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Research Article

More on (α, β)-Normal Operators in Hilbert Spaces

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We study some properties of (α, β)-normal operators and we present various inequalities between the operator norm and the numerical radius of (α, β)-normal operators on Banach algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators $T : \mathscr{H} \to \mathscr{H}$, where $\mathscr{H}$ is Hilbert space.

1. Introduction

Throughout the paper, let $\mathcal{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$, $\mathcal{B}_h(\mathscr{H})$ denote the algebra of all self-adjoint operators in $\mathcal{B}(\mathscr{H})$, and I is the identity operator. In case of $\dim \mathscr{H} = n$, we identify $\mathcal{B}(\mathscr{H})$ with the full matrix algebra $\mathscr{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathcal{B}_h(\mathscr{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ is valid for any $x \in \mathscr{H}$, and then we write $A \geq 0$. Moreover, by $A > 0$ we mean $\langle Ax, x \rangle > 0$ for any $x \in \mathscr{H}$. For $A, B \in \mathcal{B}_h(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. An operator A is majorized by B, if there exists a constant λ such that $\|Ax\| \leq \lambda \|Bx\|$ for all $x \in \mathscr{H}$ or equivalently $A^*A \leq \lambda^2 B^*B$ [1].

For real numbers α and β with $0 \leq \alpha \leq 1 \leq \beta$, an operator T acting on a Hilbert space $\mathscr{H}$ is called (α, β)-*normal* [2, 3] if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T. \tag{1.1}$$

An immediate consequence of above definition is

$$\alpha^2 \langle T^*Tx, x \rangle \leq \langle TT^*x, x \rangle \leq \beta^2 \langle T^*Tx, x \rangle, \tag{1.2}$$

from which we obtain

$$\alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|, \tag{1.3}$$

for all $x \in \mathscr{H}$.

Notice that, according to (1.1), if T is (α,β)-normal operator, then T and T^* majorize each other.

In [3], Moslehian posed two problems about (α,β)-normal operators as follows.

For fixed $\alpha > 0$ and $\beta \neq 1$,

(i) give an example of an (α,β)-normal operator which is neither normal nor hyponormal;

(ii) is there any nice relation between norm, numerical radius, and spectral radius of an (α,β)-normal operator?

Dragomir and Moslehian answered these problems in [2], as more as, they propounded a nice example of (α,β)-normal operator that is neither normal nor hyponormal, as follows.

The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathcal{B}(\mathbb{C}^2)$ is an (α,β)-normal with $\alpha = \sqrt{(3-\sqrt{5})/2}$ and $\beta = \sqrt{(3+\sqrt{5})/2}$.

The *numerical radius* $w(T)$ of an operator T on $\mathscr{H}$ is defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}. \tag{1.4}$$

Obviously, by (1.4), for any $x \in \mathscr{H}$ we have

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2. \tag{1.5}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators. Moreover, we have

$$w(T) \leq \|T\| \leq 2w(T) \quad (T \in \mathcal{B}(\mathscr{H})). \tag{1.6}$$

For other results and historical comments on the numerical radius see [4].

The *antieigenvalue* of an operator $T \in \mathcal{B}(\mathscr{H})$ defined by

$$\mu_1(T) := \inf_{Tx\neq 0} \frac{\operatorname{Re}\langle Tx, x\rangle}{\|Tx\|\|x\|}. \tag{1.7}$$

The vector $x \in \mathscr{H}$ which takes $\mu_1(T)$ is called an antieigenvector of T. We refer more study on this matter to [4].

In this paper, we prove some properties of (α,β)-normal operators and state various inequalities between the operator norm and the numerical radius of (α,β)-normal operators in Hilbert spaces.

2. Some Properties of (α, β)-Normal Operators

In this section, we establish some properties of (α, β)-normal operators. It is easy to see that if T is an (α, β)-normal $(\alpha > 0)$ then T^* is $(1/\beta, 1/\alpha)$-normal. We find numbers $z \in \mathbb{C}$ such that $z + T$ is (α, β)-normal where T is (α, β)-normal.

We know by the Cauchy-Schwartz inequality that $-1 \leq \mu_1(T) \leq 1$. Also we can write

$$\mu_1(T) = \inf_{\substack{\|x\|=1 \\ Tx \neq 0}} \frac{\operatorname{Re}\langle Tx, x\rangle}{\|Tx\|}. \tag{2.1}$$

We define

$$\mu_2(T) := \sup_{\substack{\|x\|=1 \\ Tx \neq 0}} \frac{\operatorname{Re}\langle Tx, x\rangle}{\|Tx\|}. \tag{2.2}$$

We know that if T is normal operator then $z + T$ is also normal.

Theorem 2.1. *Let T be an (α, β)-normal operator on a Hilbert space such that $0 \leq \alpha < 1 < \beta$ and $z \in \mathbb{C}$. Then $z + T$ is (α, β)-normal, if provided one of the following conditions holds:*

(i) $\mu_1(\overline{z}T) \geq 0$,

(ii) $\mu_1(\overline{z}T) < 0, |z|^2 \geq -2|z|\|T\|\mu_1(\overline{z}T)$.

Proof. In both of above cases, we show that

$$|z|^2 + 2\operatorname{Re}\langle \overline{z}Tx, x\rangle \geq 0, \quad \forall x \in \mathscr{H} \text{with } \|x\| = 1, Tx \neq 0. \tag{2.3}$$

By the assumption (i), $\mu_1(\overline{z}T) \geq 0$, we have $\operatorname{Re}\langle \overline{z}Tx, x\rangle / |z|\|Tx\| \geq 0$ for every $x \in \mathscr{H}$ with $\|x\| = 1$ and $Tx \neq 0$, consequently we get $\operatorname{Re}\langle \overline{z}Tx, x\rangle \geq 0$, and therefore (2.3) is valid. On the other hand, if (ii) holds and we set $B := \mu_1(\overline{z}T)$ then we get $B \leq \operatorname{Re}\langle \overline{z}Tx, x\rangle / |z|\|Tx\|$ for every $x \in \mathscr{H}$ with $\|x\| = 1$ and $Tx \neq 0$, consequently:

$$\inf\{B\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf\left\{ \|Tx\| \frac{\operatorname{Re}\langle \overline{z}Tx, x\rangle}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0 \right\}. \tag{2.4}$$

Since $B < 0$, we obtain

$$-B \inf\{-\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf\left\{ \|Tx\| \frac{\operatorname{Re}\langle \overline{z}Tx, x\rangle}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0 \right\}, \tag{2.5}$$

and so

$$B \sup\{\|Tx\| : \|x\| = 1, Tx \neq 0\} \leq \inf\left\{ \|Tx\| \frac{\operatorname{Re}\langle \overline{z}Tx, x\rangle}{|z|\|Tx\|} : \|x\| = 1, Tx \neq 0 \right\}. \tag{2.6}$$

Now, by using the last inequality, we have

$$\begin{aligned}
|z|^2 + 2|z|\|T\|\mu_1(\overline{z}T) &= |z|^2 + 2|z|\left(\sup_{\substack{\|x\|=1 \\ Tx\neq 0}} \|Tx\|\right)\left(\inf_{\substack{\|x\|=1 \\ Tx\neq 0}}\left\{\frac{\operatorname{Re}\langle \overline{z}Tx, x\rangle}{|z|\|Tx\|}\right\}\right) \\
&\leq |z|^2 + 2|z|\inf_{\|x\|=1}\left\{\|Tx\|\frac{\operatorname{Re}\langle \overline{z}Tx, x\rangle}{|z|\|Tx\|}\right\} \\
&= |z|^2 + 2\inf_{\|x\|=1}\{\operatorname{Re}\langle \overline{z}Tx, x\rangle\}.
\end{aligned} \tag{2.7}$$

This shows that (2.3) holds for (ii), too. Thus, for any $x \in \mathscr{H}$ with $\|x\| = 1$ we have

$$\begin{aligned}
\alpha^2\langle (z+T)^*(z+T)x, x\rangle &= \alpha^2\left[\left\langle |z|^2x, x\right\rangle + \langle \overline{z}Tx, x\rangle + \langle zT^*x, x\rangle\right] + \alpha^2\langle T^*Tx, x\rangle \\
&\leq \left\langle |z|^2x, x\right\rangle + \langle \overline{z}Tx, x\rangle + \langle zT^*x, x\rangle + \langle TT^*x, x\rangle \\
&= \langle (z+T)(z+T)^*x, x\rangle \\
&\leq \beta^2\left[\left\langle |z|^2x, x\right\rangle + \langle \overline{z}Tx, x\rangle + \langle zT^*x, x\rangle\right] + \beta^2\langle T^*Tx, x\rangle \\
&= \beta^2\langle (z+T)^*(z+T)x, x\rangle
\end{aligned} \tag{2.8}$$

and this completes the proof. □

Corollary 2.2. *Let T be an (α, β)-normal operator. We have the following.*

(i) *If $\mu_1(T) \geq 0$ then $z + T$ is (α, β)-normal operator for any $z > 0$.*

(ii) *If $\mu_2(T) \leq 0$ then $z + T$ is (α, β)-normal operator for any $z < 0$.*

Proof. (i) By the definition of the first antieigenvalue of T, for all $z > 0$ we have

$$\mu_1(\overline{z}T) = \mu_1(zT) = \mu_1(T) \geq 0. \tag{2.9}$$

By using Theorem 2.1(i) we imply that $z + T$ is an (α, β)-normal.

(ii) If $z < 0$, then

$$\mu_1(\overline{z}T) = -\mu_2(T) \geq 0. \tag{2.10}$$

By using Theorem 2.1(i) we imply that $z + T$ is an (α, β)-normal. □

Corollary 2.3. *Let T be an injective and (α, β)-normal operator with $\alpha > 0$. Then*

(i) *$\mathcal{R}(T)$ is dense,*

(ii) *T^* is injective,*

(iii) *if T is surjective then T^{-1} is also (α, β)-normal.*

Proof. Since the inequality (1.3) is valid, we obtain $\mathcal{N}(T^*) = \mathcal{N}(T)$, and therefore $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^*) = \mathcal{N}(T) = 0$, thus $\mathcal{R}(T)$ is a dense subspace of $\mathcal{H}$ and T^* is injective. This proves (i) and (ii).

To prove (iii), we note that since T is surjective, we imply that T is invertible. On the other hand we have $(T^*)^{-1} = (T^{-1})^*$. Also we know that if A and B are two positive and invertible operators with $0 < A \leq B$ then $B^{-1} \leq A^{-1}$. Since T is (α, β)-normal, by taking inverse from all sides of (1.1), we get

$$\frac{1}{\beta^2} T^{-1}(T^*)^{-1} \leq (T^*)^{-1}T^{-1} \leq \frac{1}{\alpha^2} T^{-1}(T^*)^{-1}. \tag{2.11}$$

This means that $(T^{-1})^*$ is $(1/\beta, 1/\alpha)$-normal, thus T^{-1} is (α, β)-normal. □

Example 2.4. Consider the following matrix T in $\mathcal{B}(\mathbb{C}^2)$:

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2.12}$$

T is an (α, β)-normal operator, with parameters $\alpha = \sqrt{(3 - \sqrt{5})/2}$ and $\beta = \sqrt{(3 + \sqrt{5})/2}$. Then $T^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ is (α, β)-normal.

For $T \in \mathcal{B}(\mathcal{H})$ we call

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \tag{2.13}$$

the *spectral radius* of T, where $\sigma(T)$ is the spectrum of T and it is known that $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$ [5, page 102].

Theorem 2.5. *Let T be an (α, β)-normal operator such that T^{2^n} is (α, β)-normal operator for every $n \in \mathbb{N}$, too. Then, we have*

$$\frac{1}{\beta}\|T\| \leq r(T) \leq \|T\|. \tag{2.14}$$

Proof. For any $T \in \mathcal{B}(\mathcal{H})$ we have

$$\|T^*T\| = \|T\|^2. \tag{2.15}$$

In particular, if T is a self-adjoint operator then $\|T^2\| = \|T\|^2$. Thus, by the definition of (α, β)-normal operator, we have

$$\left\| T^{*2}T^2 \right\| \geq \frac{1}{\beta^2} \left\| (T^*T)^2 \right\| = \frac{1}{\beta^2} \|T\|^4. \tag{2.16}$$

By induction on n, we imply that

$$\left\| T^{*2^n} T^{2^n} \right\| \geq \frac{1}{\beta^{2^{n+1}-2}} \|T\|^{2^{n+1}}, \tag{2.17}$$

from which we obtain

$$\begin{aligned} r(T)^2 = r(T^*)r(T) &= \lim_{n\to\infty} \left(\left\| T^{*2^n} \right\| \left\| T^{2^n} \right\| \right)^{1/2^n} \\ &\geq \lim_{n\to\infty} \left\| T^{*2^n} T^{2^n} \right\|^{1/2^n} \\ &\geq \lim_{n\to\infty} \left(\frac{1}{\beta^{2^{n+1}-2}} \|T\|^{2^{n+1}} \right)^{1/2^n} \\ &= \frac{1}{\beta^2} \|T\|^2 \lim_{n\to\infty} \frac{1}{\beta^{-2/2^n}} = \frac{1}{\beta^2} \|T\|^2. \end{aligned} \tag{2.18}$$

Therefore, we get $(1/\beta)\|T\| \leq r(T) \leq \|T\|$. This completes the proof. □

Below, we give an example of (α, β)-normal operator such that it satisfies in Theorem 2.5.

Example 2.6. Assume that $\mathscr{H}$ is a separable Hilbert space and $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathscr{H}$. We define the operator $T \in \mathcal{B}(\mathscr{H})$ as follows:

$$Te_n = \begin{cases} e_{n-1}, & n \equiv 0 \pmod 3, \\ \dfrac{1}{2} e_{n-1}, & n \equiv 1 \pmod 3, \\ 2e_{n-1}, & n \equiv 2 \pmod 3, \end{cases} \tag{2.19}$$

so

$$T^*e_n = \begin{cases} \dfrac{1}{2} e_{n+1}, & n \equiv 0 \pmod 3, \\ 2e_{n+1}, & n \equiv 1 \pmod 3, \\ e_{n+1}, & n \equiv 2 \pmod 3, \end{cases} \tag{2.20}$$

and by simple computation we get

$$TT^*e_n = \begin{cases} \dfrac{1}{4} e_n, & n \equiv 0 \pmod 3, \\ 4e_n, & n \equiv 1 \pmod 3, \\ e_n, & n \equiv 2 \pmod 3, \end{cases} \qquad T^*Te_n = \begin{cases} e_n, & n \equiv 0 \pmod 3, \\ \dfrac{1}{4} e_n, & n \equiv 1 \pmod 3, \\ 4e_n, & n \equiv 2 \pmod 3. \end{cases} \tag{2.21}$$

Consequently, T is $(1/4, 4)$-normal operator and also T^n is $(1/4, 4)$-normal operator, for any integer $n \geq 0$. Thus we have $\|T\| = 2$ and $r(T) = 1$, hence (2.14) is valid.

3. Inequalities Involving Norms and Numerical Radius

In this section we state some inequalities involving norms and numerical radius.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (α, β)-normal operator.*

(i) *For positive real numbers p and q with $p \geq 2$ and $(1/p) + (1/q) = 1$ we have*

$$\|T + T^*\|^p + \|T - T^*\|^p \geq 2(1 + \alpha^q)^{p-1}\|T\|^p. \tag{3.1}$$

(ii) *If $0 \leq p \leq 1$ or $p \geq 2$, then we have*

$$\left(\|T + T^*\|^2 + \|T - T^*\|^2\right)^p \geq \|T\|^{2p}\varphi(\alpha, p), \tag{3.2}$$

where $\varphi(\alpha, p) = 2^p[(1 + \alpha^p)^2 + (2^p - 2^2)\alpha^p]$.

(iii) *If $\mathcal{N}(T) = 0$ and for any $x \in \mathcal{H}$ with $\|x\| = 1$ we have*

$$\left\| \frac{Tx}{\|T^*x\|} - \frac{T^*x}{\|Tx\|} \right\| \leq \rho, \tag{3.3}$$

then, we obtain

$$\alpha\|T\|^2 \leq \omega\left(T^2\right) + \frac{\rho^2}{2}\beta\|T\|^2. \tag{3.4}$$

Proof. (i) We use the following known inequality:

$$\|a + b\|^p + \|a - b\|^p \geq 2\left(\|a\|^q + \|b\|^q\right)^{p-1}, \tag{3.5}$$

which is valid for any $a, b \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space.

Now, if we take $a = Tx$ and $b = T^*x$ in (3.5), then for any $x \in \mathcal{H}$ we get

$$\begin{aligned} \|Tx + T^*x\|^p + \|Tx - T^*x\|^p &\geq 2\left(\|Tx\|^q + \|T^*x\|^q\right)^{p-1} \\ &\geq 2\left(\|Tx\|^q + \alpha^q\|Tx\|^q\right)^{p-1} \\ &= 2(1 + \alpha^q)^{p-1}\|Tx\|^{q(p-1)} \\ &= 2(1 + \alpha^q)^{p-1}\|Tx\|^p. \end{aligned} \tag{3.6}$$

Taking the supremum in (3.6) over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result (3.1).

(ii) We use the following inequality [6, Theorem 8, page 551]:

$$\left(\|a + b\|^2 + \|a - b\|^2\right)^p \geq 2^p\left(\left(\|a\|^p + \|b\|^p\right)^2 + \left(2^p - 2^2\right)\|a\|^p\|b\|^p\right), \tag{3.7}$$

where a and b are two vectors in a Hilbert space and $0 \leq p \leq 1$ or $p \geq 2$.

Now, if we put $a = Tx$ and $b = T^*x$ in (3.7), then we obtain

$$\begin{aligned}
&\left(\|Tx + T^*x\|^2 + \|Tx - T^*x\|^2\right)^p \\
&\quad \geq 2^p\left(\left(\|Tx\|^p + \|T^*x\|^p\right)^2 + \left(2^p - 2^2\right)\|Tx\|^p\|T^*x\|^p\right), \\
&\quad \geq 2^p\left(\|Tx\|^{2p}(1+\alpha^p)^2 + \left(2^p - 2^2\right)\alpha^p\|Tx\|^{2p}\right) \\
&\quad = 2^p\|Tx\|^{2p}\left[(1+\alpha^p)^2 + \left(2^p - 2^2\right)\alpha^p\right] \\
&\quad = \|Tx\|^{2p}\varphi(\alpha, p).
\end{aligned} \tag{3.8}$$

Now, taking the supremum over $\|x\| = 1$ in (3.8), we get the desired result (3.2).

(iii) We use the following reverse of Schwarz's inequality:

$$(0 \leq)\|a\|\|b\| - |\langle a, b\rangle| \leq \|a\|\|b\| - \operatorname{Re}\langle a, b\rangle \leq \frac{1}{2}\rho^2\|a\|\|b\|, \tag{3.9}$$

which is valid for $a, b \in \mathscr{H} \setminus \{0\}$ and $\rho > 0$, with $\|(a/\|b\|) - (b/\|a\|)\| \leq \rho$ (see [7]). We take $a = Tx$ and $b = T^*x$ in (3.9) to get

$$\|Tx\|\|T^*x\| \leq |\langle Tx, T^*x\rangle| + \frac{1}{2}\rho^2\|Tx\|\|T^*x\|. \tag{3.10}$$

Thus, we obtain

$$\alpha\|Tx\|^2 \leq |\langle Tx, T^*x\rangle| + \frac{1}{2}\rho^2\beta\|Tx\|^2. \tag{3.11}$$

Now, taking the supremum over $\|x\| = 1$ in recent inequality, we get the desired result (3.4). □

Theorem 3.2. *Assume that T is an (α, β)-normal operator. Then, we have*

$$\left(1 + \alpha^2\right)\|T\|^2 \leq \frac{1}{2}\|T - T^*\|^2 + \omega\left(T^2\right). \tag{3.12}$$

Proof. By [2, Theorem 3.1], we have

$$2(1+\alpha^p)\|T\|^p \leq \frac{1}{2}\left[\|T + T^*\|^p + \|T - T^*\|^p\right], \tag{3.13}$$

and also

$$\left\|\frac{T^*T + TT^*}{2}\right\|^{p/2} \leq \frac{1}{4}\left[\|T + T^*\|^p + \|T - T^*\|^p\right]. \tag{3.14}$$

On the other hand, it is known [8] that for $A, B \in \mathcal{B}(\mathcal{H})$ we have

$$\left\| \frac{A+B}{2} \right\|^2 \leq \frac{1}{2}\left[\left\| \frac{A^*A + B^*B}{2} \right\| + \omega(B^*A) \right]. \tag{3.15}$$

By using this inequality we get

$$\left\| \frac{T+T^*}{2} \right\|^2 \leq \frac{1}{2}\left[\left\| \frac{T^*T + TT^*}{2} \right\| + \omega\left(T^2\right) \right]. \tag{3.16}$$

If we put $p = 2$ in (3.14), we obtain

$$\begin{aligned} \left\| \frac{T+T^*}{2} \right\|^2 &\leq \frac{1}{2}\left[\frac{1}{4}\left(\|T+T^*\|^2 + \|T-T^*\|^2 \right) + \omega\left(T^2\right) \right] \\ &= \frac{1}{2}\left[\left\| \frac{T+T^*}{2} \right\|^2 + \left\| \frac{T-T^*}{2} \right\|^2 + \omega\left(T^2\right) \right]. \end{aligned} \tag{3.17}$$

Thus we get

$$\frac{1}{2}\left\| \frac{T+T^*}{2} \right\|^2 \leq \frac{1}{2}\left\| \frac{T-T^*}{2} \right\|^2 + \frac{\omega(T^2)}{2}. \tag{3.18}$$

Now, we take $p = 2$ in (3.13) to obtain

$$\left(1+\alpha^2\right)\|T\|^2 \leq \left\| \frac{T-T^*}{2} \right\|^2 + \left\| \frac{T-T^*}{2} \right\|^2 + \omega\left(T^2\right) = \frac{1}{2}\|T-T^*\|^2 + \omega\left(T^2\right). \tag{3.19}$$

This completes the proof. □

Theorem 3.3. *Assume that T is an (α, β)-normal operator. Then for any real s with $0 \leq s \leq 1$, we have*

$$\left((1-s)\frac{1}{\beta^2} + s \right)\left((1-s) + s\frac{1}{\beta^2} \right)\|T\|^4 \leq \left[1 - s + s\beta^2 \right]\|T\|^2\|T-T^*\|^2 + \omega\left(T^2\right)^2. \tag{3.20}$$

Proof. By [9, Theorem 2.6] (see also [10, Theorem 2.4]), we have

$$\begin{aligned} &\left[(1-s)\|a\|^2 + s\|b\|^2 \right]\left[(1-s)\|b\|^2 + s\|a\|^2 \right] - |\langle a, b\rangle|^2 \\ &\quad \leq \left[(1-s)\|a\|^2 + s\|b\|^2 \right]\left[(1-s)\|b - ta\|^2 + s\|tb - a\|^2 \right], \end{aligned} \tag{3.21}$$

where $0 \le s \le 1$, $t \in \mathbb{R}$ and $a, b \in \mathscr{H}$. By taking $t = 1$, $a = Tx$, and $b = T^*x$ in (3.21), we get

$$\begin{aligned}&\left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right]\left[\|(1-s)T^*x\|^2 + s\|Tx\|^2\right] - |\langle Tx, T^*x\rangle|^2 \\ &\quad\le \left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right]\left[(1-s)\|T^*x - Tx\|^2 + s\|T^*x - Tx\|^2\right],\end{aligned} \tag{3.22}$$

thus, we have

$$\begin{aligned}&\left[\frac{(1-s)}{\beta^2}\|T^*x\|^2 + s\|T^*x\|^2\right]\left[(1-s)\|T^*x\|^2 + \frac{s}{\beta^2}\|T^*x\|^2\right] - \left|\left\langle T^2x, x\right\rangle\right|^2 \\ &\quad\le \left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right]\left[(1-s)\|T^*x\|^2 + s\|Tx\|^2\right] - \left|\left\langle T^2x, x\right\rangle\right|^2 \\ &\quad\le \left[(1-s)\|Tx\|^2 + s\|T^*x\|^2\right]\left[(1-s)\|T^*x - Tx\|^2 + s\|T^*x - Tx\|^2\right] \\ &\quad\le \left[(1-s)\|Tx\|^2 + s\beta^2\|Tx\|^2\right]\|T^*x - Tx\|^2.\end{aligned} \tag{3.23}$$

Finally, we take supremum over $\|x\| = 1$ from both sides of

$$\begin{aligned}&\left(\frac{(1-s)}{\beta^2} + s\right)\left((1-s) + \frac{s}{\beta^2}\right)\|T^*x\|^4 \\ &\quad\le \left[(1-s)\|Tx\|^2 + s\beta^2\|Tx\|^2\right]\|T^*x - Tx\|^2 + \left|\left\langle T^2x, x\right\rangle\right|^2,\end{aligned} \tag{3.24}$$

and we use triangle inequality for supremums to complete the proof. □

Corollary 3.4. *Let T be an (α, β)-normal operator. Then, we have*

$$\frac{1}{\beta}\|T\|^2 \le \|T\|\|T - T^*\| + \omega\left(T^2\right). \tag{3.25}$$

Proof. By using the inequality (3.21) we get

$$\left((1-s) + s\alpha^2\right)\left((1-s)\alpha^2 + s\right)\|T\|^4 \le \left[1 - s + s\alpha^2\right]\|T\|^2\|T - T^*\|^2 + w\left(T^2\right)^2. \tag{3.26}$$

We take $s = 0$ in inequalities (3.20) and (3.26) to imply

$$\max\left\{\frac{1}{\beta^2}, \alpha^2\right\}\|Tx\|^4 \le \|Tx\|^2\|T - T^*\|^2 + \omega\left(T^2\right)^2. \tag{3.27}$$

Thus, $\max\{1/\beta, \alpha\}\|Tx\|^2 \le \|Tx\|\|Tx - T^*x\| + \omega(T^2)$. Now, taking supremum overall x with $\|x\| = 1$, the desired inequality is obtained. □

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