Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 409237, 20 pages
doi:10.1155/2012/409237

Research Article

On the Difference Equation $x_n = a_n x_{n-k} / (b_n + c_n x_{n-1} \cdots x_{n-k})$

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Received 27 May 2012; Accepted 2 July 2012

Academic Editor: Jean Pierre Gossez

The behavior of well-defined solutions of the difference equation $x_n = a_n x_{n-k} / (b_n + c_n x_{n-1} \cdots x_{n-k})$, $n \in \mathbb{N}_0$, where $k \in \mathbb{N}$ is fixed, the sequences a_n, b_n and c_n are real, $(b_n, c_n) \neq (0,0)$, $n \in \mathbb{N}_0$, and the initial values $x_{-k}, \dots, x_{-1}$ are real numbers, is described.

1. Introduction

Recently there has been a huge interest in studying nonlinear difference equations and systems (see, e.g., [1–33] and the references therein). Here we study the difference equation

$$x_n = \frac{x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}$ is fixed, the sequences b_n and c_n are real, $(b_n, c_n) \neq (0,0)$, $n \in \mathbb{N}_0$, and the initial values $x_{-k}, \dots, x_{-1}$ are real numbers. Equation (1.1) is a particular case of the equation

$$x_n = \frac{\widehat{a}_n x_{n-k}}{\widehat{b}_n + \widehat{c}_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with real sequences $\widehat{a}_n$, $\widehat{b}_n$ and $\widehat{c}_n$. For $\widehat{a}_n = 0$, $n \in \mathbb{N}_0$, the equation is trivial, and, for $\widehat{a}_n \neq 0$, $n \in \mathbb{N}_0$, it is reduced to equation (1.1) with $b_n = \widehat{b}_n/\widehat{a}_n$ and $c_n = \widehat{c}_n/\widehat{a}_n$.

Equation

$$x_n = \frac{x_{n-k}}{b + c x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $b, c \in \mathbb{R}$, which was treated in [32], is a particular case of equation (1.1).

As in [32], here, we employ our idea of using a change of variables in equation (1.1) which extends the one in our paper [21] and is later also used, for example, in [4]. For similar methods see also [22, 25]. Equation (1.3) in the case $k = 2$ was also studied in [1, 2], in a different way. The case when the sequences b_n and c_n are two-periodic was studied in [31] (some related results are also announced in talk [3]). For related symmetric systems of difference equations, see [27, 29]. For some other recent results on difference equations and systems which can be solved, see, for example, [6, 7, 20–22, 30, 31, 33]. Some classical results can be found, for example, in [11].

Equation (1.1) is a particular case of the equation

$$y_n = f(y_{n-1}, \ldots, y_{n-k}, n) y_{n-k}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $f : \mathbb{R}^{k+1} \to \mathbb{R}$ is a continuous function. Numerous particular cases of (1.4) have been investigated, for example, in [9, 21, 23]. In this paper we adopt the customary notation $\prod_{i=k+1}^{k} g_i = 1$ and $\sum_{i=k+1}^{k} g_i = 0$.

2. Case $c_n = 0$, $n \in \mathbb{N}_0$

Here we consider the case $c_n = 0$, $n \in \mathbb{N}_0$. In this case equation (1.1) becomes

$$x_n = \frac{x_{n-k}}{b_n}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

$b_n \neq 0$, $n \in \mathbb{N}_0$, from which it follows that for each $i \in \{1, \ldots, k\}$

$$x_{km-i} = \frac{x_{-i}}{\prod_{j=1}^{m} b_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{2.2}$$

Using formula (2.2) the following theorem can be easily proved.

Theorem 2.1. *Consider equation* (1.1) *with* $c_n = 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$. *Then the following statements are true:*

(a) *if*

$$\liminf_{m\to\infty} |b_{km-i}| = p_i > 1, \tag{2.3}$$

for some $i \in \{1,\dots,k\}$, *then* $x_{km-i} \to 0$ *as* $m \to \infty$;

(b) *if, for each* $i \in \{1,\dots,k\}$, *the limits* p_i *in* (2.3) *are greater than* 1, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if* $b_{km-i} = 1$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{1,\dots,k\}$, *then* $x_{km-i} = x_{-i}$, $m \in \mathbb{N}_0$;

(d) *if* $b_{km-i} = -1$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{1,\dots,k\}$, *then* $x_{km-i} = (-1)^m x_{-i}$, $m \in \mathbb{N}_0$;

(e) *if*

$$\limsup_{m\to\infty} |b_{km-i}| = q_i \in [0,1), \tag{2.4}$$

and $x_{-i} \neq 0$, *for some* $i \in \{1,\dots,k\}$, *then* $|x_{km-i}| \to \infty$, *as* $m \to \infty$;

(f) *if, for each* $i \in \{1,\dots,k\}$, *the limits* q_i *in* (2.4) *belong to the interval* $[0,1)$ *and* $x_{-i} \neq 0$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

3. Case $b_n = 0$, $n \in \mathbb{N}_0$

In this section we consider the case $b_n = 0$, $n \in \mathbb{N}_0$. Note that in this case equation (1.1) becomes

$$x_n = \frac{x_{n-k}}{c_n x_{n-1} \cdots x_{n-k+1} x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $c_n \neq 0$, $n \in \mathbb{N}_0$. If x_n is a well-defined solution of equation (3.1) (i.e., a solution with initial values $x_{-i} \neq 0$, $i = 1,\dots,k$, which implies $x_n \neq 0$, $n \in \mathbb{N}_0$), then

$$x_n = \frac{1}{c_n x_{n-1}(x_{n-2} \cdots x_{n-k+1})} = \frac{c_{n-1} x_{n-2} \cdots x_{n-k}}{c_n x_{n-2} \cdots x_{n-k+1}} = \frac{c_{n-1}}{c_n} x_{n-k}, \quad n \in \mathbb{N}. \tag{3.2}$$

Hence for each $i \in \{0,1,\dots,k-1\}$

$$x_{km-i} = x_{-i} \prod_{j=1}^{m} \frac{c_{kj-i-1}}{c_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{3.3}$$

Using formula (3.3) we easily prove the next theorem.

Theorem 3.1. *Consider equation* (1.1) *with* $b_n = 0$, $c_n \neq 0$, $n \in \mathbb{N}_0$. *Then the following statements are true:*

(a) *if*

$$\liminf_{m\to\infty}\left|\frac{c_{km-i}}{c_{km-i-1}}\right| = \widehat{p}_i > 1, \tag{3.4}$$

for some $i \in \{0,1,\dots,k-1\}$, *then* $x_{km-i} \to 0$ *as* $m \to \infty$;

(b) *if, for each* $i \in \{0,1,\dots,k-1\}$, *the limits* $\widehat{p}_i$ *in* (3.4) *are greater than* 1, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if* $c_{km-i-1} = c_{km-i}$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{0,1,\dots,k-1\}$, *then* $x_{km-i} = x_{-i}$, $m \in \mathbb{N}_0$;

(d) *if* $c_{km-i-1} = -c_{km-i}$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{0,1,\dots,k-1\}$, *then* $x_{km-i} = (-1)^m x_{-i}$, $m \in \mathbb{N}_0$;

(e) *if*

$$\limsup_{m\to\infty}\left|\frac{c_{km-i}}{c_{km-i-1}}\right| = \widehat{q}_i \in [0,1), \tag{3.5}$$

and $x_{-i} \neq 0$ *for some* $i \in \{0,1,\dots,k-1\}$, *then* $|x_{km-i}| \to \infty$, *as* $m \to \infty$;

(f) *if, for each* $i \in \{0,1,\dots,k-1\}$, $x_{-i} \neq 0$ *and the limits* $\widehat{q}_i$ *in* (3.5) *belong to the interval* $[0,1)$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

4. Case $b_n \neq 0$ and $c_n \neq 0$

The case when $b_n \neq 0$ and $c_n \neq 0$ for every $n \in \mathbb{N}_0$ is considered in this section.

If $x_{-i_0} = 0$ for some $i_0 \in \{1,\dots,k\}$, then from (1.1) we have that

$$x_{km-i_0} = 0, \quad \text{for } m \in \mathbb{N}_0. \tag{4.1}$$

From (4.1) and (1.1) we have that for each $i \in \{1,\dots,k\} \setminus \{i_0\}$

$$x_{km-i} = \frac{x_{k(m-1)-i}}{b_{km-i}} = \frac{x_{-i}}{\prod_{j=1}^{m} b_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{4.2}$$

From (4.1) we see that, for $i = i_0$, (4.2) also holds. Hence Theorem 2.1 can be applied in this case. Note that if $x_n = 0$ for some $n \in \mathbb{N}_0$, then (1.1) implies that there is an $i_0 \in \{1,\dots,k\}$ such that $x_{-i_0} = 0$, and by the previous consideration we have that (4.2) also holds.

If $x_{-i} \neq 0$, for each $i \in \{1,\dots,k\}$, then for every well-defined solution we have $x_n \neq 0$ for $n \geq -k$ (note that there are solutions which are not well defined, that is, those for which $x_{n-1}\cdots x_{n-k} = -b_n/c_n$, for some $n \in \mathbb{N}_0$).

Multiplying equation (1.1) by $x_{n-1}\cdots x_{n-k+1}$ and using the transformation

$$y_n = \frac{1}{x_n x_{n-1}\cdots x_{n-k+1}}, \quad n \ge -1, \tag{4.3}$$

we obtain equation

$$y_n = b_n y_{n-1} + c_n, \quad n \in \mathbb{N}_0. \tag{4.4}$$

Note that from (4.3), for every well-defined solution $(x_n)_{n\ge -k}$ of equation (1.1) such that $x_{-i} \ne 0$, for each $i \in \{1,\dots,k\}$, it follows that $y_n \ne 0$, $n \ge -1$.

Since $b_n \ne 0$, $n \in \mathbb{N}_0$, we have that

$$y_n = \left(\prod_{i=0}^{n} b_i\right)\left(y_{-1} + \sum_{j=0}^{n} \frac{c_j}{\prod_{i=0}^{j} b_i}\right), \quad n \in \mathbb{N}_0. \tag{4.5}$$

From (4.3) and (4.5) we have that

$$x_n = \frac{1}{y_n x_{n-1}\cdots x_{n-k+1}} = \frac{y_{n-1}}{y_n} x_{n-k} = \frac{y_{-1} + \sum_{j=0}^{n-1}\left(c_j/\prod_{i=0}^{j} b_i\right)}{b_n\left(y_{-1} + \sum_{j=0}^{n}\left(c_j/\prod_{i=0}^{j} b_i\right)\right)} x_{n-k}, \tag{4.6}$$

for every $n \in \mathbb{N}_0$.

Hence, from (4.6), we obtain that

$$x_{mk-i} = x_{-i}\prod_{l=1}^{m} \frac{1/\alpha + \sum_{j=0}^{kl-i-1}\left(c_j/\prod_{i=0}^{j} b_i\right)}{b_{kl-i}\left(1/\alpha + \sum_{j=0}^{kl-i}\left(c_j/\prod_{i=0}^{j} b_i\right)\right)}, \tag{4.7}$$

for every $m \in \mathbb{N}_0$ and each $i = 1,2,\dots,k$, where

$$\alpha = \prod_{l=1}^{k} x_{-l}. \tag{4.8}$$

5. Case $b_n = 1$, $n \in \mathbb{N}_0$

Here we consider the case $b_n = 1$, $n \in \mathbb{N}_0$. In this case, from (4.7) we have that for each $i \in \{1,\dots,k\}$

$$x_{mk-i} = x_{-i}\prod_{l=1}^{m} \frac{1 + \alpha\sum_{j=0}^{kl-i-1} c_j}{1 + \alpha\sum_{j=0}^{kl-i} c_j}, \quad m \in \mathbb{N}_0. \tag{5.1}$$

Note that this formula includes also the case when $x_{-i_0} = 0$ for some $i_0 \in \{1,\dots,k\}$.

Now we formulate and prove a result in this case by using formula (5.1).

Theorem 5.1. *Consider equation* (1.1) *with* $b_n = 1$, $n \in \mathbb{N}_0$, $\operatorname{sign} c_n = \operatorname{sign} c_0$, $n \in \mathbb{N}$, $\alpha \neq 0$, *and*

$$\alpha \sum_{j=0}^{n} c_j \neq -1, \quad n \in \mathbb{N}_0. \tag{5.2}$$

Then the following statements hold:

(a) *if for some* $i \in \{1, \dots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = +\infty, \tag{5.3}$$

$$\lim_{l \to \infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = 0, \tag{5.4}$$

then $x_{mk-i} \to 0$ *as* $m \to \infty$;

(b) *if* (5.3) *and* (5.4) *hold for every* $i \in \{1, \dots, k\}$, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if for some* $i \in \{1, \dots, k\}$ *the sum*

$$\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} \tag{5.5}$$

converges, then the sequence x_{mk-i} *is also convergent;*

(d) *if the sum in* (5.5) *is finite for every* $i \in \{1, \dots, k\}$, *then the sequences* x_{km-i} *are convergent.*

Proof. Let $(x_n)_{n \geq -k}$ be a solution of equation (1.1). Using condition $\operatorname{sign} c_n = \operatorname{sign} c_0$, $n \in \mathbb{N}$, it is easy to see that if (5.4) holds for some $i \in \{1, \dots, k\}$, there is an $m_0 \in \mathbb{N}$ such that for $j \geq m_0 + 1$ the terms in the product in (5.1) are positive and that the following asymptotic formula

$$\ln(1 + x) = x + O\left(x^2\right) \tag{5.6}$$

can be used with x being the fraction in the limit (5.4). From (5.1) and (5.6) we have that

$$\begin{aligned}
|x_{km-i}| &= |x_{-i}|\prod_{l=1}^{m}\left|\frac{1+\alpha\sum_{j=0}^{kl-i-1}c_j}{1+\alpha\sum_{j=0}^{kl-i}c_j}\right| \\
&= |x_{-i}|c(m_0)\exp\left(\sum_{l=m_0+1}^{m}\ln\frac{1+\alpha\sum_{j=0}^{kl-i-1}c_j}{1+\alpha\sum_{j=0}^{kl-i}c_j}\right) \\
&= |x_{-i}|c(m_0)\exp\left(\sum_{l=m_0+1}^{m}\ln\left(1-\frac{\alpha c_{kl-i}}{1+\alpha\sum_{j=0}^{kl-i}c_j}\right)\right) \\
&= |x_{-i}|c(m_0)\exp\left(-\sum_{l=m_0+1}^{m}\frac{\alpha c_{kl-i}(1+o(1))}{1+\alpha\sum_{j=0}^{kl-i}c_j}\right),
\end{aligned} \tag{5.7}$$

where

$$c(m_0) = \prod_{l=1}^{m_0}\left|\frac{1+\alpha\sum_{j=0}^{kl-i-1}c_j}{1+\alpha\sum_{j=0}^{kl-i}c_j}\right|. \tag{5.8}$$

Using formula (5.7), the assumptions regarding the sum $\sum_{j=m_0+1}^{\infty}(\alpha c_{kl-i}/(1+\alpha\sum_{j=0}^{kl-i}c_j))$ and the comparison test for the series whose terms are of eventually the same sign, the results in the theorem easily follow. □

6. Case $b_n = -1$, $n \in \mathbb{N}_0$

Here we consider the case $b_n = -1$, $n \in \mathbb{N}_0$. In this case from (4.7) we have

$$x_{mk-i} = (-1)^m x_{-i}\prod_{l=1}^{m}\frac{1+\alpha\sum_{j=0}^{kl-i-1}(-1)^{j+1}c_j}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}, \tag{6.1}$$

for every $m \in \mathbb{N}_0$ and each $i = 1, 2, \ldots, k$, where α is defined by (4.8).

Theorem 6.1. *Consider equation* (1.1) *with* $\alpha \neq 0$, $b_n = -1$, $n \in \mathbb{N}_0$, *and*

$$\alpha \sum_{j=0}^{n} (-1)^{j+1} c_j \neq -1, \quad n \in \mathbb{N}_0. \tag{6.2}$$

Then the following statements hold:

(a) *if for some* $i \in \{1, \ldots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = +\infty, \tag{6.3}$$

$$\lim_{l \to \infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = 0, \tag{6.4}$$

$$\sum_{l=1}^{\infty} \frac{c_{kl-i}^2}{\left(1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j\right)^2} < +\infty, \tag{6.5}$$

then $x_{mk-i} \to 0$ *as* $m \to \infty$;

(b) *if for every* $i \in \{1, \ldots, k\}$, (6.3), (6.4), *and* (6.5) *hold, then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if for some* $i \in \{1, \ldots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = -\infty, \tag{6.6}$$

conditions (6.4) *and* (6.5) *hold, and* $x_{-i} \neq 0$, *then* $|x_{mk-i}| \to \infty$ *as* $m \to \infty$;

(d) *if for every* $i \in \{1, \ldots, k\}$, *conditions* (6.4), (6.5), *and* (6.6) *hold, and* $x_{-i} \neq 0$, $i \in \{1, \ldots, k\}$, *then* $|x_n| \to \infty$ *as* $n \to \infty$;

(e) *if for some* $i \in \{1, \ldots, k\}$ *the sum*

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \tag{6.7}$$

converges and condition (6.5) *holds, then the sequences* x_{2mk-i} *and* $x_{(2m+1)k-i}$ *are also convergent;*

(f) *if for every* $i \in \{1, \ldots, k\}$ *the sum in* (6.7) *converges and condition* (6.5) *holds, then the sequences* x_{2km-j}, $j = 1, \ldots, 2k$ *are convergent.*

Proof. Let $(x_n)_{n\geq -k}$ be a solution of equation (1.1). By (6.4) we see that irrespectively on $i \in \{1,\ldots,k\}$, there is an $m_1 \in \mathbb{N}$ such that for $j \geq m_1+1$ the terms in the product in (6.1) belong to the interval $(1/2, 3/2)$ and that asymptotic formulae

$$\ln(1+x) = x - \frac{x^2}{2} + O\left(x^3\right) \tag{6.8}$$

can be used with $-x$ being the fraction in (6.4). From this and (6.1) we have that

$$\begin{aligned}
|x_{km-i}| &= |x_{-i}|\prod_{l=1}^{m}\left|\frac{1+\alpha\sum_{j=0}^{kl-i-1}(-1)^{j+1}c_j}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}\right| \\
&= |x_{-i}|c_1(m_1)\exp\left(\sum_{l=m_1+1}^{m}\ln\frac{1+\alpha\sum_{j=0}^{kl-i-1}(-1)^{j+1}c_j}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}\right) \\
&= |x_{-i}|c_1(m_1)\exp\left(\sum_{l=m_1+1}^{m}\ln\left(1-\frac{\alpha(-1)^{kl-i+1}c_{kl-i}}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}\right)\right) \\
&= |x_{-i}|c_1(m_1)\exp\left(-\sum_{l=m_1+1}^{m}\left(\frac{\alpha(-1)^{kl-i+1}c_{kl-i}}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}+\frac{\alpha^2c_{kl-i}^2(1+o(1))}{2\left(1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j\right)^2}\right)\right),
\end{aligned} \tag{6.9}$$

where

$$c_1(m_1) = \prod_{l=1}^{m_1}\left|\frac{1+\alpha\sum_{j=0}^{kl-i-1}(-1)^{j+1}c_j}{1+\alpha\sum_{j=0}^{kl-i}(-1)^{j+1}c_j}\right|. \tag{6.10}$$

Using formula (6.9), the assumptions of the theorem and some well-known convergence tests for series, the results in (a)–(f) easily follow. □

7. Case $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$

In this section we consider equation (1.1) for the case $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$, that is, when the sequences b_n and c_n are k-periodic.

First we show the existence of k-periodic solutions of equation (4.4). If

$$(\overline{y}_0, \overline{y}_1, \ldots, \overline{y}_{k-1}) \tag{7.1}$$

is such a solution, then we have that

$$\overline{y}_1 = b_1\overline{y}_0 + c_1,\ \overline{y}_2 = b_2\overline{y}_1 + c_2, \ldots,\ \overline{y}_0 = b_k\overline{y}_{k-1} + c_k. \tag{7.2}$$

By successive elimination, or by Kronecker theorem (note that system (7.2) is linear), we get

$$\overline{y}_i = \frac{\sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}}{1 - \prod_{j=1}^{k} b_j}, \quad i = \overline{1,k}, \tag{7.3}$$

if $\prod_{j=1}^{k} b_j \neq 1$, where σ is the permutation defined by

$$\sigma(i) = i - 1, \quad i = \overline{2,k}, \ \sigma(1) = k, \tag{7.4}$$

and $\sigma^{[i]} = \sigma \circ \sigma^{[i-1]}$, $\sigma^{[0]} = \mathrm{Id}$, where Id denotes the identity.

It is easy to see that (4.4) along with k periodicity of sequences b_n and c_n implies

$$y_{km+i} = \left(\prod_{j=1}^{k} b_j\right) y_{k(m-1)+i} + \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}, \tag{7.5}$$

for every $m \in \mathbb{N}_0$ and $i \in \{1, 2, \ldots, k\}$, such that $k(m-1) + i \geq -1$.

Since (7.5) is a linear first-order difference equation, we have that when $\prod_{j=1}^{k} b_j \neq 1$, its general solution is

$$y_{km+i} = \left(\prod_{j=1}^{k} b_j\right)^m y_i + \frac{1 - \left(\prod_{j=1}^{k} b_j\right)^m}{1 - \prod_{j=1}^{k} b_j} \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}. \tag{7.6}$$

By letting $m \to \infty$ in (7.6) we obtain the following corollary.

Corollary 7.1. *Consider equation* (4.4) *with* $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$. *Assume that*

$$\left| \prod_{j=1}^{k} b_j \right| < 1. \tag{7.7}$$

Then for every solution y_n *of the equation we have that*

$$\lim_{m \to \infty} y_{km+i} = \overline{y}_i, \tag{7.8}$$

for every $i \in \{1, 2, \ldots, k\}$, *that is,* y_n *converges to the* k*-periodic solution in formula* (7.3).

Let

$$L_i := \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}, \quad i = \overline{1,k}, \qquad q := \prod_{j=1}^{k} b_j. \tag{7.9}$$

From now on we will use the following convention: if $i, j \in \mathbb{N}_0$, then we regard that $L_j = L_i$, if $i \equiv j \pmod{k}$. Also if a sequence $(m_j)_{j\in\mathbb{N}_0}$ is defined by the relation $m_j = f(L_j)$, where f is a real function, then we will assume that $m_j = m_i$, if $i \equiv j \pmod{k}$.

Using (7.6) and notation (7.9) in the relation $x_n = (y_{n-1}/y_n)x_{n-k}$ (see (4.6)), for the case $q \neq 1$, we have that

$$\begin{aligned} x_{km+i} &= x_{i-k}\prod_{j=0}^{m}\frac{(y_{i-1}-L_{i-1}/(1-q))q^j + L_{i-1}/(1-q)}{(y_i - L_i/(1-q))q^j + L_i/(1-q)} \\ &= x_{i-k}\prod_{j=0}^{m}\frac{L_{i-1}}{L_i}\cdot\frac{1+((1-q)y_{i-1}/L_{i-1}-1)q^j}{1+((1-q)y_i/L_i-1)q^j}, \end{aligned} \tag{7.10}$$

for every $m \in \mathbb{N}_0$ and each $i \in \{2, \ldots, k\}$, and

$$\begin{aligned} x_{km+1} &= x_{1-k}\prod_{j=0}^{m}\frac{(y_k - L_k/(1-q))q^{j-1} + L_k/(1-q)}{(y_1 - L_1/(1-q))q^j + L_1/(1-q)} \\ &= x_{1-k}\prod_{j=0}^{m}\frac{L_k}{L_1}\cdot\frac{1+((1-q)y_k/L_k-1)q^{j-1}}{1+((1-q)y_1/L_1-1)q^j}. \end{aligned} \tag{7.11}$$

Now we present some results, which are applications of formulae (7.10) and (7.11).

7.1. Case $q = -1$

If $q = -1$, then by (7.5) we get

$$y_{km+i} = -y_{k(m-1)+i} + L_i = L_i - \left(L_i - y_{k(m-2)+i}\right) = y_{k(m-2)+i}, \quad m \in \mathbb{N}, \tag{7.12}$$

for $k(m-2) + i \geq -1$; that is, y_{km+i} is two-periodic for each $i \in \{1, \ldots, k\}$. Hence y_n is a $2k$-periodic solution of equation (4.4), in this case.

Hence from the relation $x_n = (y_{n-1}/y_n)x_{n-k}$ (see (4.6)), for each $i \in \{1, 2, \ldots, k\}$, we have

$$x_{km-i} = \frac{y_{km-i-1}}{y_{km-i}}x_{km-i-k} = \frac{y_{km-i-1}}{y_{km-i}}\frac{y_{k(m-1)-i-1}}{y_{k(m-1)-i}}x_{km-i-2k}, \tag{7.13}$$

for $k(m-1) \geq i$.

From (7.13) and by $2k$ periodicity of y_n, we get

$$x_{2kl+j} = \left(\frac{y_{j-1}y_{j+k-1}}{y_j y_{j+k}}\right)^l x_j, \quad l \in \mathbb{N}_0, \tag{7.14}$$

for each $j \in \{-k+1, \ldots, -1, 0, 1, \ldots, k\}$.

From (7.14), the behavior of solutions of equation (1.1), in this case, easily follows. For example, if

$$p_j := \frac{y_{j-1}y_{j+k-1}}{y_j y_{j+k}} = 1, \tag{7.15}$$

for each $j \in \{-k+1, \dots, -1, 0, 1, \dots, k\}$, then the solution $(x_n)_{n \geq k}$ of (1.1) is $2k$-periodic.

7.2. *Case* $q = 1$

If $q = 1$ and $\alpha \neq 0$, then from (7.5) we obtain

$$y_{km+i} = y_{k(m-1)+i} + L_i, \quad m \in \mathbb{N}_0, \ i = \overline{1,k}, \tag{7.16}$$

when $k(m-1) + i \geq -1$, from which along with (4.6), it follows that

$$\begin{aligned} x_{km+i} &= x_i \prod_{j=1}^{m} \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i}, \quad m \in \mathbb{N}, \ i = \overline{2,k}, \\ x_{km+1} &= x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1}, \quad m \in \mathbb{N}. \end{aligned} \tag{7.17}$$

Corollary 7.2. *Consider equation* (1.1). *Let* $q = 1$, $\alpha \neq 0$, *and* $\widehat{p}_i := L_{i-1}/L_i$, $i \in \{1, \dots, k\}$. *Then the following statements hold true.*

(a) *If* $|\widehat{p}_i| < 1$, *for some* $i \in \{1, \dots, k\}$, *then* $x_{km+i} \to 0$ *as* $m \to \infty$.

(b) *If* $|\widehat{p}_i| > 1$, *or* $L_i = 0$ *and* $L_{i-1} \neq 0$, *for some* $i \in \{1, \dots, k\}$, *then* $|x_{km+i}| \to \infty$ *as* $m \to \infty$, *if* $x_i \neq 0$.

(c) *If* $\widehat{p}_i = 1$, *for some* $i \in \{2, \dots, k\}$, *and* $(y_{i-1} - y_i)/L_i > 0$, *then* $|x_{km+i}| \to \infty$ *as* $m \to \infty$, *if* $x_i \neq 0$.

(d) *If* $\widehat{p}_i = 1$, *for some* $i \in \{2, \dots, k\}$, *and* $(y_{i-1} - y_i)/L_i < 0$, *then* $x_{km+i} \to 0$ *as* $m \to \infty$.

(e) *If* $\widehat{p}_i = 1$, *for some* $i \in \{2, \dots, k\}$, *and* $y_{i-1} = y_i$, *then the sequence* $(x_{km+i})_{m \in \mathbb{N}_0}$ *is convergent.*

(f) *If* $\widehat{p}_1 = 1$, *and* $(y_k - L_1 - y_1)/\ L_1 > 0$, *then* $|x_{km+1}| \to \infty$ *as* $m \to \infty$, *if* $x_1 \neq 0$.

(g) *If* $\widehat{p}_1 = 1$, *and* $(y_k - L_1 - y_1)/L_1 < 0$, *then* $x_{km+1} \to 0$ *as* $m \to \infty$.

(h) *If* $\widehat{p}_1 = 1$, *and* $y_k = L_1 + y_1$, *then the sequence* $(x_{km+1})_{m \in \mathbb{N}_0}$ *is convergent.*

Proof. The statements in (a) and (b) follow from the facts that

$$\lim_{j \to \infty} \left| \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \right| = \left| \widehat{p}_i \right|, \quad i \in \{2, \dots, k\} \tag{7.18}$$

if $L_i \neq 0$,

$$\lim_{j\to\infty}\left|\frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}\right| = +\infty, \quad i \in \{2,\dots,k\} \tag{7.19}$$

if $L_i = 0$ and $L_{i-1} \neq 0$,

$$\lim_{j\to\infty}\left|\frac{y_k+(j-1)L_k}{y_1+jL_1}\right| = |\widehat{p}_1|, \tag{7.20}$$

if $L_1 \neq 0$, and

$$\lim_{j\to\infty}\left|\frac{y_k+(j-1)L_k}{y_1+jL_1}\right| = +\infty, \tag{7.21}$$

if $L_1 = 0$ and $L_k \neq 0$.

Now assume that $\widehat{p}_i = 1$ and let $(x_n)_{n\geq -k}$ be a solution of equation (1.1). It is easy to see that there is an $m_2 \in \mathbb{N}$ such that for $j \geq m_2+1$ the terms in the products in (7.17) are positive and that the following asymptotic formulae

$$(1+x)^{-1} = 1-x+O\left(x^2\right), \qquad \ln(1+x) = x+O\left(x^2\right) \tag{7.22}$$

can be applied with $x = (y_{i-1}-y_i)/(jL_i)$, when $i \in \{2,\dots,k\}$ or with $x = (y_k-L_1-y_1)/(jL_1)$. Using these formulae, for the case $i \in \{2,\dots,k\}$, we have that

$$\begin{aligned} x_{km+i} &= x_i\prod_{j=1}^{m}\frac{y_{i-1}+jL_{i-1}}{y_i+jL_i} \\ &= x_i c(m_2)\exp\left(\sum_{j=m_2+1}^{m}\ln\frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}\right) \\ &= x_i c(m_2)\exp\left(\sum_{j=m_2+1}^{m}\ln\left(1+\frac{y_{i-1}-y_i}{jL_i}+O\left(\frac{1}{j^2}\right)\right)\right) \\ &= x_i c(m_2)\exp\left(\sum_{j=m_2+1}^{m}\left(\frac{y_{i-1}-y_i}{jL_i}+O\left(\frac{1}{j^2}\right)\right)\right), \end{aligned} \tag{7.23}$$

where

$$c(m_2) = \prod_{j=1}^{m_2}\frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}. \tag{7.24}$$

Letting $m \to \infty$ in (7.23), using the facts that

$$\sum_{j=m_2+1}^{m} \frac{1}{j} \longrightarrow +\infty \quad \text{as } m \longrightarrow \infty \tag{7.25}$$

and that the series $\sum_{j=m_2+1}^{\infty} O(1/j^2)$ converges, we get statements (c)–(e).

If $\widehat{p}_1 = 1$, that is $L_1 = L_k \neq 0$, then by using (7.22) we get

$$\begin{aligned}
x_{km+1} &= x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1} \\
&= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln \frac{y_k + (j-1)L_k}{y_1 + jL_1} \right) \\
&= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln\left(1 + \frac{y_k - L_1 - y_1}{jL_1} + O\left(\frac{1}{j^2}\right)\right)\right) \\
&= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \left(\frac{y_k - L_1 - y_1}{jL_1} + O\left(\frac{1}{j^2}\right)\right)\right),
\end{aligned} \tag{7.26}$$

where

$$d(m_2) = \prod_{j=1}^{m_2} \frac{y_k + (j-1)L_k}{y_1 + jL_1}. \tag{7.27}$$

Letting $m \to \infty$ in (7.26), using (7.25) and the fact that the series $\sum_{j=m_2+1}^{\infty} O(1/j^2)$ converges, we get statements (f)–(h), as desired. □

7.3. *Case* $q \neq \pm 1$

If $q \neq \pm 1$, then from (7.6) we get

$$y_{km+i} = q^m s_i + t_i, \quad m \in \mathbb{N}_0, \tag{7.28}$$

where

$$s_i = y_i + \frac{L_i}{q-1}, \quad t_i = \frac{L_i}{1-q}, \quad i = \overline{1,k}, \tag{7.29}$$

from (4.6) it follows that

$$x_{km+i} = x_i \prod_{j=1}^{m} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i}, \quad m \in \mathbb{N}_0, \tag{7.30}$$

for $i \in \{2, \ldots, k\}$, and

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1} \prod_{j=2}^{m} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1}, \quad m \in \mathbb{N}_0. \tag{7.31}$$

Note that $t_i = t_j$, if $i \equiv j \pmod{k}$.

Corollary 7.3. *If $0 < |q| < 1$, $\alpha \neq 0$, and $q^j s_i + t_i \neq 0$, for every $j \in \mathbb{N}_0$ and $i \in \{1, \ldots, k\}$, then the following statements hold true.*

(a) *If $|t_{i-1}| < |t_i|$, for some $i \in \{1, \ldots, k\}$, we have that $x_{km+i} \to 0$ as $m \to \infty$.*

(b) *If $|t_{i-1}| > |t_i|$, and $s_i \neq 0$ if $t_i = 0$ for some $i \in \{1, \ldots, k\}$, we have that $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(c) *If $t_{i-1} = t_i \neq 0$, for some $i \in \{1, \ldots, k\}$, then x_{km+i} is convergent.*

(d) *If $t_{i-1} = t_i = 0$, and $|s_{i-1}| < |s_i|$ for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to 0$ as $m \to \infty$.*

(e) *If $t_1 = t_k = 0$, and $|s_k| < |q s_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(f) *If $t_{i-1} = t_i = 0$, and $|s_{i-1}| > |s_i|$ for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(g) *If $t_1 = t_k = 0$, and $|s_k| > |q s_1|$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(h) *If $t_{i-1} = t_i = 0$, and $s_{i-1} = s_i \neq 0$ for some $i \in \{2, \ldots, k\}$, then x_{km+i} is constant.*

(i) *If $t_1 = t_k = 0$, and $s_k = q s_1 \neq 0$, then x_{km+1} is constant.*

(j) *If $t_{i-1} = t_i = 0$, and $s_{i-1} = -s_i \neq 0$ for some $i \in \{2, \ldots, k\}$, then $x_{km+i} = (-1)^m x_i$.*

(k) *If $t_1 = t_k = 0$, and $s_k = -q s_1 \neq 0$, then $x_{km+1} = x_1 y_k (-1)^{m-1}/(q s_1)$.*

(l) *If $t_{i-1} = -t_i \neq 0$, for some $i \in \{1, \ldots, k\}$, then the subsequences x_{2km+i} and $x_{2km+k+i}$ are convergent.*

Proof. Since we have that

$$\lim_{j \to \infty} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{t_{i-1}}{t_i}, \quad i \in \{2, \ldots, k\} \tag{7.32}$$

when $t_i \neq 0$,

$$\lim_{j \to \infty} \left| \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} \right| = +\infty, \quad i \in \{2, \ldots, k\} \tag{7.33}$$

when $|t_{i-1}| > t_i = 0$ and $s_i \neq 0$,

$$\lim_{j \to \infty} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} = \frac{t_k}{t_1}, \tag{7.34}$$

when $t_1 \neq 0$, and

$$\lim_{j\to\infty}\left|\frac{q^{j-1}s_k + t_k}{q^j s_1 + t_1}\right| = +\infty, \tag{7.35}$$

when $|t_k| > t_1 = 0$ and $s_1 \neq 0$, the statements in (a) and (b) easily follow from (7.30)–(7.35).

(c) If $t_{i-1} = t_i \neq 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m}\left(1 + q^j\left(\frac{s_{i-1} - s_i}{t_i}\right) + o\left(q^j\right)\right), \tag{7.36}$$

for $i \in \{2,\dots,k\}$, and if $t_1 = t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1}\prod_{j=2}^{m}\left(1 + q^{j-1}\left(\frac{s_k - q s_1}{t_1}\right) + o\left(q^j\right)\right) \tag{7.37}$$

from which the statement in (c) easily follows.

(d)–(k) If $t_{i-1} = t_i = 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m}\frac{s_{i-1}}{s_i}, \tag{7.38}$$

for $i \in \{2,\dots,k\}$, and if $t_1 = t_k = 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1}\prod_{j=2}^{m}\frac{s_k}{q s_1} \tag{7.39}$$

from which the statements in (d)–(k) easily follow.

(l) If $t_{i-1} = -t_i \neq 0$, then we have that

$$x_{km+i} = x_i \prod_{j=1}^{m}\left[-\left(1 - q^j\left(\frac{s_{i-1} + s_i}{t_i}\right) + o\left(q^j\right)\right)\right] \tag{7.40}$$

for $i \in \{2,\dots,k\}$, and if $t_1 = -t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1}\prod_{j=2}^{m}\left[-\left(1 - q^{j-1}\left(\frac{s_k + q s_1}{t_1}\right) + o\left(q^j\right)\right)\right] \tag{7.41}$$

from which the statement in (l) easily follows. □

Corollary 7.4. *If $|q| > 1$ and $\alpha \neq 0$, and $q^j s_i + t_i \neq 0$, for every $j \in \mathbb{N}_0$ and $i \in \{1, \dots, k\}$, then the following statements hold true.*

(a) *If $|s_{i-1}| < |s_i|$, for some $i \in \{2, \dots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(b) *If $|s_k| < |qs_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(c) *If $|s_{i-1}| > |s_i|$, or $s_i = 0$, $s_{i-1} \neq 0$ and $t_i \neq 0$, for some $i \in \{2, \dots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(d) *If $|s_k| > |qs_1|$, or if $s_1 = 0$, $s_k \neq 0$ and $t_1 \neq 0$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(e) *If $s_{i-1} = s_i \neq 0$, for some $i \in \{2, \dots, k\}$, then the sequence $(x_{km+i})_{m\in\mathbb{N}_0}$ is convergent.*

(f) *If $s_{i-1} = s_i = 0$ and $|t_{i-1}| < |t_i|$ for some $i \in \{2, \dots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(g) *If $s_1 = s_k = 0$ and $|t_k| < |t_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(h) *If $s_{i-1} = s_i = 0$ and $|t_{i-1}| > |t_i|$ for some $i \in \{2, \dots, k\}$, then $|x_{km+i}| \to +\infty$ as $m \to \infty$, if $x_i \neq 0$.*

(i) *If $s_1 = s_k = 0$ and $|t_k| > |t_1|$, then $|x_{km+1}| \to +\infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(j) *If $s_{i-1} = s_i = 0$ and $t_{i-1} = t_i$ for some $i \in \{2, \dots, k\}$, then the sequence x_{km+i} is constant.*

(k) *If $s_1 = s_k = 0$ and $t_1 = t_k$, then the sequence x_{km+1} is constant.*

(l) *If $s_{i-1} = s_i = 0$ and $t_{i-1} = -t_i$ for some $i \in \{2, \dots, k\}$, then the sequence x_{km+i} is two-periodic.*

(m) *If $s_1 = s_k = 0$ and $t_1 = -t_k$, then the sequence x_{km+1} is two periodic.*

(n) *If $s_k = qs_1 \neq 0$, then the sequence $(x_{km+1})_{m\in\mathbb{N}_0}$ is convergent.*

(o) *If $s_{i-1} = -s_i \neq 0$, for some $i \in \{2, \dots, k\}$, then the sequences $(x_{2km+i})_{m\in\mathbb{N}_0}$ and $(x_{2km+k+i})_{m\in\mathbb{N}_0}$, are convergent.*

(p) *If $s_k = -qs_1 \neq 0$, then the sequences $(x_{2km+1})_{m\in\mathbb{N}_0}$ and $(x_{2km+k+1})_{m\in\mathbb{N}_0}$, are convergent.*

Proof. (a)–(d) These statements follow correspondingly from the next relations (which are derived using formulae (7.30) and (7.31)):

$$\lim_{j\to\infty} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{s_{i-1}}{s_i} \tag{7.42}$$

for $i \in \{2, \dots, k\}$ if $s_i \neq 0$, and

$$\lim_{j\to\infty} \left| \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} \right| = +\infty \tag{7.43}$$

for $i \in \{2, \dots, k\}$ if $s_i = 0$, $s_{i-1} \neq 0$ and $t_i \neq 0$;

$$\lim_{j\to\infty} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} = \frac{s_k}{qs_1}, \tag{7.44}$$

if $s_1 \neq 0$, and

$$\lim_{j \to \infty} \left| \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} \right| = +\infty \tag{7.45}$$

if $s_1 = 0$, $s_k \neq 0$ and $t_1 \neq 0$.

(e) If $s_{i-1} = s_i \neq 0$, then from (7.30) we get

$$x_{km+i} = x_i \prod_{j=1}^{m} \left(1 + q^{-j} \left(\frac{t_{i-1} - t_i}{s_{i-1}} \right) + o\left(q^{-j}\right) \right), \tag{7.46}$$

for $i \in \{2, \ldots, k\}$, from which (e) follows.

(f)–(m) If $s_{i-1} = s_i = 0$ for some $i \in \{2, \ldots, k\}$, then for $i \in \{2, \ldots, k\}$ we have

$$\frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{t_{i-1}}{t_i} \tag{7.47}$$

while when $s_1 = s_k = 0$, we have

$$\frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} = \frac{t_k}{t_1} \tag{7.48}$$

from which the statements (f)–(i) easily follow.

(n) If $s_k = q s_1 \neq 0$, then we have

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1} \prod_{j=2}^{m} \left(1 + q^{-j} \left(\frac{t_k - t_1}{s_1} \right) + o\left(q^{-j}\right) \right), \tag{7.49}$$

from which along with the assumption $|q| > 1$ the statement follows.

(o) and (p) If $s_{i-1} = -s_i \neq 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m} \left[-\left(1 - q^{-j} \left(\frac{t_{i-1} + t_i}{s_i} \right) + o\left(q^{-j}\right) \right) \right] \tag{7.50}$$

for $i \in \{2, \ldots, k\}$, and

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1} \prod_{j=2}^{m} \left[-\left(1 - q^{-j} \left(\frac{t_k + t_1}{s_1} \right) + o\left(q^{-j}\right) \right) \right]. \tag{7.51}$$

From (7.50) and (7.51) the statements in (o) and (p) correspondingly follow. □

Acknowledgment

The second author is supported by Grant P201/10/1032 of the Czech Grant Agency (Prague) and by the Council of Czech Government grant MSM 00 216 30519. The fourth author is supported by Grant FEKT-S-11-2-921 of Faculty of Electrical Engineering and Communication, Brno University of Technology. This paper is partially also supported by the Serbian Ministry of Science projects III 41025, III 44006, and OI 171007.

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