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Research Article

The Expression of the Generalized Drazin Inverse of $A - CB$

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We investigate the generalized Drazin inverse of $A - CB$ over Banach spaces stemmed from the Drazin inverse of a modified matrix and present its expressions under some conditions.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. In particular, we write $\mathcal{B}(\mathcal{X})$ instead of $\mathcal{B}(\mathcal{X}, \mathcal{X})$.

For any $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent its range and null space, respectively. If $A \in \mathcal{B}(\mathcal{X})$, the symbols $\sigma(A)$ and $\operatorname{acc}(\sigma(A))$ stand for its spectrum and the set of all accumulation points of $\sigma(A)$, respectively.

Recall the concept of the generalized Drazin inverse introduced by Koliha [1] that the element $T_d \in \mathcal{B}(\mathcal{X})$ is called the generalized Drazin inverse of $T \in \mathcal{B}(\mathcal{X})$ provided it satisfies

$$TT_d = T_dT, \qquad T_dTT_d = T_d, \qquad T - T^2T_d \text{ is quasinilpotent.} \tag{1.1}$$

If it exists then it is unique. The Drazin index $\operatorname{Ind}(T)$ of T is the least positive integer k if $(T - T^2T_d)^k = 0$, and otherwise $\operatorname{Ind}(T) = +\infty$.

From the definition of the generalized Drazin inverse, it is easy to see that if T is a quasinilpotent operator, then T_d exists and $T_d = 0$. It is well known that the generalized Drazin inverse of $T \in \mathcal{B}(\mathcal{X})$ exists if and only if $0 \notin \operatorname{acc}(\sigma(T))$ (see [1, Theorem 4.2]).

If T is generalized Drazin invertible, then the spectral idempotent T^{π} of T corresponding to 0 is given by $T^{\pi} = I - TT^{d}$.

The generalized Drazin inverse is widely investigated because of its applications in singular differential difference equations, Markor chains, (semi-) iterative method numerical analysis (see, for example, [1–5, 7], and references therein).

In this paper, we aim to discuss the generalized Drazin inverse of $A - CB$ over Banach spaces. This question stems from the Drazin inverse of a modified matrix (see, e.g., [6]). In [3], Deng studied the generalized Drazin inverse of $A - CB$. Here we research the problem under more general conditions than those in [3]. Our results extend the relative results in [3, 4].

In this section, we will list some lemmas. In next section, we will present the expressions of the generalized Drazin inverse of $A - CB$. In final section, we illustrate a simple example.

Lemma 1.1 (see [4, Theorem 2.3]). *Let $A, B \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible. If $AB = 0$, then $A + B$ is generalized Drazin invertible and*

$$(A + B)_{d} = B^{\pi} \sum_{n=0}^{\infty} B^{n} A_{d}^{n+1} + \left(\sum_{n=0}^{\infty} B_{d}^{n+1} A^{n} \right) A^{\pi}. \tag{1.2}$$

Lemma 1.2 (see [7, Theorem 5.1]). *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are generalized Drazin invertible and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, then*

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{1.3}$$

is also generalized Drazin invertible and

$$M^{d} = \begin{pmatrix} A^{d} & S \\ O & B^{d} \end{pmatrix}, \tag{1.4}$$

where

$$S = A_{d}^{2} \left(\sum_{n=0}^{\infty} A_{d}^{n} C B^{n} \right) B^{\pi} + A^{\pi} \left(\sum_{n=0}^{\infty} A^{n} C B_{d}^{n} \right) B_{d}^{2} - A_{d} C B_{d}. \tag{1.5}$$

2. Main Results

We start with our main result.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I-P)(A-CB)$ and AP are generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & \left[\sum_{n=0}^{\infty}(AP)_d^{n+1}\left(R^n + VR^{n-1} + V^2R^{n-2}\right)\right]R^{\pi} \\ & - (AP)_d\left[VR_d + V^2R_d^2 + (AP)_dV^2R_d\right] \\ & + (AP)^{\pi}\sum_{n=0}^{\infty}(AP)^n\left(R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}\right),\end{aligned} \tag{2.1}$$

where $V = PA - PCB - AP$ and the symbols $V^iR^j = 0, i = 1, 2$, if $j < 0$.

Proof. Let $S := AP$ and $T := (A-CB)(I-P)$. Then

$$TS = (A-CB)(I-P)AP = 0, \tag{2.2}$$

$$RP = (I-P)(A-CB)P = 0, \tag{2.3}$$

$$A - CB = AP + A(I-P) - CB(I-P) = S + T \tag{2.4}$$

since $AP = PAP$ and $BP = 0$. So, by Lemma 1.1,

$$(T+S)_d = S^{\pi}\sum_{n=0}^{\infty}S^nT_d^{n+1} + \sum_{n=0}^{\infty}S_d^{n+1}T^nT^{\pi}. \tag{2.5}$$

Next, we will give the representations of T_d, T^n, and T_d^n. In order to obtain the expression of T_d, rewrite T as

$$T = R + PA - PCB - PAP = R + V. \tag{2.6}$$

Since $VP = PAP - AP^2 = PAP(I-P)$,

$$V^2P = (PA - PCB - AP)PAP(I-P) = (PAPAP - APPAP)(I-P) = 0, \tag{2.7}$$

and then $V^n = 0$ for $n > 2$ since $V = PA - CB - AP$. So V_d exists and $V_d = 0$. By (2.3), $RV = RP(A - CB - AP) = 0$ and then $R_dV = R_dR_dRV = 0$. So, by Lemma 1.1,

$$T_d = (R+V)_d = R_d + VR_d^2 + V^2R_d^3, \tag{2.8}$$

and then

$$TT_d = RR_d + VR_d + V^2R_d^2. \tag{2.9}$$

Since $R(R+V)^k = R^{k+1}$ and $V^2(R+V)^k = V^2R^k$ for $k \geq 1$,

$$T^n = (R+V)^n = \left(R^2 + VR + V^2\right)(R+V)^{n-2} = R^n + VR^{n-1} + V^2R^{n-2}, \quad n \geq 2. \tag{2.10}$$

From $R_dV = 0$, it is easy to verify that

$$T_d^n = \left(R_d + VR_d^2 + V^2R_d^3\right)^n = R_d^n + VR_d^{n+1} + V^2R_d^{n+2}. \tag{2.11}$$

Hence,

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} S_d^{n+1}T^n\right)T^{\pi} &= (AP)_d\left[I + (AP)_d(R+V) + (AP)_d^2\left(R^2 + VR + V^2\right)\right] \\
&\quad \times \left(R^{\pi} - VR_d - V^2R_d^2\right) + \sum_{n=3}^{\infty}(AP)_d^{n+1}\left(R^n + VR^{n-1} + V^2R^{n-2}\right)R^{\pi} \\
&= (AP)_d\left[I + (AP)_d(R+V) + (AP)_d^2\left(R^2 + VR + V^2\right)\right]R^{\pi} \\
&\quad - (AP)_d\left(VR_d + V^2R_d^2 + (AP)_dV^2R_d\right) \\
&\quad + \sum_{n=3}^{\infty}(AP)_d^{n+1}\left(R^n + VR^{n-1} + V^2R^{n-2}\right)R^{\pi}, \\
S^{\pi}\sum_{n=0}^{\infty}S^nT_d^{n+1} &= (AP)^{\pi}\sum_{n=0}^{\infty}(AP)^n\left(R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}\right).
\end{aligned} \tag{2.12}$$

Therefore, we reach (2.1). □

When $\mathrm{Ind}(AP),\ \mathrm{Ind}(R) < +\infty$, we have the following corollary.

Corollary 2.2. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible. $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I-P)(A-CB)$ and AP are generalized Drazin invertible and $\mathrm{Ind}(R) = k < +\infty$ and $\mathrm{Ind}(AP) = h < +\infty$, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}
(A-CB)_d &= \left[\sum_{n=0}^{k-1}(AP)_d^{n+1}\left(R^n + VR^{n-1} + V^2R^{n-2}\right)\right]R^{\pi} \\
&\quad - (AP)_d\left[VR_d + V^2R_d^2 + (AP)_dV^2R_d\right] \\
&\quad + (AP)^{\pi}\sum_{n=0}^{h-1}(AP)^n\left(R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}\right),
\end{aligned} \tag{2.13}$$

where $V = PA - PCB - AP$ and the symbols $V^iR^j = 0, i = 1, 2,$ if $j < 0$.

If an operator T is quasinilpotent, $T_d = 0$ and $T^{\pi} = I$. So, the following corollary follows from Theorem 2.1.

Corollary 2.3. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I-P)(A-CB)$ is generalized Drazin invertible and AP is a quasinilpotent operator, then $A - CB$ is generalized Drazin invertible and*

$$(A-CB)_d = \sum_{n=0}^{\infty} (AP)^n \left(R_d^{n+1} + VR_d^{n+2} + V^2 R_d^{n+3}\right), \tag{2.14}$$

where $V = PA - PCB - AP$.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $BP = B$. If $R = P(A-CB)$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & R_d + A_d(I-P) + \sum_{n=0}^{\infty} A_d^{n+2}(I-P)(A-CB)P(A-CB)^n R^{\pi} \\ & + A^{\pi}\sum_{n=0}^{\infty} A^n (I-P)(A-CB)PR_d^{n+2} - A_d(I-P)(A-CB)R_d.\end{aligned} \tag{2.15}$$

Proof. Since $P^2 = P$, we have $\mathcal{X} = \mathcal{R}(P) \bigoplus \mathcal{N}(P)$ and can write P in the following matrix form:

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.16}$$

The condition $PA = PAP$, therefore, yields the matrix form of A as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}. \tag{2.17}$$

From $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ and the hypothesis that A_d exists, $A_1 \in \mathcal{B}(\mathcal{R}(P))$ and $A_2 \in \mathcal{B}(\mathcal{N}(P))$ are generalized Drazin invertible since $0 \notin \operatorname{acc}(\sigma(A))$ if and only if $0 \notin \operatorname{acc}(\sigma(A_1))$ and $0 \notin \operatorname{acc}(\sigma(A_2))$. And, by Lemma 1.2,

$$A_d = \begin{pmatrix} A_1^d & 0 \\ W & A_2^d \end{pmatrix}, \tag{2.18}$$

where W is some operator. Since

$$A(I-P) = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{2.19}$$

$(A(I-P))_d$ exists and

$$(A(I-P))_d = \begin{pmatrix} 0 & 0 \\ 0 & A_2^d \end{pmatrix} = A_d(I-P). \tag{2.20}$$

To use Theorem 2.1 to complete the proof, let $Q = (I - P)$. So $R = (I - Q)(A - CB)$ and AQ are generalized Drazin invertible. And from the conditions $PA = PAP$ and $BP = B$, we can obtain $AQ = QAQ$ and $BQ = 0$. Thus, by Theorem 2.1, we have

$$\begin{aligned}(A - CB)_d = {} & (AQ)_d R^{\pi} + (AQ)_d^2 (R + V) R^{\pi} + \left[\sum_{n=2}^{\infty} (AQ)_d^{n+1} \left(R^n + V R^{n-1} + V^2 R^{n-2}\right)\right] R^{\pi} \\ & - (AQ)^d \left[V R_d + V^2 R_d^2 + (AQ)_d V^2 R_d\right] + (AQ)^{\pi} \left(R_d + V R_d^2 + V^2 R_d^3\right) \\ & + (AQ)^{\pi} \sum_{n=1}^{\infty} (AP)^n \left(R_d^{n+1} + V R_d^{n+2} + V^2 R_d^{n+3}\right),\end{aligned} \tag{2.21}$$

where $V = QA - QCB - AQ$.

Since $P^2 = P$ and $Q^2 = Q$ and then $VQ = 0$ and $V = QV$. So $V^2 = 0$. Note that $QR = 0$ and then $QR_d = 0$ and $(AQ)_d R = 0$. Thus it follows from (2.21) that

$$\begin{aligned}(A - CB)_d & = (AQ)_d + (AQ)_d^2 V R^{\pi} + \left[\sum_{n=2}^{\infty} (AQ)_d^{n+1} V R^{n-1}\right] R^{\pi} - (AQ)_d V R_d \\ & \quad + R_d + (AQ)^{\pi} V R_d^2 + (AQ)^{\pi} \sum_{n=1}^{\infty} (AQ)^n V R_d^{n+2} \\ & = (AQ)_d + \left[\sum_{n=0}^{\infty} (AQ)_d^{n+2} V R^n\right] R^{\pi} - (AQ)_d V R_d + R_d \\ & \quad + (AQ)^{\pi} \sum_{n=0}^{\infty} (AQ)^n V (R_d)^{n+2}.\end{aligned} \tag{2.22}$$

Since $V = Q(A - CB) - (A - CB)Q = (A - CB)(I - Q) - (I - Q)(A - CB)$, $VR = Q(A - CB)R$ and $QV = Q(A - CB)(I - Q)$. Note that $R^n = P(A - CB)^n$ and $(AQ)^n = A^n Q$. Substituting V and $Q = I - P$ in (2.22) yields (2.15). □

Adding the condition $PC = C$ in Theorem 2.4 yields a result below.

Corollary 2.5. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$, $BP = B$, and $PC = C$. If $R = P(A - CB)$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A - CB)_d = {} & R_d + A_d (I - P) + \sum_{n=0}^{\infty} A_d^{n+2} (I - P) AP (A - CB)^n R^{\pi} \\ & + A^{\pi} \sum_{n=0}^{\infty} A^n (I - P) AP R_d^{n+2} - A_d (I - P) A R_d.\end{aligned} \tag{2.23}$$

Adding the condition $PC = 0$ in Theorem 2.4 yields $R = PA$. So similar to the proof of $(A(I - P))^d = A^d (I - P)$ in Theorem 2.4, we can gain $(PA)^d = PA^d$.

Corollary 2.6. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z},\mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$, $BP = B$, and $PC = 0$; then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & A_d + \sum_{n=0}^{\infty} A_d^{n+2}(I-P)(A-CB)PA^n A^{\pi} + A^{\pi}\sum_{n=0}^{\infty} A^n (I-P)(A-CB)PA_d^{n+2} \\ & - A_d(I-P)(A-CB)PA_d.\end{aligned} \tag{2.24}$$

Analogously, we can deduce Theorem 2.7 and Corollary 2.9 below.

Theorem 2.7. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $PC = C$. If $R = (A - CB)P$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & R_d + (I-P)A_d + \sum_{n=0}^{\infty} R_d^{n+2} P(A-CB)(I-P)A^n A^{\pi} \\ & + R^{\pi}\sum_{n=0}^{\infty}(A-CB)^n P(A-CB)(I-P)A_d^{n+2} - R_d(A-CB)(I-P)A_d.\end{aligned} \tag{2.25}$$

Remark 2.8 (see [4, Theorem 2.4]). It is a special case of Theorem 2.7.

Corollary 2.9. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z},\mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$, $PC = C$, and $BP = 0$; then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & A_d + \sum_{n=0}^{\infty} A_d^{n+2} P(A-CB)(I-P)A^n A^{\pi} \\ & + A^{\pi}\sum_{n=0}^{\infty} A^n P(A-CB)(I-P)A_d^{n+2} - A_d P(A-CB)(I-P)A_d.\end{aligned} \tag{2.26}$$

Similar to Theorem 2.1 and Corollary 2.2, we can show the following two results.

Theorem 2.10. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $PC = 0$. If $R = (A - CB)(I - P)$ and PA are generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A-CB)_d = {} & R^{\pi}\sum_{n=0}^{\infty}\left(R^n + R^{n-1}V + R^{n-2}V^2\right)(PA)_d^{n+1} \\ & - \left[R_d V + R_d^2 V^2 + R_d V^2 (PA)_d\right](PA)_d \\ & + \left[\sum_{n=0}^{\infty}\left(R_d^{n+1} + R_d^{n+2}V + R_d^{n+3}V^2\right)(PA)^n\right](PA)^{\pi},\end{aligned} \tag{2.27}$$

where $V = AP - CBP - PA$ and the symbols $R^i V^j = 0$, $j = 1, 2$, if $i < 0$.

Corollary 2.11. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible. $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $PC = 0$. If $R = (A - CB)(I - P)$ and PA are generalized Drazin invertible and* $\operatorname{Ind}(R) = k < +\infty$ *and* $\operatorname{Ind}(PA) = h < +\infty$*, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned}(A - CB)_d = {} & R^{\pi} \sum_{n=0}^{k-1} \left(R^n + R^{n-1}V + R^{n-2}V^2\right)(PA)_d^{n+1} \\ & - \left[R_d V + R_d^2 V^2 + R_d V^2 (PA)_d\right](PA)^d \\ & + \left[\sum_{n=0}^{h-1} \left(R_d^{n+1} + R_d^{n+2}V + R_d^{n+3}V^2\right)(PA)^n\right](PA)^{\pi},\end{aligned} \tag{2.28}$$

where $V = AP - CBP - PA$ and the symbols $R^i V^j = 0$, $j = 1, 2$, if $i < 0$.

When $PA = AP$ and $P^2 = P$ in Theorem 2.10, we can obtain the following result since $R^n = (A - CB)^n (I - P)$.

Corollary 2.12 (see [3, Theorem 4.3]). *Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ commuting with A such that $PC = 0$. If $R = (A - CB)(I - P)$ is generalized Drazin invertible, then $A - CB$ is the generalized Drazin invertible and*

$$(A - CB)_d = R_d + PA_d - R_d V A_d + R^{\pi} \sum_{n=0}^{\infty} (A - CB)^n V A_d^{n+2} + \sum_{n=0}^{\infty} R_d^{n+2} V A^n A^{\pi}, \tag{2.29}$$

where $V = -CBP$.

3. Example

Before ending this paper, we give an example as follows.

Example 3.1. Let

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Then

$$CB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A - CB = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

We will compute the Drazin inverse of $A - CB$. To do this, we choose the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Apparently, P is not idempotent and $PA \neq AP$. But $BP = 0$ and

$$AP = PAP = \begin{pmatrix} 1 & -2 & 8 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Obviously, $\operatorname{Ind}(AP) = 2$. Computing

$$R = (I - P)(A - CB) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

$$V = PA - PCB - AP = \begin{pmatrix} 0 & 4 & -4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

we have $\operatorname{Ind}(R) = 2$. So, by Corollary 2.2,

$$(A - CB)_d = \begin{pmatrix} 1 & -4 & 10 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

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