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On the tangential touch between the free and the fixed boundaries for the two-phase obstacle-like problem

John Andersson, Norayr Matevosyan and Hayk Mikayelyan

Abstract. In this paper we consider the following two-phase obstacle-problem-like equation in the unit half-ball

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \ \lambda_\pm > 0.$$

We prove that the free boundary touches the fixed boundary (uniformly) tangentially if the boundary data f and its first and second derivatives vanish at the touch-point.

1. Introduction

1.1. The problem

G. S. Weiss ([W2]) suggested the study of the following free boundary problem: find a weak solution $u \in W^{1,2}(D)$ of

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \tag{1}$$

in the domain D, such that $u - f \in W_0^{1,2}(D)$ for a given $f \in W^{1,2}(D)$. This is the two-phase analogue of the classical obstacle problem. It has been considered by N. N. Uraltseva in [U] and H. Shahgholian, N. N. Uraltseva and G. S. Weiss in [SUW].

In our paper we always assume $\lambda_\pm > 0$, and we consider the cases where D is a ball or a half-ball, as well as the case of the so-called global solution when $D = \mathbf{R}^n_+$.

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Equation (1) is the Euler–Lagrange equation of the energy functional

$$J(u) = \int_D (|\nabla u|^2 + 2\lambda_+ \max(u,0) + 2\lambda_- \max(-u,0))dx.$$

Note that if the boundary data f is non-negative (non-positive) then the solution u is so too, and we arrive at the classical obstacle problem (see [C]). In the two-phase case we do not have the property that the gradient vanishes on the free boundary, as it was in the classical case; this causes difficulties.

We consider the following problem: let u be a weak solution of (1) in the unit half-ball B_1^+, the free boundary

$$\Gamma_u := (\partial\{x : u(x) > 0\} \cup \partial\{x : u(x) < 0\}) \cap B_1^+$$

touches the fixed boundary at 0 and the boundary values of u on the flat part of the boundary $\overline{B}_1^+ \cap \{x : x_1 = 0\}$, denoted by f, satisfy the following conditions: $f \in C^{2,\mathrm{Dini}}(B_1 \cap \{x : x_1 = 0\})$ and

$$f(0) = |\nabla f(0)| = |D^2 f(0)| = 0. \tag{2}$$

We prove that the free boundary of u approaches the fixed boundary at 0 tangentially. Under some growth assumption, we prove that this approach is uniform (Theorem B). This growth assumption imposed in Theorem B is necessary, as is shown by an example. From (2) it obviously follows that $|f(x')|/|x'|^2 \le \omega(|x'|)$ for some Dini modulus of continuity ω, i.e., the blow-up of f is zero:

$$f_r(x') := \frac{f(rx')}{r^2} \to 0 \quad \text{as } r \to 0.$$

Let us recall the definition of $C^{2,\mathrm{Dini}}(B_1 \cap \Pi)$; these are functions from $C^2(B_1 \cap \Pi)$ such that

$$|D^2 f(x) - D^2 f(y)| \le \omega(|x-y|),$$

where $D^2 f$ is the Hessian of f and ω is a Dini modulus of continuity, i.e.,

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

1.2. Notation

In the sequel we use the following notation:

$\mathbf{R}^n_\pm$	$\{x\in\mathbf{R}^n: \pm x_1>0\}$,
$B(z,r)$	$\{x\in\mathbf{R}^n: \lvert x-z\rvert<r\}$,
B_r	$B(0,r)$,
B_r^+	$\mathbf{R}^n_+\cap B_r$,
Π	$\{x\in\mathbf{R}^n: x_1=0\}$,
x'	$(x_2,...,x_n)$,
K_ε	$\{x\in\mathbf{R}^n_+: x_1>\varepsilon\lvert x'\rvert\}$,
$\lVert\cdot\rVert_\infty$	L^∞-norm,
$e_1,...,e_n$	standard basis in $\mathbf{R}^n$,
$\nu,\ e$	arbitrary unit vectors,
$D_\nu,\ D_{\nu e}$	first and second directional derivatives,
$v^+,\ v^-$	$\max(v,0),\ \max(-v,0)$,
χ_D	characteristic function of the set D,
∂D	boundary of the set D,
$\Omega_u^\pm$	$\{x\in D: \pm u(x)>0\}$,
Λ_u	$\{x\in B_1^+: u(x)=\lvert\nabla u(x)\rvert=0\}$,
Γ_u	$(\partial\Omega_u^+\cup\partial\Omega_u^-)\cap D$, the free boundary,
$\mathcal{P}(...)$	see Definition 2.

1.3. "Typical" examples

We show here, with some examples, how the situation near a touch point between the free and fixed boundaries can look like.

Let us fix the ball B_R and consider the function $\lambda|x|^2/2n$, $\lambda>0$. Then we take the radial fundamental solution U of the Laplace equation multiplied with a constant C_R, such that $C_R\partial_r U(R)=\lambda R/n$. Then for an appropriate constant C the function

$$V(x)=\frac{\lambda}{2n}|x|^2-C_RU(|x|)+C$$

is non-negative in $\mathbf{R}^n$, $\Delta V=\lambda-C_R\delta_0$ and $V=|\nabla V|=0$ on ∂B_R (see Figure 1).

Thus we can construct solutions of (1) in $B_R\backslash\{0\}$ and in $\mathbf{R}^n\backslash B_R$. For instance in $\mathbf{R}^2$ we can illustrate some solutions considered in rectangles (see Figure 2). The dashed curves denote free boundaries Γ_u, $\pm$ denote regions $\Omega_u^\pm$ and 0 the region Λ_u. Figure 2 (a) shows a solution with sub-quadratic growth at the touch point. In

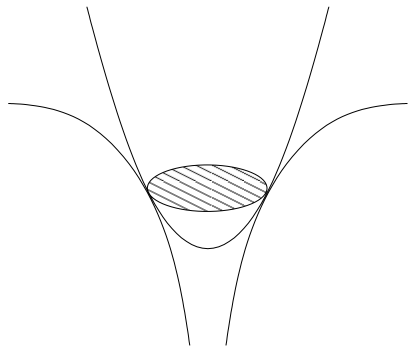

Figure 1. “Ball solutions”.

this case the blow-up of the solution is zero. Figure 2 (b) shows that even if we have a non-negative boundary data near the touch point, the blow-up still can be negative. Figure 2 (c) shows that condition (2) is essential for having a tangential touch.

Let us take the boundary data f on ∂B_1^+ to be odd-symmetric with respect to x_2. Then the solution u will be odd-symmetric too. One might expect that the free boundary of a symmetric solution has orthogonal touch. Figure 2 (a) indicates another possibility, when the zero set Λ_u is large near the contact point. This is indeed the case, as we show in Section 2. A similar argument works also in higher dimensions for every plane-symmetric domain.

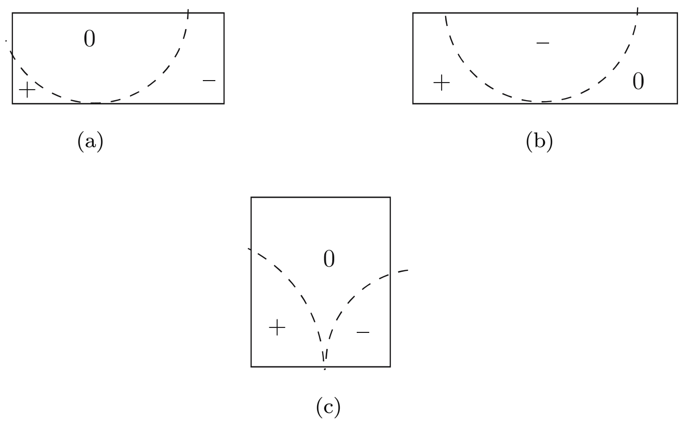

Figure 2. “Typical” examples.

2. Main results

In this section we state two theorems. The first one says that if the boundary data satisfies condition (2), then the free boundary can approach the fixed boundary only tangentially. In the second theorem we assert that this approach is uniform for a certain class of solutions.

Theorem A. *Let u be a solution of* (1) *in B_1^+ with boundary data f on Π, such that condition* (2) *is satisfied and $0\in\overline{\Gamma}_u$. Then the free boundary approaches Π at the point 0 tangentially.*

Corollary 1. *Let u be as in Theorem* A, *then one of the following limits holds*

$$\frac{|\Omega_u^+\cap B_r^+|}{|B_r^+|}\to 1,\quad \frac{|\Omega_u^-\cap B_r^+|}{|B_r^+|}\to 1,\quad \frac{|\Lambda_u\cap B_r^+|}{|B_r^+|}\to 1,\qquad \textit{as } r\to 0.$$

Moreover, one of the first two cases is possible only if condition (4), *see below, is satisfied for some c_0 and r_0.*

Definition 2. Let ω be a Dini modulus of continuity and M, c_0 and r_0 be positive constants. We define $\mathcal{P}(M,R,c_0,r_0)$ to be the class of solutions u of (1) in B_1^+, $\|u\|_{L^\infty(B_1^+)}\leq M$, $0\in\overline{\Gamma}_u$ such that the boundary data $f=u|_\Pi\in C^{2,\mathrm{Dini}}(B_1\cap\Pi)$ satisfies condition (2),

$$\|f\|_{C^2(\overline{B}_1\cap\Pi)}\leq R \quad\text{and}\quad \int_0^1\frac{\omega(s)}{s}ds\leq R. \tag{3}$$

Further, we assume

$$\sup_{B_r^+}|u|\geq c_0r^2 \quad \text{for } 0<r<r_0, \tag{4}$$

for all $u\in\mathcal{P}(M,R,c_0,r_0)$.

Remark 3. If u solves (1) in B_1^+, $0\in\overline{\Gamma}_u$ and $u|_\Pi\equiv 0$, then condition (4) is fulfilled with the constant $c_0=\max(\lambda_\pm)C$, for $0<r<1$, where C is a dimension dependent constant (see Lemma 6 and Corollary 7).

Theorem B. *There exists a modulus of continuity $\sigma(r)$ and $\tilde{r}>0$ such that if $u\in\mathcal{P}(M,R,c_0,r_0)$, then*

$$\Gamma_u\cap B_{\tilde{r}}\subset\{x: x_1<|x'|\sigma(|x'|)\}.$$

In other words the free boundaries of the functions from $\mathcal{P}$ approach Π at the point 0 uniformly tangentially.

Here σ and $\tilde{r}$ depend on the dimension, $\lambda_\pm$, c_0, r_0, M and R.

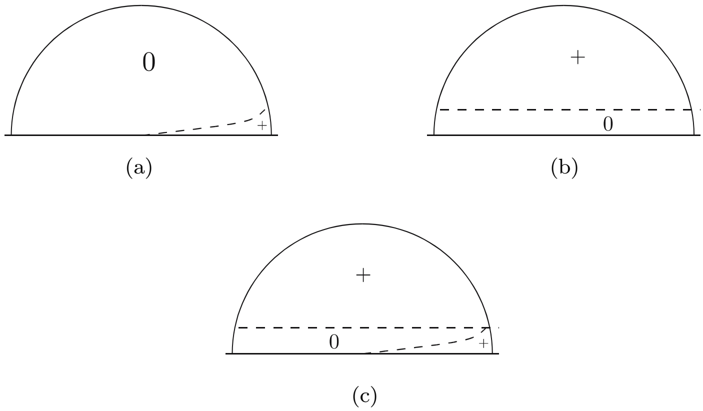

Figure 3. Non-uniform approach.

Remark 4. Since the main tool we use proving the Theorems A and B is the blow-up argument, these results can be generalised for domains with smooth enough boundaries.

Let us, with an example, indicate the necessity of condition (4). In this example (Figure 3) the boundary data is positive, so we here treat the classical obstacle problem in $\mathbf{R}^2$. Consider a solution with small boundary data f_ε supported to the right of the origin as is shown on the Figure 3 (a). This can be done using the "ball solutions" discussed above. Next consider the function $u(x)=\frac{1}{2}\lambda_+(x_1-\varepsilon)_+^2$. We will get it as a solution of our problem if we take its boundary data on ∂B_1^+ (Figure 3 (b)). Consider now the solution u to the problem with boundary data which is the sum of the boundary data of the previous two examples 3 (a) and 3 (b). Ω_u^+ will look like on Figure 3 (c). So we see that when ε tends to zero we are getting free boundary points on the x_1-axis arbitrarily near to the origin, while the boundary data remain bounded and satisfy conditions (2) and (3) with appropriate uniform constants M and R.

Remark 5. In order to get uniform tangential touch for a class of solutions we impose condition (4). This condition, however, can be replaced by the following one, which is considered in [KKS] for a different problem;

$$\frac{|\Omega_u^+\cap B_r^+|}{|B_r^+|}\geq c_0>0 \quad \text{for } r<r_0.$$

From Lemma 8 and Corollary 1 it follows that both conditions are equivalent in our case.

3. Technicalities

3.1. Non-degeneracy

In this section we introduce some (modified) results from [W2], [U] and [SUW] as well as prove growth estimates at the boundary (Lemmas 8 and 9).

Lemma 6. *Let u solve* (1) *in B_1. There exists a dimension dependent constant C such that $\|f^\pm\|_\infty < \lambda_\pm C$ implies that $0 \notin \overline{\Omega}_u^\pm$.*

Proof. Consider the "+"-case. Due to the comparison principle a similar argument is true (and well-known) for the obstacle problem, i.e. it is known in our case if the boundary data f is non-negative or non-positive. Let us now consider the related (one-phase) obstacle problem in B_1 with boundary data f^+, denote its solution by v. It is enough to show that $\Omega_u^+ \subset \Omega_v^+$. Consider the function $w = u - v$ in Ω_u^+. We have $w|_{\partial\Omega_u^+} \leq 0$ and $\Delta w \geq 0$ in Ω_u^+, hence $u - v \leq 0$ and we are done. □

Corollary 7. *Let u be the solution of* (1)*, $x_0 \in \overline{\Omega}^\pm$ and $B_{r_0}(x_0) \subset D$. Then*

$$\sup_{\partial B_r(x_0)} u^\pm \geq \lambda_\pm C r^2 \quad \textit{for } r < r_0. \tag{5}$$

Here the constant C is the same as in the previous lemma.

In other words if $x_0 \notin \operatorname{int} \Lambda_u$ and $B_{r_0}(x_0) \subset D$, then

$$\sup_{\partial B_r(x_0)} |u| \geq \min(\lambda_\pm) C r^2 \quad \textit{for } r < r_0. \tag{6}$$

Proof. Let us restrict the function u to $B_r(x_0)$ and scale it

$$u_r(x) = \frac{u(rx + x_0)}{r^2}.$$

Then u_r is a solution of (1) in B_1 with boundary data $u_r|_{\partial B_1}$. Since $0 \in \overline{\Omega}^+$ we must have

$$\sup_{\partial B_1} u_r \geq \lambda_+ C,$$

which in turn implies (5). □

Lemma 8. *Let u be the solution of* (1) *in B_1^+ and suppose that for given constants c_0 and r_0, we have*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \geq c_0 > 0 \quad \textit{for } r < r_0.$$

Then there exists a constant c depending on c_0, $\lambda_\pm$ and the dimension such that

$$\sup_{B_r^+} |u| \geq c r^2 \quad \textit{for } r < r_0. \tag{7}$$

The same is also true for Ω^-.

Proof. Let $\widetilde{B}_r^+ = B_r^+ \cap \{x : x_1 > \varepsilon r\}$. We can fix an $\varepsilon > 0$ such that

$$\frac{|\Omega_u^+ \cap \widetilde{B}_r^+|}{|\widetilde{B}_r^+|} \geq \frac{c_0}{2} \quad \text{for } r < r_0.$$

Hence for each $r > 0$ there exists $x_r \in \Omega_u^+ \cap \widetilde{B}_{r/2}^+$. Applying the previous corollary to the ball $B_{d_r}(x_r)$, where d_r is the e_1 component of x_r, we get:

$$\sup_{B_r^+} |u| \geq \sup_{B_{d_r}(x_r)} |u| \geq \lambda_+ C d_r^2 \geq \varepsilon^2 \lambda_+ \frac{C}{4} r^2.$$

Thus the lemma is proved with $c = \varepsilon^2 \lambda_+ C/4$. $\square$

In the proof of the next lemma we use the technique from [A] (Lemma 5), see also [CKS]. A similar estimate in the interior was proved by Uraltseva in [U].

Lemma 9. *Let u solve* (1) *in B_1^+, $\|u\|_\infty \leq M$ and assume that its boundary data $f = u|_\Pi$ and the Dini modulus of continuity ω satisfy conditions* (2) *and* (3). *Then there exists a constant $C = C(M, R)$ such that*

$$\sup_{B_r^+} |u - D_{e_1} u(0) x_1| \leq C r^2, \quad 0 < r < \tfrac{1}{2}.$$

Proof. Let us denote by

$$S_j(u) := \sup_{B_{2^{-j}}^+} |u - D_{e_1} u(0) x_1|$$

and $\mathbf{M}(u) := \{j : S_j(u) \leq 4 S_{j+1}(u)\}$. We want to show that $S_j(u) \leq C 2^{-2j}$. First let us show this for all $j \in \mathbf{M}(u)$. The proof is done by contradiction: assume there exists a sequence $\{u_j\}_{j=1}^\infty$ of solutions of (1) in B_1^+ such that

$$S_{k_j}(u_j) \geq j 2^{-2k_j}$$

for some $k_j \in \mathbf{M}(u_j)$. Letting $w_j(x) := u_j(x) - D_{e_1} u_j(0) x_1$ and

$$\widetilde{w}_j(x) := \frac{w_j(2^{-k_j} x)}{S_{k_j+1}(u_j)},$$

we get

$$\|\Delta \widetilde{w}_j\|_\infty \leq \max(\lambda_\pm) \frac{2^{-2k_j}}{S_{k_j+1}(u_j)} \leq \max(\lambda_\pm) \frac{2^{-2k_j}}{\frac{1}{4} S_{k_j}(u_j)} \leq \max(\lambda_\pm) \frac{4}{j} \to 0.$$

We also have

$$\sup_{B^+_{1/2}} |\widetilde{w}_j| = 1. \tag{8}$$

The condition $(D_e f(x'))^\pm \le |x'|\omega(|x'|)$, for any unit vector $e\in\Pi$, implies that

$$\sup_{B^+_r} |D_e w_j| \le Cr, \tag{9}$$

where C depends on M and R. To check this one should consider harmonic functions $v_j^\pm$ in $B^+_{1/2}$ with the same boundary data as $(D_e w_j)^\pm$. Inequality (9) then follows from the subharmonicity of $(D_e w_j)^\pm$ (see [U]) and standard estimates on Green's function for the half-ball (see [Wi]). From (9) we have

$$\sup_{B^+_r} |D_e \widetilde{w}_j| \le \frac{4Cr}{j}. \tag{10}$$

A subsequence of $\widetilde{w}_j$ converges in $C^1(B^+_{1/2})$ to a harmonic function u_0. Due to (10) we get $D_e u_0 = 0$ for all $e\in\Pi$, thus $u_0 = ax_1$. On the other hand $D_{e_1}\widetilde{w}_j(0)=0$ and by C^1-convergence (up to Π) the same holds for u_0. Hence $u_0\equiv 0$, which contradicts (8).

Next let us show that $S_j(u)\le 4C2^{-2j}$ for all j. Suppose j is the first integer for which the inequality fails to hold, then

$$S_{j-1}(u) \le 4C2^{-2(j-1)} \le 4S_j(u),$$

i.e. $j-1\in\mathbf{M}(u)$ and

$$S_j(u) \le S_{j-1}(u) \le C2^{-2(j-1)} = 4C2^{-2j},$$

a contradiction. $\square$

3.2. Monotonicity formulae

Here we introduce two monotonicity formulae, which play crucial roles in our proofs. The first one was presented by H. W. Alt, L. A. Caffarelli and A. Friedman in [ACF] and was developed in [CKS]. The second one is due to G. S. Weiss [W1], [SUW]. Andersson ([A]) adapted it to the half-space case, our representation is analogous. See also [M] for the formula in the parabolic case.

Lemma 10. (The Alt–Caffarelli–Friedman monotonicity formula) *Let* h_1 *and* h_2 *be two non-negative continuous subsolutions of* $\Delta u=0$ *in* B_R. *Assume further that* $h_1 h_2=0$ *and* $h_1(0)=h_2(0)=0$. *Then the following function is non-decreasing in* $r\in(0,R)$,

$$\varphi(r)=\frac{1}{r^4}\left(\int_{B_r}\frac{|\nabla h_1|^2dx}{|x|^{n-2}}\right)\left(\int_{B_r}\frac{|\nabla h_2|^2dx}{|x|^{n-2}}\right). \tag{11}$$

More exactly, if any of the sets $\mathrm{spt}(h_j)\cap\partial B_r$ *digresses from a spherical cap by a positive area, then either* $\varphi'(r)>0$ *or* $\varphi(r)=0$.

Lemma 11. (Weiss' monotonicity formula) *Assume that* u *solves* (1) *in* B_R^+ *and* $u|_{\Pi\cap B_R}=0$. *Then the function*

$$\Phi(r)=r^{-n-2}\int_{B_r\cap\mathbf{R}^n_+}(|\nabla u|^2+2\lambda_+u^++2\lambda_-u^-)-r^{-n-3}\int_{\partial B_r\cap\mathbf{R}^n_+}2u^2d\mathcal{H}^{n-1} \tag{12}$$

is non-decreasing for $r\in(0,R)$. *Moreover, if* $\Phi(\rho)=\Phi(\sigma)$ *for any* $0<\rho<\sigma<R$, *then* Φ *is homogeneous of degree two in* $(B_\sigma\setminus B_\rho)\cap\mathbf{R}^n_+$.

The proof is analogous to the proof of Lemma 1 in [A].

4. Global solutions

In this section we will classify all solutions of (1) in $\mathbf{R}^n_+$ with zero boundary data and quadratic growth. We will see that the only possible solutions are

$$u(x)=\pm\frac{\lambda_\pm}{2}(x_1-a)_+^2,\ \ a\geq 0,\quad \text{or}\quad u(x)=\pm\frac{\lambda_\pm}{2}x_1^2\pm\alpha x_1,\ \ \alpha\geq 0. \tag{13}$$

The proofs of the next two lemmas adapt the proofs of analogous results in [SU] to our case.

First let us prove that u is two-dimensional.

Lemma 12. *Let* u *solve* (1) *in* $\mathbf{R}^n_+$ *with boundary data* $u|_\Pi=0$. *Then the function* u *is two-dimensional, i.e., in some system of coordinates*

$$u(x)=u(x_1,x_2),$$

where the e_1 *direction is orthogonal to* Π.

Proof. Let us take any direction e orthogonal to e_1 and consider functions $(D_e u)^{\pm}$. In [U] Uraltseva proved that these functions are subharmonic. Note that they will remain so if we extend them by zero to $\mathbf{R}^n_-$. Now we can apply the Alt–Caffarelli–Friedman monotonicity formula to $(D_e u)^{\pm}$. For $r<s$ we have

$$\varphi(r, D_e u) \le \varphi(s, D_e u) \le \lim_{s\to\infty} \varphi(s, D_e u) =: C_e.$$

In [U] it is shown that the second derivatives of u are bounded, thus we can find a sequence $u_{r_j} = u(r_j x)/r_j^2 \to u_\infty$, uniformly on compact subsets and in $(W^{2,p}_{\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}})(\mathbf{R}^n_+ \cup \Pi)$, for any $1<p<\infty$ and $0<\alpha<1$. Then we have

$$C_e = \lim_{r_j\to\infty} \varphi(s r_j, D_e u) = \lim_{r_j\to\infty} \varphi(s, D_e u_{r_j}) = \varphi(s, D_e u_\infty) \quad \text{for all } s>0.$$

From $\{x : x_1<0\} \subset \{x : D_e u(x)=0\}$ and Lemma 10 it follows that $\varphi(r, D_e u_\infty) \equiv 0$ or $\varphi'(r, D_e u_\infty) > 0$ for all $r>0$. Thus $C_e=0$ and we get $D_e u \ge 0$ or $D_e u \le 0$.

For $e_2 \in \Pi$ assume that $D_{e_2} u \ge 0$ and let $e_3 \in \Pi$ be orthogonal to e_2. Consider the unit vector $e(\phi) = \cos\phi\, e_2 + \sin\phi\, e_3 \in \Pi$, $\phi \in [0,\pi]$. From the C^1-continuity we have that the sets $\{\phi : \Omega^{\pm}_{D_{e(\phi)} u} \ne \varnothing\}$ are relatively open in $[0,\pi]$. On the other hand, they are both non-empty and have empty intersection; this means that there exists $\phi_0 \in (0,\pi)$ such that $D_{e(\phi_0)} u \equiv 0$. Rotating the coordinate system we get $D_{e_2} u \ge 0$ and $D_{e_3} u \equiv 0$. Repeating the above argument for e_k, $k=4, ..., n$, we get that u is two-dimensional. $\square$

We prove now the main result of this section under the assumption of homogeneity.

Proposition 13. *Let u be homogeneous of degree two solving* (1) *in $\mathbf{R}^n_+$ with boundary data $u|_\Pi = 0$. Then either $u(x) = \frac{1}{2}\lambda_+ x_1^2$ or $u(x) = -\frac{1}{2}\lambda_- x_1^2$.*

Proof. We can consider only two-dimensional functions u. So let us rewrite u in radial coordinates as

$$u(x) = u(r,\theta) = r^2 \phi(\theta), \quad r \in [0,\infty),\ \theta \in [0,\pi].$$

Then we get the following ordinary differential equation

$$\phi'' + 4\phi = \lambda_+ \chi_{\{\phi>0\}} - \lambda_- \chi_{\{\phi<0\}}$$

in the interval $[0,\pi]$ with boundary data $\phi(0)=\phi(\pi)=0$. It can be checked that the only solutions of this ordinary differential equation are $\phi(\theta) = \pm\frac{1}{2}\lambda_\pm \sin^2\theta$. $\square$

Lemma 14. *Let u solve* (1) *in $\mathbf{R}^n_+$ with boundary data $u|_\Pi = 0$ and be quadratically bounded at infinity. Then u has one of the representations in* (13).

Proof. If the function u is non-negative or non-positive, then the result we want to prove follows from Theorem B in [SU]. So let us show that u does not change sign. We do this by contradiction; assume that $u^\pm$ are both non-trivial.

Next we consider the shrink down of u; $\tilde{u}:=\lim_{j\to\infty} u_j$, where $u_j(x)=u(r_jx)/r_j^2$, $r_j\to\infty$. It is homogeneous of degree two. To verify this we need to use Weiss' monotonicity formula

$$\Phi(s,\tilde{u})=\lim_{j\to\infty}\Phi(s,u_j)=\lim_{j\to\infty}\Phi(sr_j,u)=\Phi(\infty,u).$$

Thus $\tilde{u}$ equals to one of $\pm\frac{1}{2}\lambda_\pm x_1^2$ by Proposition 13 above. Assume for definiteness that we have the "+"-sign.

This means that for any $\delta>0$ there exists R_δ such that

$$\Omega_u^-\setminus B_{R_\delta}^+\subset\{x:x_1<\delta|x_2|\}. \tag{14}$$

Let us now take the barrier function

$$U(x_1,x_2)=x_1^4+x_2^4-6x_1^2x_2^2+C.$$

For large enough C we have $\Omega_u^-\Subset\Omega_U^+$. Since u is quadratically bounded, we get from the comparison principle that $u^-(x)\le\varepsilon U(x)$ for any $\varepsilon>0$, and thus $\Omega_u^-=\varnothing$. □

5. Proofs

Proof of Theorem A. Here we consider only the case when (4) fails to hold. It follows from Lemma 9 that $D_{x_1}u(0)=0$ and

$$\sup_{B_r^+}|u|\le c_0r^2\quad\text{for } r<r_0. \tag{15}$$

Now assume that we do not have a tangential touch at 0, i.e., there is an $\varepsilon>0$ and a sequence $x^j\in K_\varepsilon\cap\Gamma_u$, $x^j\to0$. Repeating the proof of Lemma 8 we obtain

$$\sup_{B_{2d_j}^+}|u|\ge Cd_j^2\quad\text{for } r<r_1, \tag{16}$$

where $d_j=|x^j|$. Consider the blow up sequence

$$\tilde{u}_j(x)=\frac{u(2d_jx)}{4d_j^2},$$

which is bounded by (15). Therefore there is a subsequence converging in $C^{1,\alpha}$ to a global solution u_0 with zero boundary data. This solution is non-trivial (due to (16)). As in the proof of Lemma 14, using Weiss' monotonicity formula we get that u_0 is homogeneous of degree 2. This implies that $u_0(x)=\pm\frac{1}{2}\lambda_\pm x_1^2$, which contradicts the fact that $x^j\in K_\varepsilon$. □

Proof of Theorem B. The proof is done by contradiction. Assume there exist an $\varepsilon>0$, functions u_j satisfying the conditions of the theorem and a sequence $x^j\to 0$ such that $x^j\in K_\varepsilon\cap\Gamma_{u_j}$. Let us consider the blow-up sequence

$$\tilde{u}_j(x)=\frac{u_j(d_jx)}{\sup_{B^+_{d_j}}|u_j|},$$

where $d_j:=|x^j|$. We have that

$$\Delta\tilde{u}_j=\frac{d_j^2}{\sup_{B^+_{d_j}}|u_j|}\Delta u_j.$$

Two cases are possible: either

$$\frac{d_j^2}{\sup_{B^+_{d_j}}|u_j|}\to 0 \tag{17}$$

for some subsequence or

$$\frac{d_j^2}{\sup_{B^+_{d_j}}|u_j|}\not\to 0 \tag{18}$$

for all subsequences.

Let us consider the first case. From Lemma 9 it follows that

$$-Cr^2+|D_{e_1}u_j(0)|r\le\sup_{B^+_r}|u_j|\le Cr^2+|D_{e_1}u_j(0)|r. \tag{19}$$

This together with (17) gives that $|D_{e_1}u_j(0)|d_j^{-1}\to\infty$, thus we can assume that

$$|D_{e_1}u_j(0)|>jd_j. \tag{20}$$

From here and (19) we obtain

$$\left|\frac{\sup_{B^+_{d_j}}|u_j|}{d_j|D_{e_1}u_j(0)|}-1\right|\le\frac{C}{j}\to 0.$$

We arrive at

$$\sup_{B^+_r}|\tilde{u}_j|=\frac{\sup_{B^+_{rd_j}}|u_j|}{\sup_{B^+_{d_j}}|u_j|}\le\frac{Cr^2d_j^2+|D_{e_1}u_j(0)|rd_j}{\sup_{B^+_{d_j}}|u_j|}\to r. \tag{21}$$

There is a subsequence of $\tilde{u}_j$ converging to a function u_0 in $C^{1,\alpha}$, which is harmonic in $\mathbf{R}^n_+$ (due to (17)), linearly bounded (due to (21)) and has zero boundary data at Π. Extending u_0 by odd reflection to $\mathbf{R}^n_-$ and using Liouville's theorem we get that $u_0(x)=D_{e_1}u_0(0)x_1$ which contradicts the existence of zeros in K_ε.

In the case (18) without loss of generality we can assume

$$\frac{d_j^2}{\sup_{B^+_{d_j}}|u_j|}\to d>0. \tag{22}$$

Then we have that a subsequence of $\tilde{u}_j$ converges to a function u_0 in C^1 and (22) implies that u_0 is a global solution with $d\lambda_\pm$ instead of $\lambda_\pm$ and zero boundary data. Condition (4) and Lemma 14 give us that u_0 is strictly positive or negative in $\mathbf{R}^n_+$, which contradicts the fact that $x^j\in K_\varepsilon\cap\Gamma_{u_j}$. More precisely, the functions $\tilde{u}_j$ vanish at $\tilde{x}_j:=d_j^{-1}x_j\in K_\varepsilon\cap\Gamma_{u_j}\cap\partial B_1$. Thus we can always choose the subsequence of $\tilde{u}_j$ in such a way that the corresponding subsequence $\tilde{x}_j\to x_0\in K_\varepsilon\cap\Gamma_{u_j}\cap\partial B_1$ and then $u_0(x_0)=0$. □

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References

[ACF] Alt, H. W., Caffarelli, L. A. and Friedman, A., Variational problems with two phases and their free boundaries, *Trans. Amer. Math. Soc.* **282** (1984), 431–461.

[A] Andersson, J., On the regularity of a free boundary near contact points with a fixed boundary, to appear in *J. Differential Equations*.

[C] Caffarelli, L. A., The obstacle problem revisited, *J. Fourier Anal. Appl.* **4** (1998), 383–402.

[CKS] Caffarelli, L. A., Karp, L. and Shahgholian, H., Regularity of a free boundary with application to the Pompeiu problem, *Ann. of Math.* **151** (2000), 269–292.

[KKS] Karakhanyan, A. L., Kenig, C. E. and Shahgholian, H., The behaviour of the free boundary near the fixed boundary for a minimisation problem, *Calc. Var. Partial Differential Equations*, DOI 10.1007/s00526-006-0029-x.

[M] Matevosyan, N., *Tangential Touch Between Free and Fixed Boundaries*, Ph.D. thesis, Kungliga Tekniska högskolan, Stockholm, 2003.

[SU] Shahgholian, H. and Uraltseva, N., Regularity properties of a free boundary near contact points with the fixed boundary, *Duke Math. J.* **116** (2003), 1–34.

[SUW] Shahgholian, H., Uraltseva, N. and Weiss, G. S., Global solutions of an obstacle-problem-like equation with two phases, *Monatsh. Math.* **142** (2004), 27–34.

[U] Uraltseva, N. N., Two-phase obstacle problem. Function theory and phase transitions, *J. Math. Sci.* (*New York*) **106** (2001), 3073–3077.

[W1] Weiss, G. S., Partial regularity for weak solutions of an elliptic free boundary problem, *Commun. Partial Differential Equations* **23** (1998), 439–455.

[W2] Weiss, G. S., An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary, *Interfaces Free Bound.* **3** (2001), 121–128.

[Wi] Widman, K.-O., Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations, *Math. Scand.* **21** (1967), 17–37.

John Andersson
Institutionen för Matematik
Kungliga Tekniska högskolan
SE-100 44 Stockholm
Sweden
johnan@e.kth.se

Norayr Matevosyan
Johann Radon Institut für Angewandte Mathematik
Österreichische Akademie der Wissenschaften
Altenbergerstrasse 69
AT-4040 Linz
Austria
norayr.matevosyan@oeaw.ac.at

Hayk Mikayelyan
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
DE-04109 Leipzig
Germany
hayk@math.uni-leipzig.de

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