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Research Article

The Complex Dynamics of a Stochastic Predator-Prey Model

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A modified stochastic ratio-dependent Leslie-Gower predator-prey model is formulated and analyzed. For the deterministic model, we focus on the existence of equilibria, local, and global stability; for the stochastic model, by applying Itô formula and constructing Lyapunov functions, some qualitative properties are given, such as the existence of global positive solutions, stochastic boundedness, and the global asymptotic stability. Based on these results, we perform a series of numerical simulations and make a comparative analysis of the stability of the model system within deterministic and stochastic environments.

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. In recent years, one of important predator-prey models is Leslie-Gower model [2, 3], which has been extensively studied [4, 5]. And, more and more obvious evidences of biology and physiology show that in many conditions, especially when the predators have to search for food (consequently, have to share or compete for food), a more realistic and general predator-prey system should rely on the theory of ratio-dependence, this is confirmed by lots of experimental results [4, 6]. A ratio-dependent Leslie-Gower predator-prey model [7], takes the form

$$\begin{aligned} \frac{dx}{dt} &= ax(1-bx) - p\left(\frac{x}{y}\right)y, \\ \frac{dy}{dt} &= ey\left(1-\frac{fy}{x}\right), \end{aligned} \tag{1.1}$$

where $x = x(t)$, $y = y(t)$ stand for the population (the density) of the prey and the predator at time t, respectively, and $p(x/y)$ is the predator functional response to prey. And we assumed that the prey grows logistically with growth rate a and carrying capacity $1/b$ in the absence of predation. The predator consumes the prey according to the functional response $p(x/y)$ and grow logistically with growth rate e and carrying capacity x/f proportional to the population size of prey (or prey abundance). The parameter f is a measure of the food quality that the prey provides for conversion into predator birth. The term fy/x of this equation is called the Leslie-Gower term.

On the other hand, the predator y can switch over to other population when the prey population is severely scarce, but its growth will be limited, because we cannot forget the fact that its most favorite food, the prey x, is not in abundance. In this situation, a positive constant m can be added to the denominator, m measures the extent to which the environment provides protection to the predator [8, 9], and the second equation of model (1.1) becomes

$$\frac{dy}{dt} = ey\left(1 - \frac{fy}{m+x}\right). \tag{1.2}$$

Based on the above discussions, in the paper, we will focus on the following ratio-dependent Leslie-Gower model:

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1 - bx - \frac{cy}{x+ny}\right) \triangleq f(x,y), \\ \frac{dy}{dt} &= ey\left(1 - \frac{fy}{m+x}\right) \triangleq g(x,y). \end{aligned} \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, we give some theorems about the stability property of the equilibria of model (1.3). In Section 3, we establish a stochastic model based on model (1.3) and focus on the existence of global positive solutions, stochastic boundedness, and the global asymptotic stability of the stochastic model. In Section 4, we give some numerical examples and make a comparative analysis of the stability of the model system within deterministic and stochastic environments.

2. The Dynamics of Model (1.3)

2.1. Dissipativeness

By standard simple arguments, one can show that the solution of model (1.3) always exists and stays positive. In fact, from the first equation of model (1.3), we can get

$$\limsup_{t\to\infty} x(t) \le \frac{1}{b}. \tag{2.1}$$

Hence, there exists a $T > 0$ such that, for $t > T$, $x \le 1/b$.

From the second equation of model (1.3), we see that, for $t > T$, we have $dy/dt \leq ey(1 - bfy/(mb+1))$. A standard comparison argument shows that

$$\limsup_{t\to\infty} y(t) \leq \frac{(1+bm)}{bf}. \tag{2.2}$$

Thus, we have the following conclusion.

Lemma 2.1. *Model* (1.3) *is dissipative.*

Lemma 2.2. *If $c < n$, then model* (1.3) *is permanent.*

Proof. If $c < n$, from the first equation of model (1.3), we have $dx/dt \geq ax(1 - bx - c/n)$. Therefore, by standard comparison argument, we have

$$\liminf_{t\to\infty} x(t) \geq \frac{n-c}{bn}. \tag{2.3}$$

Hence, for any $\varepsilon > 0$ and large t, $x(t) > (n-c)/bn - \varepsilon$, and

$$\frac{dy}{dt} \geq ey\left(\frac{bmn + nc - bn\varepsilon - bfny}{bmn + nc - bn\varepsilon}\right). \tag{2.4}$$

From the arbitrariness of $\varepsilon > 0$, we can get that

$$\liminf_{t\to\infty} y(t) \geq \frac{bmn + n - c}{bfn}. \tag{2.5}$$

□

2.2. Stability Analysis of the Equilibria

In this section, we will focus on the existence of equilibria and their stabilities of model (1.3).

It is easy to find that model (1.3) always has three boundary equilibria $E_0 = (0,0)$, $E_1 = (1/b, 0)$, $E_2 = (0, m/f)$. And the positive equilibria (x, y) satisfies the equations

$$ax\left(1 - bx - \frac{cy}{x+ny}\right) = 0, \qquad ey\left(1 - \frac{fy}{m+x}\right) = 0, \tag{2.6}$$

which yields

$$b(f+n)x^2 - (f - c - bmn + n)x + (c-n)m = 0. \tag{2.7}$$

For simplicity, we define

$$A_1 = \frac{f - c - bmn + n}{b(f + n)},$$
$$B_1 = \frac{(c - n)m}{b(f + n)}, \tag{2.8}$$
$$C_1 = \sqrt{\frac{(f - c - bmn + n)^2}{b^2(f + n)^2} - \frac{4(c - n)m}{b(f + n)}} = \sqrt{A_1^2 - 4B_1},$$

then (2.7) can be rewritten as

$$x^2 - A_1 x + B_1 = 0. \tag{2.9}$$

Lemma 2.3 (existence of equilibria). (a) *Suppose $A_1 > 0$ and $B_1 > 0$, in* (2.9), *then one has*

(a1) *If $C_1^2 > 0$, then it has two positive roots given by*

$$x_{e1} = \frac{A_1 + C_1}{2}, \qquad x_{e2} = \frac{A_1 - C_1}{2}. \tag{2.10}$$

Therefore, model (1.3) *has two positive equilibria $E_3 = (x_{e1}, y_{e1}) = ((A_1 + C_1)/2, (2m + A_1 + C_1)/2f)$ and $E_4 = (x_{e2}, y_{e2}) = ((A_1 - C_1)/2, (2m + A_1 - C_1)/2f)$.*

(a2) *If $C_1^2 = 0$,* (2.9) *has a unique positive root of multiplicity-2 given by $x_e = \sqrt{m(c - n)/b(f + n)}$. Thus, model* (1.3) *has a unique positive equilibrium*

$$E_e = \left(\sqrt{\frac{m(c - n)}{b(f + n)}}, \frac{bm(f + n) + \sqrt{bm(f + n)(c - n)}}{bf(f + n)}\right). \tag{2.11}$$

(a3) *If $C_1^2 < 0$, there are no positive roots that exist and then, model* (1.3) *has no positive equilibrium.*

(b) $B_1 = 0$, $A_1 > 0$, (2.9) *has a unique positive root $\overline{x} = (f - bmn)/b(f + n)$, model* (1.3) *has a unique positive equilibrium $\overline{E} = (\overline{x}, \overline{y}) = ((f - bmn)/b(f + n), (bm + 1)/b(f + n))$.*

(c) $B_1 < 0$, (2.9) *has a unique positive root $x^* = (A_1 + C_1)/2$, model* (1.3) *has a unique positive equilibrium $E^* = (x^*, y^*) = ((A_1 + C_1)/2, (2m + A_1 + C_1)/2f)$.*

Next, we discuss the local stabilities of these equilibria. Easy to obtain the following results:

Lemma 2.4. (i) *$E_1 = (1/b, 0)$ is a saddle point;*

(ii) *if $c > n$, $E_2 = (0, m/f)$ is a stable node point. If $c \leq n$, $E_2 = (0, m/f)$ is a saddle point.*

Following, let $E = (x, y)$ be an arbitrary positive equilibrium. The Jacobian matrix for $E = (x, y)$ is given by

$$J(E) = \begin{pmatrix} ax\left(-b + \dfrac{cy}{(x+ny)^2}\right) & ax\left(-\dfrac{c}{x+ny} + \dfrac{cny}{(x+ny)^2}\right) \\ \dfrac{e}{f} & -e \end{pmatrix}. \tag{2.12}$$

Then we can get the following:

$$\begin{aligned} \operatorname{tr}(J(E)) &= ax\left(-b + \frac{cy}{(x+ny)^2}\right) - e, \\ \det(J(E)) &= \frac{aex\left[b(f+n)^2x^2 + 2bmn(f+n)x - cfm + bm^2n^2\right]}{(fx+mn+nx)^2}. \end{aligned} \tag{2.13}$$

So the sign of $\det(J(E))$ is determined by

$$F(x) \triangleq b(f+n)^2x^2 + 2bmn(f+n)x - cfm + bm^2n^2. \tag{2.14}$$

Theorem 2.5. *For model* (1.3), *the stabilities of E_3 and E_4 are as follows.*

(a) *The positive equilibrium $E_3 = (x_{e1}, (m+x_{e1})/f) = ((A_1+C_1)/2, (2m+A_1+C_1)/2f)$ is*

(a1) *stable if and only if*

$$e > ax_{e1}\left(-b + \frac{cf(m+x_{e1})}{(fx_{e1}+n(m+x_{e1}))^2}\right); \tag{2.15}$$

(a2) *unstable if and only if*

$$e < ax_{e1}\left(-b + \frac{cf(m+x_{e1})}{(fx_{e1}+n(m+x_{e1}))^2}\right). \tag{2.16}$$

(b) *The positive equilibrium $E_4 = (x_{e2}, (m+x_{e2})/f) = ((A_1-C_1)/2, (2m+A_1-C_1)/2f)$ is a saddle point.*

Proof. (a) At the point E_3, we have the following:

$$\begin{aligned} F(x_{e1}) &= \frac{b(f+n)^2(A_1+C_1)^2}{4} + bmn(f+n)(A_1+C_1) + bm^2n^2 - cfm \\ &= \frac{b(f+n)^2C_1^2}{2} + C_1\left(bmn(f+n) + \frac{b(f+n)^2A_1}{2}\right) > 0, \end{aligned} \tag{2.17}$$

thus, $\det(J(E_3)) > 0$, and the nature of singularity E_3 depends on the trace given by

$$\operatorname{tr}(J(E_3)) = ax_{e1}\left(-b + \frac{cf(m + x_{e1})}{(fx_{e1} + n(m + x_{e1}))^2}\right) - e. \tag{2.18}$$

Clearly,

(a1) If $e > ax_{e1}(-b + cf(m + x_{e1})/(fx_{e1} + n(m + x_{e1}))^2)$, then $\operatorname{tr}(J(E_3)) < 0$, and $E_3 = (x_{e1}, (m + x_{e1})/f)$ is stable.

(a2) If $e < ax_{e1}(-b + cf(m + x_{e1})/(fx_{e1} + n(m + x_{e1}))^2)$, then $\operatorname{tr}(J(E_3)) > 0$, and $E_3 = (x_{e1}, (m + x_{e1})/f)$ is unstable.

(b) At the point E_4, we have the following:

$$\begin{aligned} F(x_{e2}) &= \frac{b(f+n)^2(A_1 - C_1)^2}{4} + bmn(f+n)(A_1 - C_1) + bm^2n^2 - cfm \\ &= \frac{b(f+n)^2}{2}C_1(A_1 - C_1) - bmn(f+n)C_1 - 2fm(f+n)B_1 < 0, \end{aligned} \tag{2.19}$$

then $\det(J(E_4)) < 0$, and E_4 is a saddle point. □

Figure 1 shows the dynamics of model (1.3). In this case, $E_0 = (0,0)$ is a nodal source, $E_1 = (4,0)$ is a saddle point, $E_2 = (0,071429)$ is a nodal sink, $E_4 = (0.31772, 1.5846)$ is a saddle point, and $E_3 = (0.96454, 2.3932)$ is locally asymptotically stable. There exists a separatrix curve determined by the stable manifold of E_4, which divides the behavior of trajectories, that is, the stable manifold of saddle E_4 split the feasible region into two parts such that orbits initiating inside tend to the positive equilibrium E_3, while orbits initiating outside tend to E_3 except for the stable manifolds of E_4.

Theorem 2.6. *The singularity* $\overline{E} = (\overline{x}, (m + \overline{x})/f) = ((f - bmn)/b(f+n), (bm+1)/b(f+n))$ *is*

(a) *stable if and only if*

$$e > a\overline{x}\left(-b + \frac{cf(m + \overline{x})}{(f\overline{x} + n(m + \overline{x}))^2}\right); \tag{2.20}$$

(b) *unstable if and only if*

$$e < a\overline{x}\left(-b + \frac{cf(m + \overline{x})}{(f\overline{x} + n(m + \overline{x}))^2}\right). \tag{2.21}$$

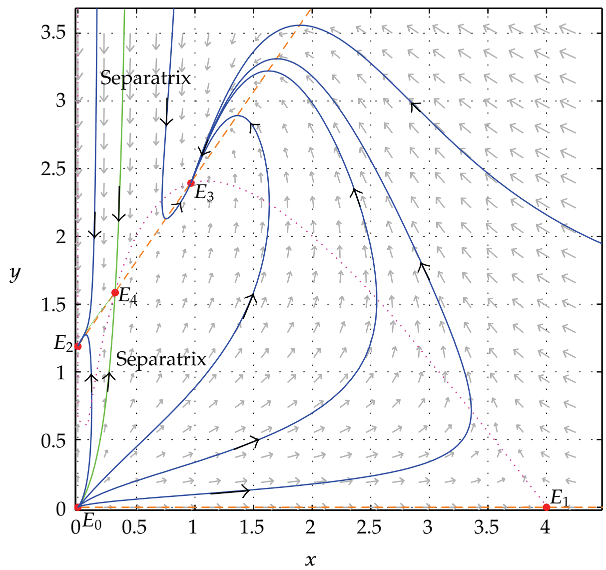

Figure 1: The phase portraits of model (1.3). The parameters are taken as $a = 2$, $b = 0.25$, $c = 0.875$, $e = 1$, $m = 0.95$, and $n = 0.75$, $f = 0.8$. In this case, $E_0 = (0, 0)$ is a source, $E_1 = (4, 0)$ and $E_4 = (0.31772, 1.5846)$ are saddle points, and $E_2 = (0, 071429)$ is a nodal sink, which is stable. The positive equilibrium $E_3 = (0.96454, 2.3932)$ is locally asymptotically stable. The dashed curve is the x-nullcline $f(x, y) = 0$, and the dotted curve is the y-nullcline $g(x, y) = 0$.

Proof. At the point $\overline{E} = (\overline{x}, (m + \overline{x})/f)$, we have

$$F(\overline{x}) = b(f+n)^2 A_1^2 + 2bmn(f+n)A_1 + bm^2n^2 - cfm = b(f+n)^2 A_1^2 + bmn(n+f)A_1 > 0, \tag{2.22}$$

as $F(\overline{x}) > 0$, then $\det(J(\overline{E})) > 0$, and the nature of singularity $\overline{E} = (\overline{x}, (m + \overline{x})/f)$ depends on the trace given by

$$\operatorname{tr}\left(J\left(\overline{E}\right)\right) = a\overline{x}\left(-b + \frac{cf(m+\overline{x})}{\left(f\overline{x} + n(m+\overline{x})\right)^2}\right) - e. \tag{2.23}$$

Clearly,

(a) If $e > a\overline{x}(-b + cf(m+\overline{x})/(f\overline{x} + n(m+\overline{x}))^2)$, then $\operatorname{tr}(J(\overline{E})) < 0$, and $\overline{E} = (\overline{x}, (m+\overline{x})/f)$ is stable.

(b) If $e < a\overline{x}(-b + cf(m+\overline{x})/(f\overline{x} + n(m+\overline{x}))^2)$, then $\operatorname{tr}(J(\overline{E})) > 0$, and $\overline{E} = (\overline{x}, (m+\overline{x})/f)$ is unstable. □

Theorem 2.7. *Singularity $E^* = (x^*, (m + x^*)/f)$ is*

(a) *stable if and only if*

$$e > ax^*\left(-b + \frac{cf(m+x^*)}{\left(fx^* + n(m+x^*)\right)^2}\right); \tag{2.24}$$

(b) *unstable if and only if*

$$e < ax^*\left(-b + \frac{cf(m+x^*)}{(fx^* + n(m+x^*))^2}\right). \tag{2.25}$$

Proof. At the point $E^* = (x^*, (m+x^*)/f)$, as $B_1 < 0$, $C_1^2 = A_1^2 - 4B$, so

$$\begin{aligned} F(x^*) &= \frac{b(f+n)^2(A_1+C_1)^2}{4} + bmn(f+n)(A_1+C_1) + bm^2n^2 - cfm \\ &= \frac{b(f+n)^2(A_1+C_1)^2}{4} + bmn(f+n)C_1 + m(n+f)(n-c) > 0, \end{aligned} \tag{2.26}$$

as $F(x^*) > 0$, then $\det(J(E^*)) > 0$, and the nature of singularity $E^* = (x^*, (m+x^*)/f)$ depends on the trace given by

$$\operatorname{tr}(J(E^*)) = ax^*\left(-b + \frac{cf(m+x^*)}{(fx^* + n(m+x^*))^2}\right) - e. \tag{2.27}$$

Clearly,

(a) if $e > ax^*(-b + cf(m+x^*)/(fx^* + n(m+x^*))^2)$, then $\operatorname{tr}(J(E^*)) < 0$, $E^* = (x^*, (m+x^*)/f)$ is stable;

(b) if $e < ax^*(-b + cf(m+x^*)/(fx^* + n(m+x^*))^2)$, then $\operatorname{tr}(J(E^*)) > 0$, $E^* = (x^*, (m+x^*)/f)$ is unstable. □

Theorem 2.8. *Singularity $E_e = (x_e, (m+x_e)/f)$ is*

(a) *a non-hyperbolic attractor node if and only if*

$$e > ax_e\left(-b + \frac{cf(m+x_e)}{(fx_e + n(m+x_e))^2}\right); \tag{2.28}$$

(b) *a non-hyperbolic repellor node if and only if*

$$e < ax_e\left(-b + \frac{cf(m+x_e)}{(fx_e + n(m+x_e))^2}\right). \tag{2.29}$$

Proof. At the point $(x_e, (m+x_e)/f)$, we have

$$F(x_e) = \frac{b(f+n)^2A_1^2}{4} + bmn(f+n)A_1 + bm^2n^2 - cfm = 0, \tag{2.30}$$

then, $\det(J(E_e)) = 0$, and $\operatorname{tr}(J(E_e)) = ax_e(-b + cf(m+x_e)/(fx_e + n(m+x_e))^2) - e$.

Hence we can conclude (2.28) and (2.29). □

Theorem 2.9. *If $n < c < n(bm+1)$ and $fn + (n-c)(bmn+n-c) < 0$ hold the boundary equilibria $E_2 = (0, m/f)$ of model (1.3) is globally asymptotically stable.*

Proof. Since $\limsup_{t\to\infty} x(t) \le 1/b$ and $\liminf_{t\to\infty} y(t) \ge (bmn+n-c)/bfn$, from the first equation of model (1.3), for any $\mu > 0$, there exists a $T_1 > 0$, for all $t \ge T_1$, we have

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1 - bx - \frac{c}{x/y+n}\right) \\ &< ax\left(1 - bx - \frac{c(bmn+n-c-bf\mu)}{fn(1+b\mu)+n(bmn+n-c-bf\mu)}\right). \end{aligned} \tag{2.31}$$

From the arbitrariness of $\mu > 0$, we can get that

$$\frac{dx}{dt} < ax\left(1 - bx - \frac{c(bmn+n-c)}{n(bmn+f+n-c)}\right). \tag{2.32}$$

As $1 - c(bmn+n-c)/n(bmn+f+n-c) < 0$, by standard comparison arguments, it follows that

$$\limsup_{t\to\infty} x(t) \le 0, \tag{2.33}$$

thus,

$$\lim_{t\to\infty} x(t) = 0. \tag{2.34}$$

As a result, using the second equation of model (1.3), one can easily know that $\lim_{t\to\infty} y(t) = m/f$. The proof is complete. □

Figure 2 shows the dynamics of model (1.3). In this case, $E_0 = (0,0)$ is a nodal source, $E_1 = (4,0)$ is a saddle point, and $E_2 = (0, 1.1875)$ is globally asymptotically stable, that is, all orbits tend to the equilibrium E_2 for any initial values.

Theorem 2.10. *Assume $B_1 < 0$, $E^* = (x^*, y^*)$ is globally asymptotically stable, if the following conditions hold*

(i) $1 - bx^* < b(\zeta + n\delta)$;

(ii) $(e(\eta + n\varsigma) - (x^*/y^*)(1-bx^*)(\zeta+m))^2 < 4aef(b(\zeta+n\delta) - (1-bx^*))(\eta+m)(\eta+n\varsigma)$, *where* $\eta = 1/b$, $\varsigma = (1+bm)/bf$, $\zeta = (n-c)/bn$, $\delta = (bmn+n-c)/bnf$.

Proof. Define a Lyapunov function:

$$V(x,y) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi} d\xi + \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta, \tag{2.35}$$

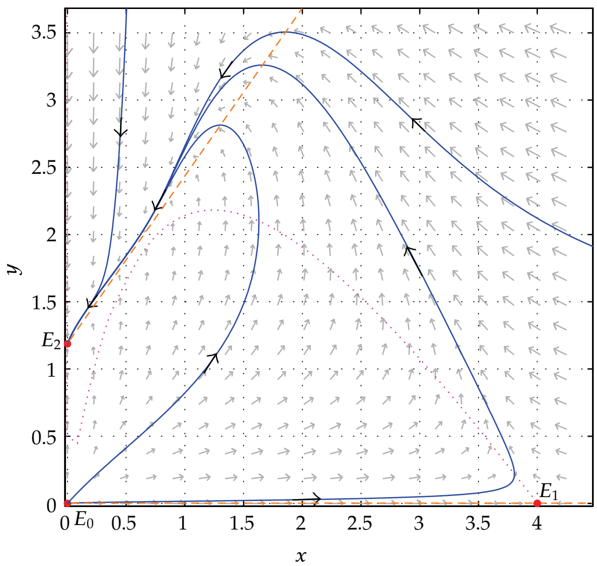

Figure 2: The phase portraits of model (1.3). The parameters are taken as $a = 2$, $b = 0.25$, $c = 0.875$, $e = 1$, $m = 0.95$, and $n = 0.7$, $f = 0.8$. In this case, $E_0 = (0, 0)$ is a nodal source, $E_1 = (4, 0)$ is a saddle point. The boundary equilibria $E_2 = (0, 1.1875)$ is globally asymptotically stable. The dashed curve is the x-nullcline $f(x, y) = 0$, and the dotted curve is the y-nullcline $g(x, y) = 0$.

so,

$$\begin{aligned}
\frac{dV}{dt} &= \frac{x-x^*}{x}\frac{dx}{dt} + \frac{(y-y^*)}{y}\frac{dy}{dt} \\
&= a(x-x^*)\left(1-bx-\frac{cy}{x+ny}\right) + e(y-y^*)\left(1-\frac{fy}{x+m}\right) \\
&= a\left(-b+\frac{cy^*}{(x^*+ny^*)(x+ny)}\right)(x-x^*)^2 \\
&\quad + \left(-\frac{acx^*}{(x^*+ny^*)(x+ny)} + \frac{efy^*}{(x^*+m)(x+m)}\right)(x-x^*)(y-y^*) - \frac{ef(y-y^*)^2}{x+m} \\
&= -A_2(x-x^*)^2 + B_2(x-x^*)(y-y^*) - C_2(y-y^*)^2,
\end{aligned} \tag{2.36}$$

let

$$\begin{gathered}
A_2 = a\left(b - \frac{cy^*}{(x^*+ny^*)(x+ny)}\right) = a\left(b - \frac{1-bx^*}{x+ny}\right), \\
B_2 = -\frac{acx^*}{(x^*+ny^*)(x+ny)} + \frac{efy^*}{(x^*+m)(x+m)} = -\frac{ax^*(1-bx^*)}{y^*(x+ny)} + \frac{e}{x+m}, \\
C_2 = \frac{ef}{x+m}.
\end{gathered} \tag{2.37}$$

If

$$A_2 > 0, \text{ that is, } 1 - bx^* < b(x + ny) < b(\zeta + n\delta),$$

$$B_2^2 < 4A_2C_2, \text{ that is, } \left(-\frac{ax^*(1-bx^*)}{y^*(x+ny)} + \frac{e}{x+m}\right)^2 < \frac{4aef}{x+m}\left(b - \frac{1-bx^*}{x+ny}\right) \text{ hold, the } dV/dt < 0. \tag{2.38}$$

Hence,

$$\left(e(x+ny) - \frac{ax^*}{y^*}(1-bx^*)(x+m)\right)^2 < 4aef\left(b(x+ny) - (1-bx^*)\right)(x+m)(x+ny), \tag{2.39}$$

according to Lemma 2.2, we can obtain that

$$\left(e(x+ny) - \frac{ax^*}{y^*}(1-bx^*)(x+m)\right)^2 < \left(e(\eta+n\varsigma) - \frac{ax^*}{y^*}(1-bx^*)(\zeta+m)\right)^2,$$
$$4aef(b(\zeta+n\delta) - (1-bx^*))(\eta+m)(\eta+n\varsigma) < 4aef\left(b(x+ny) - (1-bx^*)\right)(x+m)(x+ny), \tag{2.40}$$

so,

$$\left(e(\eta+n\varsigma) - \frac{x^*}{y^*}(1-bx^*)(\zeta+m)\right)^2 < 4aef(b(\zeta+n\delta) - (1-bx^*))(\eta+m)(\eta+n\varsigma). \tag{2.41}$$

Considering $A_2 > 0, B_2^2 < 4A_2C_2$, we obtain $dV/dt < 0$. This ends the proof. □

Figure 3 shows the dynamics of model (1.3) for the case of $B1 < 0$. In this case, $E_0 = (0,0)$ is a nodal source, $E_1 = (4,0)$ is a saddle point, $E_2 = (0,1.1875)$ is a saddle point, and $E^* = (1.4761, 3.0326)$ is globally asymptotically stable, that is, all orbits tend to the equilibrium E^* for any initial values.

3. The Stochastic Model

Those important and useful works on deterministic models provide a great insight into the effect of the pollution. In the real world, population dynamics is inevitably subjected to environmental noise (see e.g., [10, 11]), which is an important component in an ecosystem. May [12] pointed out the fact that due to environmental noise, the birth rates, carrying capacity, competition coefficients, and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent.

In this part, we focus on the stochastic stability analysis of model (1.3).

Taking into account the effect of randomly fluctuating environment, we incorporate white noise in each equations of model (1.3). We assume that fluctuations in the environment

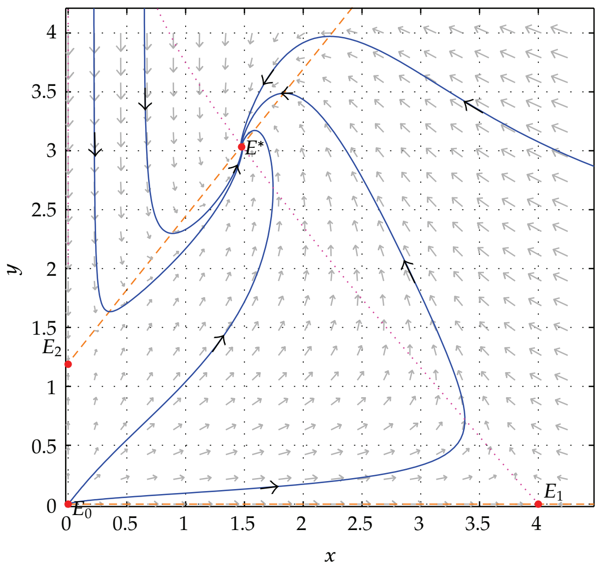

Figure 3: The phase portraits of model (1.3). The parameters are taken as $a = 2$, $b = 0.25$, $c = 0.875$, $e = 1$, $m = 0.95$, $n = 0.9$, and $f = 0.8$. In this case, $E_1 = (4, 0)$ and $E_2 = (0, 1.1875)$ are saddle points, the positive interior equilibrium point $E^* = (1.4761, 3.0326)$ is globally asymptotically stable. The dashed curve is the x-nullcline $f(x, y) = 0$, and the dotted curve is the y-nullcline $g(x, y) = 0$.

will manifest themselves mainly as fluctuations in the growth rates of the prey population and the predator population, set

$$a \longrightarrow a + \alpha \dot{B}_1(t), \qquad e \longrightarrow e + \beta \dot{B}_2(t), \tag{3.1}$$

then the stochastic version of model (1.3) is given by the following Itô type

$$\begin{aligned} dx &= x\left(1 - bx - \frac{cy}{x + ny}\right)(a\,dt + \alpha\, dB_1(t)), \\ dy &= y\left(1 - \frac{fy}{x + m}\right)(e\,dt + \beta\, dB_2(t)), \end{aligned} \tag{3.2}$$

where $B_i(t)$, $i = (1, 2)$ are the 1-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t \in R_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while F_0 contains all $\mathcal{P}$-null sets) and $\dot{B}_1(t)$, $\dot{B}_2(t)$ are, respectively, white noises with possible intensity α^2, β^2.

3.1. Existence of Global Positive Solutions

Lemma 3.1. *There is a unique local positive solution* $(x(t), y(t))$ *for* $t \in [0, \tau_e)$ *to model* (3.2) *almost surely (a.s.) for the initial value* $x_0 > 0$, $y_0 > 0$, *where* τ_e *is the explosion time.*

Proof. Consider the equations

$$du = \left\{a\left[1 - be^u - \frac{ce^v}{ne^v + e^u}\right] - \frac{1}{2}\alpha^2\left[1 - be^u - \frac{ce^v}{ne^v + e^u}\right]^2\right\}dt$$
$$+ \alpha\left[1 - be^u - \frac{ce^v}{ne^v + e^u}\right]dB_1(t),$$
$$dv = \left\{e\left[1 - \frac{fe^v}{m + e^u}\right] - \frac{1}{2}\beta^2\left[1 - \frac{fe^v}{m + e^u}\right]^2\right\}dt$$
$$+ \beta\left[1 - \frac{fe^v}{m + e^u}\right]dB_2(t), \tag{3.3}$$

on $t \geq 0$ with initial value $u(0) = \ln x_0$, $v(0) = \ln y_0$.

It is easy to see that the coefficients of model (3.3) satisfy the local Lipschitz condition, then there is a unique local solution $u(t), v(t)$ on $[0, \tau_e)$ [13]. Therefore, by Itô formula, $x(t) = e^u(t), y(t) = e^v(t)$ are the unique positive local solutions to model (3.3) with initial value $x_0 > 0$, $y_0 > 0$. □

Lemma 3.1 only tells us that there has a unique local positive solution to model (3.2). Next, we show this solution is global, that is, $\tau_e = \infty$.

Theorem 3.2. *Consider model* (3.2)*, for any given initial value* $(x_0, y_0) \in R_+^2$*, there is a unique solution* $(x(t), y(t))$ *on* $t \geq 0$ *and the solution will remain in* R_+^2 *with probability 1, where* $R_+^2 = \{x \in R^2 \mid x_i > 0,\ i = 1, 2\}$.

Proof. For convenience of statement, we introduce some notations. Define

$$F(x, y) = \frac{cy}{ny + x}, \qquad G(x, y) = \frac{fy}{m + x}. \tag{3.4}$$

Let $n_0 > 0$ be sufficiently large for x_0 and y_0 lying within the interval $[1/n_0, n_0]$. For each integer $n > n_0$, define the stopping times

$$\tau_n = \inf\left\{t \in [0, \tau_e] : x(t) \notin \left(\frac{1}{n}, n\right) \text{ or } y(t) \notin \left(\frac{1}{n}, n\right)\right\}. \tag{3.5}$$

Assume inf $\emptyset = \infty$ (as usual $\emptyset$ = the empty set). Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty}\tau_n$, then $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., and $N(t) \in R_+$ a.s., for all $t \geq 0$. In other words, to complete the proof all we need to prove that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathcal{P}\{\tau_\infty \leq T\} > \varepsilon$. There is an integer $n_1 \geq n_0$, such that

$$\mathcal{P}\{\tau_n \leq T\} \geq \varepsilon, \quad n \geq n_1. \tag{3.6}$$

Define a C^2 function $V : R_+^2 \to R_+$ by

$$V(x,y) = \left(\sqrt{x} - 1 - \frac{\ln x}{2}\right) + \left(\sqrt{y} - 1 - \frac{\ln y}{2}\right). \tag{3.7}$$

Obviously, $V(x,y)$ is nonnegative. And

$$\begin{aligned}
dV &= \frac{a(\sqrt{x}-1)}{2}(1 - bx - F(x,y))dt + \frac{\alpha^2(2-\sqrt{x})}{8}[1 - bx - F(x,y)]^2 dt \\
&\quad + \frac{\alpha(\sqrt{x}-1)}{2}[1 - bx - F(x,y)]dB_1 t + \frac{e(\sqrt{y}-1)}{2}[1 - G(x,y)]dt \\
&\quad + \frac{\beta^2(2-\sqrt{y})}{8}[1 - G(x,y)]^2 dt + \frac{\beta(\sqrt{y}-1)}{2}[1 - G(x,y)]dB_2 t \\
&\le \frac{a}{2}[\sqrt{x} + bx + F(x,y)]dt + \frac{e}{2}[\sqrt{y} + G(x,y)]dt \\
&\quad + \frac{\alpha^2}{8}\left[2bx^{3/2} - b^2x^{5/2} + 2\sqrt{x}F(x,y) + 2 + 2b^2x^2 + 4bxF(x,y) + 2F^2(x,y)\right]dt \\
&\quad + \frac{\beta^2}{8}\left[2\sqrt{y}G(x,y) + 2 + 2G(x,y)^2\right]dt + \frac{\alpha}{2}(\sqrt{x}-1)\left[1 - \frac{bx}{a} - F(x,y)\right]dB_1(t) \\
&\quad + \frac{\beta}{2}(\sqrt{y}-1)[1 - G(x,y)]dB_2(t) \\
&\le k_1 + k_2 + \frac{\alpha}{2}(\sqrt{x}-1)[1 - bx - F(x,y)]dB_1(t) + \frac{\beta}{2}(\sqrt{y}-1)[1 - G(x,y)]dB_2(t),
\end{aligned} \tag{3.8}$$

where k_1, k_2 are positive numbers. Integrating both sides of the above inequality from 0 to $\tau_n \wedge T$ and then taking expectations, yields

$$EV(x(\tau_n \wedge T), y(\tau_n \wedge T)) \le V(x_0, y_0) + (K_1 + K_2)T. \tag{3.9}$$

Set $\Omega_n = \{\tau_n \le T\}$ for $n \ge n_1$ and by (3.6), we have $\mathcal{P}(\Omega_n) \ge \varepsilon$. Note that for every $\omega \in \Omega_n$, there is some i such that $x_i(\tau_n, \omega)$ equals either n or $1/n$ for $i = 1, 2$, hence $V(x(\tau_n, \omega), y(\tau_n, \omega))$ is no less than $\min\{(\sqrt{n} - 1 - \ln n/2), (\sqrt{1/n} - 1 - \ln(1/n)/2)\}$. It then follows from (3.9) that

$$\begin{aligned}
V(x_0, y_0) + (K_1 + K_2)T &\ge E\left[I_{\Omega_n}(\omega)V(x(\tau_n), y(\tau_n))\right] \\
&\ge \varepsilon \min\left\{\left(\sqrt{n} - 1 - \frac{\ln n}{2}\right), \left(\sqrt{\frac{1}{n}} - 1 - \frac{\ln n}{2}\right)\right\},
\end{aligned} \tag{3.10}$$

where I_{Ω_n} is the indicator function of Ω_n. Let $n \to \infty$, then

$$\infty > V(x_0, y_0) + (K_1 + K_2)T = \infty \quad \text{a.s.} \tag{3.11}$$

This completes the proof. □

3.2. Stochastic Boundedness

Let us now recall the definition of stochastically ultimate boundedness [14–16].

Definition 3.3. The solution of model (3.2) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $\chi = \chi(\omega)$, such that for any initial data $(x_0, y_0) \in R_+^2$, the solution of model (3.2) has the property that

$$\limsup_{t \to \infty} P\{|x(t), y(t)| > \chi\} \leq \varepsilon. \tag{3.12}$$

Lemma 3.4. *For any $\theta \in (0,1)$, there is a positive constant $H = H(\theta) > 0$, which is independent of the initial data $(x_0, y_0) \in R_+^2$, such that the solution of model* (3.2) *has the property that*

$$\limsup_{t \to \infty} \left(E|(x(t), y(t))|^{\theta} \right) \leq H. \tag{3.13}$$

Proof. Define

$$V(x,y) = x^{\theta} + y^{\theta}, \quad (x,y) \in R_+^2. \tag{3.14}$$

For the sake of discussion, we rewrite the above as

$$V(x,y) = V_1 + V_2, \qquad V_1 = x^{\theta}, \qquad V_2 = y^{\theta}. \tag{3.15}$$

By using the Itô formula, we have

$$\begin{aligned} dV(x,y) &= LV(x,y)dt + \theta\alpha x^{\theta}[1 - bx - F(x,y)]dB_1(t) \\ &\quad + \theta\beta y^{\theta}[1 - G(x,y)]dB_2(t), \\ LV(x,y) &= a\theta x^{\theta}[1 - bx - F(x,y)] + \frac{\theta(\theta-1)\alpha^2 x^{\theta}}{2}[1 - bx - F(x,y)]^2 \\ &\quad + e\theta y^{\theta}[1 - G(x,y)] + \frac{\theta(\theta-1)\beta^2 y^{\theta}}{2}[1 - G(x,y)]^2, \end{aligned} \tag{3.16}$$

as $0 < \theta < 1$, we have

$$\begin{aligned}
LV(x,y) &\le a\theta x^{\theta} - \frac{\theta(1-\theta)\alpha^2 x^{\theta}}{2}\left[1-bx-F(x,y)\right]^2 \\
&\quad + e\theta y^{\theta}\left[1-G(x,y)\right] - \frac{\theta(1-\theta)\beta^2 y^{\theta}}{2}\left[1-G(x,y)\right]^2 \\
&\le a\theta x^{\theta} - \frac{\theta(1-\theta)\alpha^2 x^{\theta}}{2}\left[b^2x^2 - 2bx - \frac{2c}{n}\right] \\
&\quad + e\theta y^{\theta} - \frac{\theta(1-\theta)\beta^2 y^{\theta}}{2}\left[\frac{-2fy}{m}\right] \\
&= a\theta x^{\theta} - \frac{\theta(1-\theta)\alpha^2 x^{\theta}}{2}\left[b^2x^2 - 2bx - \frac{2c}{n}\right] + x^{\theta} \\
&\quad + e\theta y^{\theta} - \frac{\theta(1-\theta)\beta^2 y^{\theta}}{2}\left[\frac{-2fy}{m}\right] + y^{\theta} - V(x,y) \\
&\le H_1 - V(x,y),
\end{aligned} \tag{3.17}$$

here H_1 is an integer. So

$$dV(x,y) \le (H_1 - V(x,y))dt + \theta\alpha x^{\theta}\left[1-bx-F(x,y)\right]dB_1(t). \tag{3.18}$$

Now, by using the Itô formula again, we have

$$\begin{aligned}
d(e^t V(x,y)) &= e^t\left[V(x,y)dt + dV(x,y)\right] \\
&\le e^t H_1 dt + e^t\theta\alpha x^{\theta}\left[1-bx-F(x,y)\right]dB_1(t) \\
&\quad + e^t\theta\beta\, y^{\theta}\left[1-G(x,y)\right]dB_2(t).
\end{aligned} \tag{3.19}$$

According to the above, we can easily get

$$\begin{gathered}
e^t E(V(x,y)) \le V(x_0,y_0) + H_1(e^t - 1), \\
\limsup_{t\to\infty} EV(x(t),y(t)) \le H.
\end{gathered} \tag{3.20}$$

On the other hand,

$$\begin{aligned}
\left|(x,y)\right|^{\theta} &\le 2^{\theta/2}\max\left\{x^{\theta}, y^{\theta}\right\} \\
&\le 2^{\theta/2} V(x,y),
\end{aligned} \tag{3.21}$$

so, we can obtain

$$\begin{aligned}
&\limsup_{t\to\infty} E\left|\left(x(t), y(t)\right)\right|^{\theta} \le 2^{\theta/2}, \\
&\limsup_{t\to\infty} EV\left(x(t), y(t)\right) \le 2^{\theta/2} H.
\end{aligned} \tag{3.22}$$

Set $K(\theta) = 2^{\theta/2}H$, this completes the proof. □

Theorem 3.5. *The solutions of model* (3.2) *is stochastically ultimately bounded.*

Proof. By Lemma 3.4, let $\theta = 1/2$, there is a $K_1 > 0$ such that

$$\limsup_{t\to\infty} E\left|\left(x(t), y(t)\right)\right|^{1/2} \le K_1. \tag{3.23}$$

Now, for any $\varepsilon > 0$, let $\chi = K_1^2/\varepsilon^2$. Then by Chebyshevs inequality

$$P\left\{\left|x(t), y(t)\right| > \chi\right\} \le \frac{E\left|\left(x(t), y(t)\right)\right|^{1/2}}{\sqrt{\chi}}. \tag{3.24}$$

Hence,

$$\limsup_{t\to\infty} P\left\{\left|x(t), y(t)\right| > \chi\right\} \le \varepsilon. \tag{3.25}$$

This ends the proof. □

Next, we study the asymptotic properties of the moment solutions of model (3.2).

Theorem 3.6. *For any given $\theta \in (0,1)$, there is a $K = K(\theta) > 0$, let $(x(t), y(t))$ be the solution of model* (3.2) *with any initial value (x_0, y_0), then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E\left[x^{2+\theta}(s) + y^{2+\theta}(s)\right] ds \le K. \tag{3.26}$$

Proof. Set $V(x,y) : R_+^2 \to R_+$, from Lemma 3.4, we have

$$\begin{aligned}
LV(x,y) \le{}& a\theta x^{\theta} - \frac{\theta(1-\theta)\alpha^2 x^{\theta}}{2}\left[b^2x^2 - 2bx - \frac{2c}{n}\right] \\
&+ e\theta y^{\theta} - \frac{\theta(1-\theta)\beta^2 y^{\theta}}{2}\left(\frac{-2fy}{m}\right),
\end{aligned} \tag{3.27}$$

so, we get

$$\begin{aligned}
&LV + \frac{\theta(1-\theta)\alpha^2 b^2 x^{\theta+2}}{4} + \frac{\theta(1-\theta)\beta^2 f y^{\theta+2}}{4m} \\
&\quad \le a\theta x^{\theta} - \frac{\theta(1-\theta)\alpha^2 b^2 x^{\theta+2}}{4} + \theta(1-\theta)\alpha^2 x^{\theta}\left(bx + \frac{c}{n}\right) \\
&\qquad + e\theta y^{\theta} + \frac{\theta(1-\theta)\beta^2 f y^{\theta+1}}{m} + \frac{\theta(1-\theta)\beta^2 f y^{\theta+2}}{4m} \\
&\quad \le M,
\end{aligned} \tag{3.28}$$

here M is a positive number, so

$$\begin{aligned}
&LV + \frac{\theta(1-\theta)\gamma^2\sigma^2\left[x^{\theta+2} + y^{\theta+2}\right]}{4} \\
&\quad \le \frac{\theta(1-\theta)\alpha^2 b^2 x^{\theta+2}}{4} + \frac{\theta(1-\theta)\beta^2 f y^{\theta+2}}{4m} \\
&\quad \le M.
\end{aligned} \tag{3.29}$$

Let $\gamma^2\sigma^2 = \min\{\alpha^2 b^2, \beta^2 f/m\}$, we can get

$$\begin{aligned}
&EV(x,y) + \frac{\theta(1-\theta)\gamma^2\sigma^2}{4}\int_0^t E\left[x^{2+\theta}(s) + y^{2+\theta}(s)\right]ds \\
&\quad \le V(x_0 + y_0) + Mt.
\end{aligned} \tag{3.30}$$

That is,

$$\int_0^t E\left[x^{2+\theta}(s) + y^{2+\theta}(s)\right]ds \le \frac{4\left[V(x_0 + y_0) + Mt\right]}{\theta(1-\theta)\gamma^2\sigma^2}. \tag{3.31}$$

Set $K = 4M/\theta(1-\theta)\gamma^2\sigma^2$, so

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E\left[x^{2+\theta}(s) + y^{2+\theta}(s)\right]ds \le K. \tag{3.32}$$

□

3.3. Stochastic Asymptotic Stability

Note that a solution of model (1.3) is also a solution of model (3.2), so, in the following, we will focus on stochastic asymptotic stability of the positive equilibria of model (3.2). As an example, we only give the proof of the unique positive equilibrium $E^* = (x^*, y^*)$ of model (3.2).

Theorem 3.7. *Let*

$$A_3 = -ab + \frac{acy^*}{(ny^* + x^*)} + \frac{\alpha^2 b^2 x^*}{2} + \frac{\alpha^2 c^2 x^* (y^*)^2}{2(ny^* + x^*)^2} + \frac{\beta^2 f^2 (y^*)^3}{2(m + x^*)^2},$$
$$B_3 = \frac{\alpha^2 c^2 (x^*)^3}{2(ny^* + x^*)^2} + \frac{\beta^2 (fm + x^*)^2 y^*}{2(m + x^*)^2}, \tag{3.33}$$
$$C_3 = \frac{\alpha^2 bc (x^*)^2}{(ny^* + x^*)} + \frac{\alpha^2 c^2 y^* (x^*)^2}{(ny^* + x^*)^2} + \frac{fy^*}{(m + x^*)} + \frac{\beta^2 (fm + x^*) f (y^*)^2}{(m + x^*)^2},$$

if

$$(bm+1)n^2 + \left[(1-b)f - c\right]n - cf \ge 0, \qquad [(m-1)b + 1]n - c \ge 0,$$
$$A_3 < 0, \qquad C_3^2 - 4A_3 B_3 < 0, \tag{3.34}$$

when $B_1 < 0$, then the equilibrium position $E^* = (x^*, y^*)$ of model (3.2) is stochastically asymptotically stable in the large, that is, for any initial data $(x(0), y(0))$, the solution of model (3.2) has the property that

$$\lim_{t\to\infty} (x(t)) = x^*, \qquad \lim_{t\to\infty} (y(t)) = y^*, \quad a.s. \tag{3.35}$$

Proof. From the theory of stability of stochastic differential equations [13], we only need to establish a Lyapunov function $V(z)$ satisfying $LV(z) \le 0$ and the identity holds if and only if $z = z^*$, where $z = z(t)$ is the solution of the stochastic differential equation

$$dz = f(z(t), t)dt + g(z(t), t)dB(t), \tag{3.36}$$

z^* is the equilibrium position of model (3.36), and

$$LV(z,t) = V_z(z,t) f(z,t) + \frac{1}{2} \operatorname{tr}\left[g^T(z,t) V_{zz}(z,t) g(z,t)\right]. \tag{3.37}$$

Define Lyapunov functions

$$V_1(x) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi} d\xi, \qquad V_2(x) = \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta, \tag{3.38}$$

the nonnegativity of this function can be observed from $u - 1 - \ln u \geq 0$ on $u > 0$. Applying Itô formula leads to

$$
\begin{aligned}
LV_1(x) &= a(x-x^*)\left[1-bx-F(x,y)\right] + \frac{\alpha^2 x^*}{2}\left[1-bx-F(x,y)\right]^2 \\
&= a(x-x^*)\left[bx^*-bx+F(x^*,y^*)-F(x,y)\right] + \frac{\alpha^2 x^*}{2}\left[bx^*-bx+F(x^*,y^*)-F(x,y)\right]^2 \\
&= \Bigg\{-ab+\frac{acy^*}{(ny^*+x^*)(ny+x)}+\frac{\alpha^2 b^2 x^*}{2}-\frac{bc\alpha^2 x^* y^*}{(ny^*+x^*)(ny+x)} \\
&\quad +\frac{c^2\alpha^2 x^*(y^*)^2}{2(ny^*+x^*)^2(ny+x)^2}\Bigg\}(x-x^*)^2+\frac{c^2\alpha^2 (x^*)^3}{2(ny^*+x^*)^2(ny+x)^2}(y-y^*)^2 \\
&\quad -\frac{acx^*}{(ny^*+x^*)(ny+x)}(x-x^*)(y-y^*) \\
&\quad +\frac{bc\alpha^2 (x^*)^2}{(ny^*+x^*)(ny+x)}(x-x^*)(y-y^*)-\frac{c^2\alpha^2 y^*(x^*)^2}{(ny^*+x^*)^2(ny+x)^2}(x-x^*)(y-y^*),
\end{aligned}
\tag{3.39}
$$

when $ny + x \geq 1$, according to Lemma 2.2, we have $(bm+1)n^2 + [(1-b)f - c]n - cf \geq 0$, so,

$$
\begin{aligned}
LV_1(x) \leq{}& \left\{-ab+\frac{acy^*}{(ny^*+x^*)}+\frac{\alpha^2 b^2 x^*}{2}+\frac{c^2\alpha^2 x^*(y^*)^2}{2(ny^*+x^*)^2}\right\}(x-x^*)^2 \\
&+\frac{c^2\alpha^2 (x^*)^3}{2(ny^*+x^*)^2}(y-y^*)^2 \\
&+\left\{\frac{bc\alpha^2 (x^*)^2}{(ny^*+x^*)}+\frac{c^2\alpha^2 y^*(x^*)^2}{(ny^*+x^*)^2}\right\}|(x-x^*)|\left|(y-y^*)\right|.
\end{aligned}
\tag{3.40}
$$

Similarly, by using the Itô formula, we can obtain

$$
LV_2(y) = e(y-y^*)\left[1-G(x,y)\right] + \frac{\beta^2 y^*}{2}\left[1-G(x,y)\right]^2,
\tag{3.41}
$$

when $m + x \geq 1$, according to Lemma 2.2, we have $[(m-1)b+1]n - c \geq 0$, so

$$
\begin{aligned}
LV_2(y) \leq{}& \frac{\beta^2 (fm+x^*)^2 y^*}{2(m+x^*)^2}(y-y^*)^2+\frac{\beta^2 f^2 (y^*)^3}{2(m+x^*)^2}(x-x^*)^2 \\
&+\left\{\frac{fy^*}{(m+x^*)}+\frac{\beta^2 (fm+x^*)f(y^*)^2}{(m+x^*)^2}\right\}|(x-x^*)|\left|(y-y^*)\right|,
\end{aligned}
\tag{3.42}
$$

hence,

$$
\begin{aligned}
LV(x,y) &= LV_1(x) + LV_2(y) \\
&\leq \left\{ -ab + \frac{acy^*}{(ny^*+x^*)} + \frac{\alpha^2 b^2 x^*}{2} + \frac{c^2\alpha^2 x^* (y^*)^2}{2(ny^*+x^*)^2} + \frac{\beta^2 f^2 (y^*)^3}{2(m+x^*)^2} \right\} (x-x^*)^2 \\
&\quad + \left\{ \frac{c^2\alpha^2 (x^*)^3}{2(ny^*+x^*)^2} + \frac{\beta^2 (fm+x^*)^2 y^*}{2(m+x^*)^2} \right\} (y-y^*)^2 \\
&\quad + \left\{ \frac{bc\alpha^2 (x^*)^2}{a^2(ny^*+x^*)} + \frac{c^2\alpha^2 y^* (x^*)^2}{(ny^*+x^*)^2} + \frac{fy^*}{(m+x^*)} + \frac{\beta^2 (fm+x^*) f (y^*)^2}{(m+x^*)^2} \right\} \\
&\quad |(x-x^*)| \left|(y-y^*)\right|.
\end{aligned}
\tag{3.43}
$$

So,

$$
LV(x,y) \leq A_3 (x-x^*)^2 + B_3 (y-y^*)^2 + C_3 |(x-x^*)| \left|(y-y^*)\right|. \tag{3.44}
$$

Let

$$
|z-z^*| = \left(|(x-x^*)|, \left|(y-y^*)\right|\right)^T, \tag{3.45}
$$

then we get

$$
\begin{aligned}
LV(x,y) &\leq \frac{1}{2} (|z-z^*|)^T \begin{pmatrix} 2A_3 & C_3 \\ C_3 & 2B_3 \end{pmatrix} |z-z^*| \\
&= A_3 |(x-x^*)|^2 + C_3 |(x-x^*)| \left|(y-y^*)\right| + B_3 \left|(y-y^*)\right|^2.
\end{aligned}
\tag{3.46}
$$

Clearly, if (3.34) hold, then the above inequality implies $LV(x,y) < 0$ along all trajectories in the first quadrant except (x^*, y^*). Then the desired assertion (3.35) follows immediately. This completes the proof. □

4. Conclusions and Remarks

In this paper, we consider a modified stochastic Leslie-Gower predator-prey model. The value of this study lies in two aspects. First, it presents the analysis of stability for the equilibria of model (1.3). Second, it verifies some relevant properties of the stochastic model (3.2) with white noise, which shows that the existence of global positive solutions, stochastic boundedness, and stochastic asymptotic stability.

Next, we give some numerical examples to illustrate the dynamical behaviors of model (3.2) by using the method mentioned in [17].

In Figure 4(a), we choose $\alpha = \beta = 0$, that is, without noise, we observe that the positive equilibrium $E^* = (1.4761, 3.0326)$ is globally stable. In Figure 4(b), with noise densities

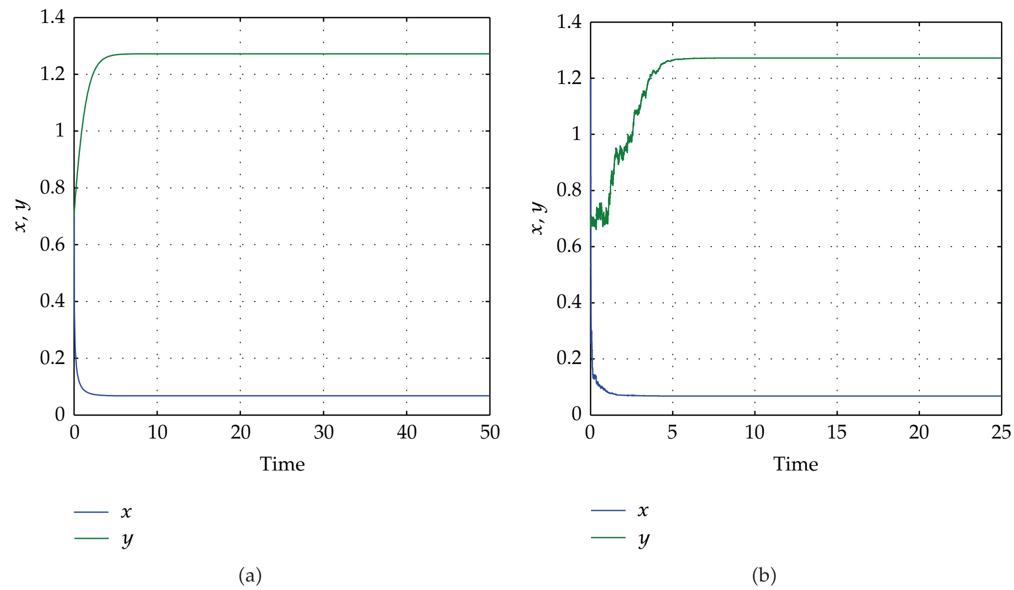

Figure 4: The solution of the stochastic model (3.2). The parameters are taken as $a = 2$, $b = 0.25$, $c = 0.875$, $e = 1$, $m = 0.95$, $n = 0.9$, and $f = 0.8$. (a) $\alpha = \beta = 0$; (b) $\alpha = 0.8$ and $\beta = 0.4$.

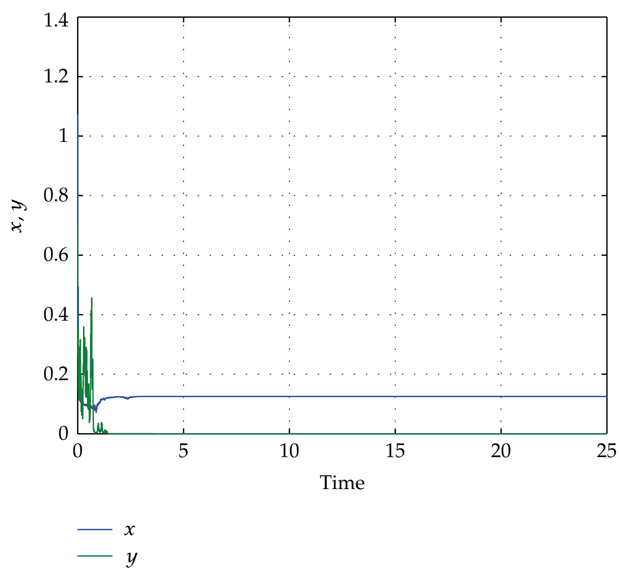

Figure 5: The solution of the stochastic model (3.2). The parameters are taken as $\alpha = 2$, $\beta = 4$, $a = 2$, $b = 0.25$, $c = 0.875$, $e = 1$, $m = 0.95$, $n = 0.9$, and $f = 0.8$.

$\alpha = 0.8$, $\beta = 0.4$, starting with a homogeneous state $E^* = (1.4761, 3.0326)$, the random white noise leads to a slight oscillations, and the later random noise makes the oscillations decay, ending with the time-independent stability. Comparing Figure 4(a) with Figure 4(b), one can realize that, if the white noise is not strong, the stochastic perturbation does not cause sharp changes of the dynamics of the system. However, in Figure 5, we choose that $\alpha = 2$ and $\beta = 4$, which violates condition (3.34), we find that the stochastic model (3.2) is not permanent. This shows that strong white noise might make a permanent system be nonpersistent.

We note that, when the noise is not large, the stochastic model preserves the property of the global stability, that is to say, when the noise is not sufficiently large, the populations may be stochastic permanence and stochastic persistent in mean. In this case, we can ignore the noise and use the deterministic model to describe the population dynamics. But, when the noise is sufficiently large, the noise can force the population to become extinct. In this case, we cannot ignore the effect of the noise. That's to say, in the case of sufficiently large noise, we cannot use deterministic model but stochastic model to describe the population dynamics. Our complete analysis of the model will give new suggestions to the studies of the population dynamics.

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