\nabla)\phi\cdot\partial_t u\,dx = -\int\Delta\phi\nabla\phi\cdot\partial_t u\,dx, \tag{3.27}$$

we reach

$$\begin{aligned}
\|\partial_t u\|_{L^2} &\le \|\theta\|_{L^2} + \|u\cdot\nabla u\|_{L^2} + \|\Delta\phi\nabla\phi\|_{L^2} \\
&\le C + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^\infty} \\
&\le C + C\|u\|_{L^2}^{2/3}\|u\|_{H^3}^{1/3}\cdot\|u\|_{L^2}^{1/2}\|u\|_{H^3}^{1/2} + C\|\nabla\phi\|_{L^\infty} \\
&\le C + C\|u\|_{H^3}^{5/6} + C\|\nabla\phi\|_{L^\infty}.
\end{aligned} \tag{3.28}$$

Here, we have used the Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^3}^2 \leq C\|u\|_{L^2}\|u\|_{H^3}. \tag{3.29}$$

Taking ∂_t to (1.3), we see that

$$\partial_t^2\theta + u \cdot \nabla\partial_t\theta - \Delta\partial_t\theta = -\partial_t u \cdot \nabla\theta. \tag{3.30}$$

Testing the above equation by $\partial_t\theta$, using (1.2), (1.6) and (3.3), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\partial_t\theta|^2 dx + \int |\nabla\partial_t\theta|^2 dx &= -\int \partial_t u \cdot \nabla\theta \cdot \partial_t\theta dx \\
&= \int \partial_t u \cdot \theta\nabla\partial_t\theta dx \\
&\leq \|\theta\|_{L^\infty}\|\partial_t u\|_{L^2}\|\nabla\partial_t\theta\|_{L^2} \\
&\leq C\|\partial_t u\|_{L^2}\|\nabla\partial_t\theta\|_{L^2} \\
&\leq \frac{1}{2}\|\nabla\partial_t\theta\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2,
\end{aligned} \tag{3.31}$$

whence

$$\frac{d}{dt}\int |\partial_t\theta|^2 dx + \int |\nabla\partial_t\theta|^2 dx \leq C + C\|u\|_{H^3}^2 + C\|\nabla\phi\|_{L^\infty}^2. \tag{3.32}$$

By the classical regularity theory of elliptic equation, it follows from (1.4), (1.5), (1.6), (3.16), and (3.10) that

$$\begin{aligned}
\|\phi\|_{H^5} &\leq C\|f'(\phi) - \mu\|_{H^3} \\
&\leq C\|f'(\phi)\|_{H^3} + C\|\mu\|_{H^3} \\
&\leq C\|f'(\phi)\|_{H^3} + C\|\partial_t\phi + u \cdot \nabla\phi\|_{H^1} \\
&\leq C + C\|\phi\|_{H^3} + C\|\partial_t\phi\|_{H^1} + C\|\nabla u\|_{L^\infty}\|\nabla\phi\|_{L^2} + C\|u\|_{L^2}\|\Delta\phi\|_{L^\infty} \\
&\leq C + C\|\phi\|_{H^4} + C\|\partial_t\phi\|_{H^1} + C\|u\|_{H^3} \\
&\leq C + C\|\phi\|_{H^2}^{1/3}\|\phi\|_{H^5}^{2/3} + C\|\partial_t\phi\|_{H^1} + C\|u\|_{H^3},
\end{aligned} \tag{3.33}$$

which implies

$$\|\phi\|_{H^5} \leq C + C\|\partial_t\phi\|_{H^1} + C\|u\|_{H^3}. \tag{3.34}$$

Here, we have used the Gagliardo-Nirenberg inequality:

$$\|\phi\|_{H^4} \le C\|\phi\|_{H^2}^{1/3}\|\phi\|_{H^5}^{2/3}. \tag{3.35}$$

Taking ∂_t to (1.4) and (1.5), we have

$$\partial_t^2\phi + u\cdot\nabla\partial_t\phi = \Delta\partial_t\mu - \partial_t u\cdot\nabla\phi, \tag{3.36}$$

$$-\Delta\partial_t\phi + \partial_t\left(\phi^3-\phi\right) = \partial_t\mu. \tag{3.37}$$

Testing (3.36) by $\partial_t\phi$, using (3.37), (1.4), (1.6), (3.16), and (3.28), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\partial_t\phi|^2dx + \int|\Delta\partial_t\phi|^2dx \\
&\quad= \int\partial_t\left(\phi^3-\phi\right)\cdot\Delta\partial_t\phi\,dx - \int\partial_t u\cdot\nabla\phi\cdot\partial_t\phi\,dx \\
&\quad= \int\left(3\phi^2-1\right)\partial_t\phi\cdot\Delta\partial_t\phi\,dx + \int\partial_t u\cdot\phi\nabla\partial_t\phi\,dx \\
&\quad\le \left\|3\phi^2-1\right\|_{L^\infty}\|\partial_t\phi\|_{L^2}\|\Delta\partial_t\phi\|_{L^2} + \|\partial_t u\|_{L^2}\|\phi\|_{L^\infty}\|\nabla\partial_t\phi\|_{L^2} \\
&\quad\le C\|\partial_t\phi\|_{L^2}\|\Delta\partial_t\phi\|_{L^2} + C\|\partial_t u\|_{L^2}\|\nabla\partial_t\phi\|_{L^2} \\
&\quad\le C\|\partial_t\phi\|_{L^2}\|\Delta\partial_t\phi\|_{L^2} + C\left(1+\|u\|_{H^3}^{5/6}+\|\nabla\phi\|_{L^\infty}\right)\|\nabla\partial_t\phi\|_{L^2},
\end{aligned} \tag{3.38}$$

whence

$$\frac{d}{dt}\int|\partial_t\phi|^2dx + \int|\Delta\partial_t\phi|^2dx \le C\|\partial_t\phi\|_{L^2}^2 + C + C\|u\|_{H^3}^2 + C\|\nabla\phi\|_{L^\infty}^2. \tag{3.39}$$

Here, we have used

$$\|\nabla\partial_t\phi\|_{L^2}^2 \le C\|\partial_t\phi\|_{L^2}\|\Delta\partial_t\phi\|_{L^2}. \tag{3.40}$$

Combining (3.17), (3.21), (3.26), (3.32), (3.34), and (3.39), using Lemmas 2.1, 2.3, and 2.5, we conclude that

$$\|u\|_{L^\infty(0,T;H^3)} \le C, \tag{3.41}$$

$$\|\theta\|_{L^\infty(0,T;H^1)} + \|\partial_t\theta\|_{L^\infty(0,T;L^2)} + \|\partial_t\theta\|_{L^2(0,T;H^1)} \le C, \tag{3.42}$$

$$\|\partial_t\phi\|_{L^\infty(0,T;L^2)} \le C. \tag{3.43}$$

It follows from (1.3), (3.41), and (3.42) that

$$\theta \in L^{\infty}\left(0, T; H^{2}\right) \cap L^{2}\left(0, T; H^{3}\right). \tag{3.44}$$

It follows from (1.4), (1.5), (3.41), and (3.43) that

$$\|\phi\|_{L^{\infty}(0,T;H^{4})} \leq C. \tag{3.45}$$

This completes the proof.

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