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Research Article

Some Properties of a Generalized Class of Analytic Functions Related with Janowski Functions

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We define a class $\widetilde{T}_k\ [A,\ B,\ \alpha,\ \rho]$ of analytic functions by using Janowski's functions which generalizes a number of classes studied earlier such as the class of strongly close-to-convex functions. Some properties of this class, including arc length, coefficient problems, and a distortion result, are investigated. We also discuss the growth of Hankel determinant problem.

1. Introduction

Let A be the class of analytic functions satisfying the condition $f(0) = 0$, $f'(0) - 1 = 0$ in the open unit disc $E = \{z : |z| < 1\}$. Let $f(z)$ and $g(z)$ be analytic in E. Then the function $f(z)$ is said to be subordinate to $g(z)$, written as $f(z) \prec g(z)$ if there exists an analytic function $w(z)$ in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ in E. If $g(z)$ is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

A function $p(z)$, analytic in E with $p(0) = 1$ is said to be in the class $P[A, B, \rho]$, $-1 \le B < A \le 1$, $0 \le \rho < 1$, if and only if

$$p(z) \prec \frac{1 + [(1-\rho)A + \rho B]z}{1 + Bz}, \quad z \in E. \tag{1.1}$$

It is noted that for $\rho = 0$, the class $P[A, B, \rho]$ reduces to the class $P[A, B]$ which was introduced by Janowski [1], and for $\rho = 0$, $A = 1$, and $B = -1$, we obtain the well-known class P of functions with positive real part. Now, we consider the generalized class $P_k[A, B, \rho]$ of Janowski functions which is defined as follows.

A function $p(z) \in P_k[A,B,\rho]$ if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \tag{1.2}$$

where $p_1(z),\ p_2(z) \in P[A,B,\rho]$, $-1 \leq B < A \leq 1$, $k \geq 2$, and $0 \leq \rho < 1$. It is clear that $P_2[A,B,\rho] \equiv P[A,B,\rho]$ and $P_k[1,-1,0] \equiv P_k$, the well-known class given and studied by Pinchuk [2].

We define the following classes as

$$\begin{aligned} R_k[A,B,\rho] &= \left\{ f(z) : f(z) \in A,\ \frac{zf'(z)}{f(z)} \in P_k[A,B,\rho],\ z \in E \right\}, \\ V_k[A,B,\rho] &= \left\{ f(z) : f(z) \in A,\ \frac{(zf'(z))'}{f'(z)} \in P_k[A,B,\rho],\ z \in E \right\}. \end{aligned} \tag{1.3}$$

For $A = 1$, $B = -1$, and $\rho = 0$, we obtain the well-known classes of bounded boundary rotation V_k and bounded radius rotation R_k, for details [3–8]. The classes $V_k[A,B,0]$ and $R_k[A,B,0]$ have been extensively studied by Noor in [9–11]. Also $V_2[A,B,\rho] \equiv S^*[A,B,\rho]$ and $R_2[A,B,\rho] \equiv C[A,B,\rho]$, where $S^*[A,B,\rho]$ and $C[A,B,\rho]$ are the classes studied by Polatoğlu in [12].

Throughout in this paper, we assume that $k \geq 2$, $-1 \leq B < A \leq 1$, and $0 \leq \rho < 1$ unless otherwise mentioned.

Definition 1.1. Let $f(z) \in A$, then $f(z) \in \widetilde{T}_k[A,B,\alpha,\rho]$ if and only if, for $\alpha \geq 0$, there exists a function $g(z) \in V_k[A,B,\rho]$ such that

$$\left| \arg \frac{f'(z)}{g'(z)} \right| \leq \frac{\alpha\pi}{2}, \quad z \in E. \tag{1.4}$$

For $k = 2$, $\rho = 0$, $A = 1$, and $B = -1$, $\widetilde{T}_2(1,-1,\alpha,0)$ is the class of strongly close-to-convex functions of order α in the sense of Pommerenke [13]. Also $\widetilde{T}_2(1,-1,1,0)$ is the class of close-to-convex functions, see [14].

In [15], the qth Hankel determinant $H_q(n)$, $q \geq 1$, $n \geq 1$, for a function $f(z) \in A$ is stated by Noonan and Thomas as follows.

Definition 1.2. Let $f(z) \in A$, then the qth Hankel determinant of $f(z)$ is defined for $q \geq 1$, $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.5}$$

The Hankel determinant plays an important role, for instance, in the study of the singularities by Hadamard, see [16, page 329], Edrei [17] and in the study of power series with integral coefficients by Pólya [18, page 323], Cantor [19], and many others.

In this paper, we will determine the rate of growth of the Hankel determinant $H_q(n)$ for $f(z) \in \widetilde{T}_k[A, B, \alpha, \rho]$, as $n \to \infty$. This determinant has been considered by several authors. That is, Noor [20] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for a function $f(z)$ belongs to the class V_k. Pommerenke in [21] studied the Hankel determinant for starlike functions. The Hankel determinant problem for other interesting classes of analytic functions was discussed by Noor [22–24].

Lemma 1.3. *Let $f(z) \in A$. Let the qth Hankel determinant of $f(z)$ for $q \geq 1$, $n \geq 1$ be defined by* (1.5). *Then, writting $\Delta_j(n) = \Delta_j(n, z_1, f(z))$, we have*

$$H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \cdots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \cdots & \Delta_{q-2}(n+q-2) \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q-2) & \cdots & \Delta_{q}(n+2q-2) \end{vmatrix}, \tag{1.6}$$

where with $\Delta_0(n) = a_n$, one defines, for $j \geq 1$,

$$\Delta_j(n, z_1, f(z)) = \Delta_{j-1}(n, z_1, f(z)) - \Delta_{j-1}(n+1, z_1, f(z)). \tag{1.7}$$

Lemma 1.4. *With $z_1 = (n/(n+1))y$ and $v \geq 0$ any integer,*

$$\Delta_j(n+v, z_1, zf'(z)) = \sum_{m=0}^{j} \binom{j}{m} \frac{y^m(v-(m-1)n)}{(n+1)^m} \Delta_{j-m}(n+m+v, f(z)). \tag{1.8}$$

Lemmas 1.3 and 1.4 are due to Noonan and Thomas [15].

Lemma 1.5. *A function $v(z) \in V_k[A, B, \rho]$ if and only if there exist two functions $v_1(z)$, $v_2(z) \in S^*[A, B]$ and $v_3(z)$, $v_4(z) \in C[A, B, \rho]$ such that*

$$v'(z) = \frac{(v_1(z)/z)^{((k/4)+(1/2))(1-\rho)}}{(v_2(z)/z)^{((k/4)-(1/2))(1-\rho)}}, \tag{1.9}$$

$$v'(z) = \frac{(v_3'(z))^{((k/4)+(1/2))}}{(v_4'(z))^{((k/4)-(1/2))}}. \tag{1.10}$$

Using the definition of class $P_k[A, B, \rho]$ and simple calculations yields the above result.

Lemma 1.6. *Let $f(z) \in V_k[A, B, \rho]$, then*

$$\left.\begin{array}{ll} \dfrac{(1+Br)^{\eta_1}}{(1-Br)^{\eta_2}}, & B \neq 0, \\ e^{-(k/2)(1-\rho)Ar}, & B = 0, \end{array}\right\} \leq |f'(z)| \leq \begin{cases} \dfrac{(1-Br)^{\eta_1}}{(1+Br)^{\eta_2}}, & B \neq 0, \\ e^{(k/2)(1-\rho)Ar}, & B = 0, \end{cases} \tag{1.11}$$

with

$$\eta_1 = \left(1 - \frac{A}{B}\right)\left(\frac{k}{4} - \frac{1}{2}\right)(1-\rho), \qquad \eta_2 = \left(1 - \frac{A}{B}\right)\left(\frac{k}{4} + \frac{1}{2}\right)(1-\rho). \tag{1.12}$$

This result follows easily by using Lemma 1.5 and a result for the class $S^*[A,B]$ due to Polatoğlu et al. [12]. This result is best possible.

2. Some Properties of the Class $\widetilde{T}_k[A,B,\alpha,\rho]$

Theorem 2.1. *The function $f(z) \in \widetilde{T}_k[A,B,\alpha,\rho]$ if and only if there exist two functions $f_1(z)$, $f_2(z) \in \widetilde{T}_2[A,B,\alpha,\rho]$ such that*

$$f'(z) = \frac{(f_1'(z))^{(k/4)+(1/2)}}{(f_2'(z))^{(k/4)-(1/2)}}. \tag{2.1}$$

Proof. From (1.4), we have

$$f'(z) = g'(z)p^{\alpha}(z), \tag{2.2}$$

where $g(z) \in V_k[A,B,\rho]$ and $p(z) \in P$. Using (1.10), we obtain

$$f'(z) = \frac{(g_1'(z))^{(k/4)+(1/2)}p^{\alpha}(z)}{(g_2'(z))^{(k/4)-(1/2)}} = \frac{(g_1'(z)p^{\alpha}(z))^{(k/4)+(1/2)}}{(g_2'(z)p^{\alpha}(z))^{(k/4)-(1/2)}} = \frac{(f_1'(z))^{(k/4)+(1/2)}}{(f_2'(z))^{(k/4)-(1/2)}}, \tag{2.3}$$

with $g_1(z),\ g_2(z) \in S^*[A,B]$ and $f_1(z),\ f_2(z) \in \widetilde{T}_2[A,B,\alpha,\rho]$, which completes the required result. □

Theorem 2.2. *Let $f(z) \in \widetilde{T}_k[A,B,\alpha,\rho]$ then $f(z) \in C$ for $|z| < r_0$, where r_0 is the root of*

$$1 - (A_1 + 2\alpha)r - (1 + B_1)r^2 + (A_1 + 2\alpha B)r^3 + B_1 r^4 = 0, \tag{2.4}$$

with $A_1 = (k/2)(1-\rho)(A-B)$ and $B_1 = \rho B^2 + (1-\rho)AB$.

Proof. From (1.4), we have

$$f'(z) = g'(z)p^{\alpha}(z), \quad g(z) \in V_k\left[A,B,\rho\right],\ p(z) \in P. \tag{2.5}$$

Since $g(z) \in V_k[A,B,\rho]$, therefore using (1.9), we have

$$f'(z) = \left[\frac{(s_1(z)/z)^{((k/4)+(1/2))}}{(s_2(z)/z)^{((k/4)-(1/2))}}\right]^{(1-\rho)} p^{\alpha}(z). \tag{2.6}$$

Differentiating logarithmically (2.6) with respect to z, we obtain

$$\frac{(zf'(z))'}{f'(z)} = \rho + (1-\rho)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}\right\} + \alpha\frac{zp'(z)}{p(z)}. \tag{2.7}$$

Using the well-known results for the classes P and $S^*[A,B]$, we have

$$\begin{aligned}\operatorname{Re}\frac{(zf'(z))'}{f'(z)} &> \rho + (1-\rho)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1-Ar}{1-Br} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1+Ar}{1+Br}\right\} - \frac{2\alpha r}{1-r^2} \\ &= \frac{1-(A_1+2\alpha)r-(1+B_1)r^2+\left(A_1+2\alpha B^2\right)r^3+B_1r^4}{(1-B^2r^2)(1-r^2)}, \quad B\neq 0,\end{aligned} \tag{2.8}$$

where $A_1 = (k/2)(1-\rho)(A-B)$ and $B_1 = \rho B^2 + (1-\rho)AB$. Let

$$P(r) = 1-(A_1+2\alpha)r-(1+B_1)r^2+\left(A_1+2\alpha B^2\right)r^3+B_1r^4, \tag{2.9}$$

then $P(0) = 1 > 0$ and $P(1) = -2\alpha(1-B^2) < 0$ for $-1 < B < 1$ and therefore, there exists a root $r_0 \in (0,1)$. This completes the proofs. □

Theorem 2.3. *Let $f(z) \in \widetilde{T}_k[A,B,\alpha,\rho]$, then for $-1 \le B < 0$, $-1 < A \le 1$, and $(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha > 1$,*

$$L_r f(z) = C(\alpha,\rho,k,A,B)\left(\frac{1}{1-r}\right)^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-1}, \tag{2.10}$$

where $C(\alpha,\rho,k,A,B)$ is a constant depending upon $\alpha,\rho,k,A,$ and B only.

Proof. With $z = re^{i\theta}$,

$$\begin{aligned}L(r,f(z)) &= \int_0^{2\pi}\left|zf'(z)\right|d\theta \\ &= \int_0^{2\pi}\left|zg'(z)p^{\alpha}(z)\right|d\theta, \quad g(z)\in V_k\left[A,B,\rho\right],\ p(z)\in P.\end{aligned} \tag{2.11}$$

Since $g(z) \in V_k[A,B,\rho]$, therefore by using (1.9) with $s_1(z), s_2(z) \in S^*[A,B]$, we have

$$\begin{aligned}L(r,f(z)) &\le \int_0^{2\pi}\left|\frac{z^{\rho}(s_1(z))^{((k/4)+(1/2))(1-\rho)}(p(z))^{\alpha}}{(s_2(z))^{((k/4)-(1/2))(1-\rho)}}\right|d\theta \\ &\le r^{\rho-((k/4)-(1/2))(1-\rho)}(1-B)^{(1-(A/B))((k/4)-(1/2))(1-\rho)} \\ &\quad\times\int_0^{2\pi}|s_1(z)|^{((k/4)+(1/2))(1-\rho)}\left|p(z)\right|^{\alpha}d\theta, \quad B\neq 0.\end{aligned} \tag{2.12}$$

Using the well-known Holder's inequality, with $m_1 = 2/(2-\alpha)$ and $m_2 = 2/\alpha$ such that

$(1/m_1) + (1/m_2) = 1$ and $0 < \alpha < 2$, we can write

$$L_r(f(z)) \le 2\pi r^{\rho-((k/4)-(1/2))(1-\rho)}(1-B)^{(1-(A/B))((k/4)-(1/2))(1-\rho)} \times \left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2 d\theta\right)^{\alpha/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{((k/2)+1)(1-\rho)/(2-\alpha)}d\theta\right)^{(2-\alpha)/2}. \tag{2.13}$$

Also, it is known [13] that, for $p(z) \in P,\ z \in E$,

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2 d\theta \le \frac{1+3r^2}{1-r^2}. \tag{2.14}$$

Therefore,

$$\begin{aligned}
L_r(f(z)) &\le 2\pi r^{\rho-((k/4)-(1/2))(1-\rho)}(1-B)^{(1-(A/B))((k/4)-(1/2))(1-\rho)} \\
&\quad\times \left(\frac{1+3r^2}{1-r^2}\right)^{\alpha/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{((k/2)+1)(1-\rho)/2-\alpha}d\theta\right)^{(2-\alpha)/2} \\
&\le \frac{\pi r(1-B)^{(1-(A/B))((k/4)-(1/2))(1-\rho)}2^{1+(\alpha/2)}}{(1-r)^{\alpha/2}} \\
&\quad\times \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|1+\mathrm{B}re^{i\theta}\right|^{(1-(A/B))((k/2)+1)(1-\rho)/(2-\alpha)}}d\theta\right)^{(2-\alpha)/2} \\
&\le \frac{\pi r(1-B)^{(1-(A/B))((k/4)-(1/2))(1-\rho)}2^{1+(\alpha/2)}}{(1-r)^{\alpha/2}} \\
&\quad\times \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left(1-\left|\mathrm{B}re^{i\theta}\right|\right)^{(1-(A/B))((k/2)+1)(1-\rho)/(2-\alpha)}}d\theta\right)^{(2-\alpha)/2}.
\end{aligned} \tag{2.15}$$

Therefore, we have

$$L_r(f(z)) \le C(\alpha,\rho,k,A,B)\left(\frac{1}{1-r}\right)^{\alpha/2}\left(\left(\frac{1}{1-|B|r}\right)^{((1-(A/B))((k/2)+1)(1-\rho)/(2-\alpha))-1}\right)^{(2-\alpha)/2}. \tag{2.16}$$

Since $1/(1-|B|r) \le 1/(1-r)$, for $-1 \le B < 0$, therefore

$$L_r(f(z)) \le C(\alpha,\rho,k,A,B)\left(\frac{1}{1-r}\right)^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-1}, \tag{2.17}$$

which is the required result. □

Theorem 2.4. *Let* $f(z) \in \widetilde{T}_k[A, B, \alpha, \rho]$, *then for* $-1 \leq B < 0$, $-1 < A \leq 1$, *and* $(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha > 1$,

$$|a_n| \leq C_1(\alpha, \rho, k, A, B)n^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-2}. \tag{2.18}$$

Proof. By Cauchy's theorem, we have

$$\begin{aligned} n|a_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi}\left|zf'(z)\right|d\theta \\ &= \frac{1}{2\pi r^n}L_r(f(z)) \\ &\leq \frac{1}{2\pi r^n}C(\alpha, \rho, k, A, B)\left(\frac{1}{1-r}\right)^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-1}. \end{aligned} \tag{2.19}$$

Now putting $r = 1-(1/n)$, we have

$$|a_n| = C_1(\alpha, \rho, k, A, B)n^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-2}, \tag{2.20}$$

which is required. □

Theorem 2.5. *Let* $f(z) \in \widetilde{T}_k[A, B, \alpha, \rho]$, *then*

$$\begin{aligned} &\left.\begin{matrix} \dfrac{(1+Br)^{(1-(A/B))((k/4)-(1/2))(1-\rho)}}{(1-Br)^{(1-(A/B))((k/4)+(1/2))(1-\rho)}}\left(\dfrac{1-r}{1+r}\right)^{\alpha}, & B \neq 0, \\ e^{-(k/2)(1-\rho)Ar}\left(\dfrac{1-r}{1+r}\right)^{\alpha}, & B = 0, \end{matrix}\right\} \leq \left|f'(z)\right| \\ &\leq \begin{cases} \dfrac{(1-Br)^{(1-(A/B))((k/4)-(1/2))(1-\rho)}}{(1+Br)^{(1-(A/B))((k/4)+(1/2))(1-\rho)}}\left(\dfrac{1+r}{1-r}\right)^{\alpha}, & B \neq 0, \\ e^{(k/2)(1-\rho)Ar}\left(\dfrac{1+r}{1-r}\right)^{\alpha}, & B = 0. \end{cases} \end{aligned} \tag{2.21}$$

Proof. Since $f(z) \in \widetilde{T}_k[A, B, \alpha, \rho]$, therefore

$$f'(z) = g'(z)p^{\alpha}(z), \quad g(z) \in V_k[A, B, \rho],\ p(z) \in P. \tag{2.22}$$

Using Lemma 1.5 and the well-known distortion result of class P, we obtain the required result. □

Theorem 2.6. *Let* $f(z) \in \widetilde{T}_k[A, B, \alpha, \rho]$, *then for* $-1 \leq B < 0$, $-1 < A \leq 1$, *and* $(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha > 1$,

$$H_q(n) = O(1)\begin{cases} n^{(1-(A/B))((k/2)+1)(1-\rho)+\alpha-2}, & q = 1, \\ n^{\{((k/2)+1)(1-\rho)+\alpha-1\}q-q^2}, & q \geq 2, \end{cases} \quad k \geq \frac{8(q-1)}{(1-(A/B))(1-\rho)} - 2, \tag{2.23}$$

where $k > (2/(1-\rho))((B(2-\alpha)/(B-A))+2j)-2$, *and* $O(1)$ *is a constant depending on* k, α, β, ρ, γ, *and* j *only.*

Proof. From (1.4), we have

$$zf'(z) = z\left(g'(z)\right)p^{\alpha}(z), \tag{2.24}$$

where $g(z) \in V_k[A,B,\rho]$, $p(z) \in P$. It follows easily from Alexander type relation that

$$zf'(z) = g_1(z)p^{\alpha}(z), \quad g(z) \in R_k\left[A,B,\rho\right]. \tag{2.25}$$

Using (1.9) with $s_1(z), s_2(z) \in S^*[A,B]$, we have

$$g(z) = \left[\frac{(s_1(z))^{((k/4)+(1/2))}}{(s_2(z))^{((k/4)-(1/2))}}\right]^{(1-\rho)}. \tag{2.26}$$

Therefore,

$$zf'(z) = \left[\frac{(s_1(z))^{((k/4)+(1/2))}}{(s_2(z))^{((k/4)-(1/2))}}\right]^{(1-\rho)} p^{\alpha}(z). \tag{2.27}$$

Let $F(z) = zf'(z)$, then for $j \geq 1$, z_1 any nonzero complex and $z = re^{i\theta}$, consider $\Delta_j(n, z_1, F(z))$ as defined by (1.7). Then,

$$\left|\Delta_j(n,z_1,F(z))\right| = \frac{1}{2\pi r^{n+j}}\left|\int_0^{2\pi}(z-z_1)^j F(z)e^{i(n+j)\theta}d\theta\right|, \tag{2.28}$$

and by using (2.27), we have

$$\begin{aligned}
\left|\Delta_j(n,z_1,F(z))\right| &\leq \frac{1}{2\pi r^{n+j}}\int_0^{2\pi}\left(|z-z_1|s_1(z)\right)^j \frac{|s_1(z)|^{((k/4)+(1/2))(1-\rho)-j}}{|s_2(z)|^{((k/4)-(1/2))(1-\rho)}}\left|p(z)\right|^{\alpha}d\theta \\
&\leq \frac{2^j(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi r^{((k/4)-(1/2))(1-\rho)n-j}}\left(\frac{1}{1-r}\right)^j \\
&\quad\times\int_0^{2\pi}\left|(s_1(z))\right|^{((k/4)+(1/2))(1-\rho)-j}\left|p(z)\right|^{\alpha}d\theta,
\end{aligned} \tag{2.29}$$

where we have used the result proved in [25]. The well-known Holder's inequality will give us

$$\begin{aligned}
\left|\Delta_j(n,z_1,F(z))\right| \leq{}& \frac{2^j(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi r^{((k/4)-(1/2))(1-\rho)n-j}}\left(\frac{1}{1-r}\right)^j\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p(z)\right|^2 d\theta\right)^{\alpha/2} \\
&\times\left(\frac{1}{2\pi}\int_0^{2\pi}\left|(s_1(z))\right|^{((k/2)+1)(1-\rho)-2j/2-\alpha}d\theta\right)^{(2-\alpha)/2}.
\end{aligned} \tag{2.30}$$

Using (2.14) in (2.30), we obtain

$$\left|\Delta_j(n, z_1, F(z))\right| \le \frac{2^j(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi r^{((k/4)-(1/2))(1-\rho)n-j}}\left(\frac{1}{1-r}\right)^j \times \left(\frac{1+3r^2}{1-r^2}\right)^{\alpha/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|(s_1(z))|^{((k/2)+1)(1-\rho)-2j/(2-\alpha)}d\theta\right)^{(2-\alpha)/2}. \tag{2.31}$$

Therefore, we can write

$$\left|\Delta_j(n, z_1, F(z))\right| \le \frac{2^{\alpha+j}(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi r^{1-\rho+n}}\left(\frac{1}{1-r}\right)^{j+(\alpha/2)} \times \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left(1-\left|\mathrm{B}re^{i\theta}\right|\right)^{(1-(A/B))((k/2)+1)(1-\rho)-2(1-(A/B))j/(2-\alpha)}}d\theta\right)^{(2-\alpha)/2}. \tag{2.32}$$

Now, using a subordination result for starlike functions, we have

$$\begin{aligned}\left|\Delta_j(n, z_1, F(z))\right| &\le \frac{2^{\alpha+j}(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi \mathrm{r}^{1-\rho+n}}\left(\frac{1}{1-r}\right)^{j+(\alpha/2)} \\ &\quad\times\left[\left(\frac{1}{1-r}\right)^{((1-(A/B))((k/2)+1)(1-\rho)-2(1-(A/B))j/(2-\alpha))-1}\right]^{(2-\alpha)/2} \\ &= \frac{2^{\alpha+j}(1-B)^{((B-A)/B)((k/4)-(1/2))(1-\rho)}}{2\pi r^{1-\rho+n}}\left(\frac{1}{1-r}\right)^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-1+(A/B)j},\end{aligned} \tag{2.33}$$

where c_2 is a constant depending on $k,\ \alpha,\ \beta,\ \rho,\ \gamma,$ and j only and $((1-(A/B))[((k/2)+1)(1-\rho)-2j])/(2-\alpha) > 1$. Applying Lemma 1.4 and putting $z_1 = (n/(n+1))e^{i\theta_n},\ (n \to \infty),\ r = 1-(1/n)$, we have for $k \ge (2/(1-\rho))((B(2-\alpha)/(B-A))+2j)-2$,

$$\left|\Delta_j\left(n, e^{i\theta_n}, f(z)\right)\right| = O(1)n^{(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha+(A/B)j-2}, \tag{2.34}$$

where $O(1)$ is a constant depending on $k,\ \alpha,\ \beta,\ \rho,\ \gamma,$ and j only. We now estimate the rate of growth of $H_q(n)$. For $q=1,\ H_q(n) = a_n = \Delta_0(n)$ and

$$H_1(n) = a_n = O(1)n^{(1-(A/B))((k/2)+1)(1-\rho)+\alpha-2}. \tag{2.35}$$

For $q \geq 2$, we use similar argument due to Noonan and Thomas [15] together with Lemma 1.3 to have

$$H_q(n) = O(1)n^{[(1-(A/B))((k/4)+(1/2))(1-\rho)+\alpha-1]q-q^2}, \quad k \geq \frac{8(q-1)}{(1-(A/B))(1-\rho)} - 2, \tag{2.36}$$

and $O(1)$ depends only on $k,\ \alpha,\ \beta,\ \rho,\ \gamma$, and j. □

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