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The concentration function of additive functions on shifted primes

by

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In memory of Rolando Chaqui

1.

A real valued function f defined on the positive integers is *additive* if it satisfies $f(rs)=f(r)+f(s)$ whenever r and s are coprime. Such functions are determined by their values on the prime-powers.

For an additive arithmetic function f, let C_h denote the frequency amongst the integers n not exceeding x of those for which $h<f(n)\leqslant h+1$. Estimates for C_h that are uniform in h, f and x play a vital rôle in the study of the value distribution of additive functions. They can be employed to develop criteria necessary and sufficient that a suitably renormalised additive function possess a limiting distribution, as well as to elucidate the resulting limit law. They bear upon problems of algebraic nature, such as the product and quotient representation of rationals by rationals of a given type. In that context their quantitative aspect is important.

It is convenient to write $a\ll b$ *uniformly in* α if on the values of α being considered, the functions a, b satisfy $|a(\alpha)|\leqslant cb(\alpha)$ for some absolute constant c. When the uniformity is clear, I do not declare it.

Let

$$W(x)=4+\min_{\lambda}\left(\lambda^2+\sum_{p\leqslant x}\frac{1}{p}\min(1,|f(p)-\lambda\log p|)^2\right),$$

where the sum is taken over prime numbers. Improving upon an earlier result of Halász, Ruzsa proved that $C_h\ll W(x)^{-1/2}$, uniformly in h, f and $x\geqslant 2$ [15]. This result is best possible in the sense that for each of a wide class of additive functions there is a value of h so that the inequality goes the other way.

From a number theoretical point of view it is desirable to possess analogs of Ruzsa's result in which the additive function f is confined to a particular sequence of integers of arithmetic interest. In this paper I consider shifted primes.

Let a be a non-zero integer. Let Q_h denote the frequency amongst the primes p not exceeding x of those for which $h<f(p+a)\leqslant h+1$.

THEOREM 1. *The estimate* $Q_h \ll W(x)^{-1/2}$ *holds uniformly in* h, f *and* $x\geqslant 2$.

If for an integer $N\geqslant 3$ we define S_h to be the frequency amongst the primes p less than N of those for which $h<f(N-p)\leqslant h+1$, and set

$$Y(N)=4+\min_{\lambda}\Bigg(\lambda^2+\sum_{\substack{p<N\\(p,N)=1}}\frac{1}{p}\min(1,|f(p)-\lambda\log p|)^2\Bigg),$$

then there is an analogous result.

THEOREM 2. *The estimate* $S_h \ll Y(N)^{-1/2}$ *holds uniformly in* h, f *and* $N\geqslant 3$.

The estimates given in these two theorems are of the same quality as Ruzsa's, and again best possible. In particular, Theorem 1 improves the bound

$$Q_h \ll W(x)^{-1/2}(\log W(x))^2$$

of Timofeev [17].

Following Ruzsa, Timofeev gave a number of important applications of his bound. Qualitative applications are typified by a second proof of the following result, conjectured by Erdős and Kubilius, and first established by Hildebrand using a quite different method [14]. The frequencies

$$\nu_x(p;\, f(p+a)\leqslant z)=\pi(x)^{-1}\sum_{\substack{p\leqslant x\\ f(p+a)\leqslant z}}1$$

converge weakly to a distribution function as $x\to\infty$ if and only if the three series

$$\sum_{|f(p)|>1}\frac{1}{p},\quad \sum_{|f(p)|\leqslant 1}\frac{f(p)}{p},\quad \sum_{|f(p)|\leqslant 1}\frac{f(p)^2}{p}$$

converge. The latter is the classical condition of Erdős and Wintner, appropriate to the weak convergence of the frequencies $\nu_x(n;\, f(n)\leqslant z)$ on the natural numbers [10].

As a quantitative application of his estimate, Timofeev shows that if

$$E(x)=4+\sum_{\substack{p\leqslant x\\ f(p)\neq 0}}\frac{1}{p},$$

then the number of primes not exceeding x for which $f(p+a)$ assumes any (particular) value is $\ll \pi(x)E(x)^{-1/2}(\log E(x))^2$. Employing the present Theorem 1, the logarithmic factor may be stripped from this bound. The improved inequality is then analogous to an estimate of Halász concerning additive functions on the natural numbers [12] and, in a sense, best possible.

The concentration function estimate of Theorem 2 also has many applications, in particular to the study of the value distribution of additive functions. These are new and of a new type. They involve not only the primes but also the length of the interval on which the additive function is considered. Thus the frequencies

$$\nu_N(p;\, f(N-p)\leqslant z)=(\pi(N-1))^{-1}\sum_{\substack{p<N\\ f(N-p)\leqslant z}}1$$

converge weakly to a distribution function as $N\to\infty$, if and only if the three series condition of Erdős and Wintner is satisfied. More complicated examples involving unbounded renormalisations can be successfully treated, but I leave the details to another occasion.

Also left to another occasion is the application of the method of this paper to the study of the representation of rationals by products and quotients of shifted primes.

To gain an upper bound for the frequency Q_h, Timofeev counts those integers m, not exceeding x, which have no prime factor in an interval $(z_1, z_2]$, and for which $h< f(m+1)\leqslant h+1$. By Fourier analysis he reduces himself directly to estimating sums of multiplicative functions over the sequence $m+1$, $m\leqslant x$. Such sums he treats by the dispersion method of Linnik.

The dispersion method becomes unwieldy if small prime factors of numbers are to be considered. To avoid this z_1 is to be taken perhaps up to a power of $\log x$. Moreover, in his application of the dispersion method, Timofeev employs asymptotic estimates for the frequency of integers m, up to x, which satisfy various non-trivial side conditions. These estimates he derives from a result in the present author's book [2, Chapter 2, Lemma 2.1, pp. 79–89] which in turn is obtained by applications of the Selberg sieve. For this methodology to effect reasonably small error terms, it is practically necessary that z_2 be appreciably smaller than a power of x, possibly down to $\exp(\log x(\log\log x)^{-2})$. Owing to such restrictions upon z_1 and z_2, the method of Timofeev tends intrinsically to the loss of two logarithms in the final bound.

My present method also applies Fourier analysis, but is otherwise conceptually different. It makes no application of the dispersion method. It is of interest for itself. The probability device of stepping between a distribution function and its characteristic function allows concentration functions to be estimated using integrals derived by Fourier analysis. I start by regarding a suitable such integral as the action of a linear operator which is real non-negative on exponential functions. This non-negativity allows the application of Selberg's sieve method, and the operator will then commute with the summations that arise in connection with the sieve.

Several features of the overall method call for comment. Lemmas 3, 4, 6 and 7 represent analogs of the Bombieri–Vinogradov theorem for multiplicative functions. Typical is Lemma 7. What is important in the present context is not only the condition $D \leqslant x^{1/2-\varepsilon}$ on the moduli, but the quality of the error term. Over an interval of length x an arbitrary power of a logarithm is saved. In order to gain this the estimate applies not to the multiplicative function g, but to a modified function $g-\beta_1-\beta_2$, with functions β_j defined immediately preceding the statement of Lemma 6. This makes manifest a remark in [7, p. 408] already in view in [5], that the construction of such analogs for unrestricted multiplicative functions would involve a change in form. The function β_2 is largely supported on the primes and without further information concerning the value distribution of $g(p)$ on residue classes cannot be generally removed. It is as if there were an underlying spectral decomposition, whose first eigenfunction gives rise to β_2. The effect of successive eigenfunctions might be generated from β_1 by induction.

Another feature, arising in connection with the functions β_j, is the casting of Selberg-sieve square functions $(\sum_{d|m} \lambda_d)^2$ on the multiplicative integers in a rôle which on the additive group of reals is usually played by the Fejér kernel. In part, five simultaneous applications of the Selberg sieve are contrived.

It will transpire that in the definition of $W(x)$ we may confine λ to the range $|\lambda| \leqslant (\log x)^2$, and in the definition of $Y(N)$, to $|\lambda| \leqslant (\log N)^2$.

Unless otherwise stated g will denote a complex valued multiplicative function of at most 1 in absolute value. Only in the specific application to the proofs of Theorems 1 and 2 will g have the form $\exp(itf(n))$ and depend upon the additive function f under consideration.

The proof of Theorem 1 is given in complete detail in four stages, the first three summarized by the inequalities (2), (10) and (14). The proof of Theorem 2 follows the same lines, and only significant changes are described in detail. I close the paper with a discussion of several examples to indicate senses in which Theorems 1 and 2 may be viewed best possible.

I thank the referee for a careful reading of the text.

2. Proof of Theorem 1, beginning

For every real u,

$$\int_{-1}^{1}(1-|t|)e^{itu}\,dt=\left(\frac{\sin\frac{1}{2}u}{\frac{1}{2}u}\right)^2$$

is non-negative. Moreover, for $|u|\leqslant 1$ this integral is as large as $4/\pi^2$. Let $3|a|\leqslant y\leqslant z\leqslant x$. Those primes in the interval $(z,x]$ for which $h<f(p+a)\leqslant h+1$ certainly lie amongst the integers n in the same interval which have no prime factor in the range $(y,z]$ and satisfy $h<f(n+1)\leqslant h+1$. Define the multiplicative function $g(m)=\exp(itf(m))$. Then

$$Q_h\leqslant\frac{3}{\pi(x)}\sum_{\substack{n\leqslant x\\(n,P_{y,z})=1}}\int_{-1}^{1}(1-|t|)e^{-ith}g(n+a)\,dt+\frac{z}{\pi(x)},\tag{1}$$

where $P_{y,z}$ denotes the product of the primes in the interval $(y,z]$. In the subsequent argument z will be chosen a fixed power of x not exceeding $x^{3/4}$, y to be a (small) constant. It is convenient to introduce a further parameter w, $y\leqslant w\leqslant z$, which will ultimately be chosen a power of $\log x$, and the corresponding products $P_{y,w},P_{w,z}$. The condition $(n,P_{y,z})=1$ will be introduced in two stages; $(n,P_{w,z})=1$ by one method, $(n,P_{y,w})=1$ by another.

Since the integrals are non-negative, the direction of the inequality (1) is preserved if we replace the condition $(n,P_{y,z})=1$ by $(n,P_{y,w})=1$, and introduce real numbers λ_d, $1\leqslant d\leqslant z$, with $\lambda_1=1$, in the style of Selberg's sieve method. The first of the two terms in the bound for Q_h does not exceed

$$\frac{3}{\pi(x)}\sum_{\substack{n\leqslant x\\(n,P_{y,w})=1}}\Bigg(\sum_{\substack{d|(n,P_{w,z})\\d\leqslant z}}\lambda_d\Bigg)^2\int_{-1}^{1}(1-|t|)e^{-ith}g(n+a)\,dt$$

so that

$$Q_h\leqslant\frac{3}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{d_j|P_{w,z}\\d_j\leqslant z}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leqslant x,\,(n,P_{y,w})=1\\n\equiv 0\ (\mathrm{mod}\ [d_1,d_2])}}g(n+a)\,dt+\frac{z}{\pi(x)}.\tag{2}$$

We are reduced to the study of multiplicative functions on arithmetic progressions with the adjoined condition $(n,P_{y,w})=1$. Estimates for their sums will be developed in the next section. In the interests of clarity of exposition, the best possible values of various parameters arising will not be pursued here. It will be convenient to denote $\log x$ by l, $\log\log x$ by $\log l$.

It may seem curious to split the condition $(n, P_{y,z})=1$. However, an expected estimate of the type

$$\sum_{\substack{n\leqslant x\\ n\equiv a \pmod D}} g(n)=\frac{1}{\phi(D)}\sum_{\substack{n\leqslant x\\ (n,D)=1}} g(n)+\text{'small'},$$

certainly fails if g is (near to) a non-principal character defined $(\mathrm{mod}\ D)$, as may be the case. In [4], [6], [9] I showed that the moduli for which such an estimate fails are effectively multiples of a single modulus D_0. In the present situation it will be arranged that any such D_0 may be assumed to divide $P_{y,w}$.

3. Large moduli

LEMMA 1. *The inequality*

$$\sum_{d\leqslant Q}\frac{Dd}{\phi(Dd)}\sum_{\chi\ (\mathrm{mod}\ Dd)}{}^{*}\left|\sum_{n=M}^{M+N} a_n\chi(n)\right|^2 \ll (N+DQ^2)\sum_{n=M}^{M+N}|a_n|^2$$

*is valid for integers M, N and D, $N\geqslant 1$, $D\geqslant 1$, all complex a_n, $n=M, \ldots, M+N$, and real $Q\geqslant 1$. Here * denotes summation over primitive characters.*

Proof. This version of the Large Sieve may be found as Lemma (6.5), on p. 111 of [3].

A multiplicative function g is exponentially multiplicative if it satisfies $g(p^k)= g(p)^k/k!$ on the powers of primes. This is only a new requirement if $k\geqslant 2$.

For $b\geqslant 2$ define

$$k_1(n)=\sum_{\substack{pm=n\\ p\leqslant b}} g(m)g(p)\log p,\quad k_2(n)=\sum_{\substack{pr=n\\ r\leqslant b}} g(r)g(p)\log p, \tag{3}$$

and set $\alpha(n)=g(n)\log n-k_1(n)-k_2(n)$.

LEMMA 2. *If $A\geqslant 0$, $0<\delta<\frac{1}{2}$, $b=(\log x)^{6A+15}$, $\sqrt{3}\leqslant y\leqslant x^{\delta}$, and ε is sufficiently small in terms of δ, then for exponentially multiplicative g*

$$\sum_{y<d\leqslant y^{1+\varepsilon}}\frac{1}{d}\sum_{\chi\ (\mathrm{mod}\ d)}{}^{*}\max_{v\leqslant x}\left|\sum_{n\leqslant v}\alpha(n)\chi(n)\right| \ll y^{-1}xl^{2A+8}+xl^{-A}.$$

The implied constant depends at most upon A and δ.

Proof. This is Theorem 2, p. 178 of [5].

LEMMA 3. *In the notation of Lemma 2*

$$\sum_{D_1D_2\leqslant x^6}\max_{(r,D_1D_2)=1}\Bigg|\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1D_2}}}\alpha(n)-\frac{1}{\phi(D_2)}\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1}\\ (n,D_2)=1}}\alpha(n)\Bigg|$$

$$\ll xl^{1-A}(\log l)^2+w^{-1}xl^{2A+9}(\log l)^2+w^{-1/2}xl^{7/2}\log l,$$

where D_1 is confined to those integers whose prime factors do not exceed w, and D_2 to integers whose prime factors exceed w. The implied constant depends at most upon A.

Proof. Without loss of generality $w\leqslant x^6$, otherwise there will be no D_2. If $(D_1,D_2)=1$, then the characters (mod D_1D_2) may be uniquely expressed in the form $\chi_1\chi_2$ where χ_j is induced by a character (mod D_j), $j=1,2$. For if χ_j is induced by a primitive character (mod m_j), then for $j=1,2$, $m_j|D_j$ and so $(m_1,m_2)=1$. From the orthogonal properties of Dirichlet characters

$$\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1D_2}}}\alpha(n)-\frac{1}{\phi(D_1D_2)}\sum_{\substack{n\leqslant x\\ (n,D_1D_2)=1}}\alpha(n)$$

$$=\frac{1}{\phi(D_1D_2)}\sum_{\substack{\chi_1\chi_2\\ \text{not principal}}}\overline{\chi_1\chi_2}(r)\sum_{n\leqslant x}\alpha(n)\chi_1\chi_2(n).$$

The difference

$$\frac{1}{\phi(D_2)}\Bigg(\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1}\\ (n,D_2)=1}}\alpha(n)-\frac{1}{\phi(D_1)}\sum_{\substack{n\leqslant x\\ (n,D_1)=1\\ (n,D_2)=1}}\alpha(n)\Bigg)$$

has a similar representation with the characters restricted by: χ_2 principal, χ_1 not principal. Subtraction gives

$$\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1D_2}}}\alpha(n)-\frac{1}{\phi(D_2)}\sum_{\substack{n\leqslant x\\ n\equiv r \pmod{D_1}\\ (n,D_2)=1}}\alpha(n)=\frac{1}{\phi(D_1D_2)}\sum_{\chi}{}'\,\bar{\chi}(r)\sum_{n\leqslant x}\alpha(n)\chi(n)$$

where χ runs through the characters $\chi_1\chi_2$ (mod D_1D_2) with χ_2 induced by a non-principal character (mod D_2). In particular, if $\chi_1\chi_2$ is induced by a primitive character (mod d), then d has at least one prime factor in common with D_2, and so satisfies $d>w$.

The sum to be estimated does not exceed

$$\sum_{D_1D_2\leqslant x^6}\frac{1}{\phi(D_1D_2)}\sum_{\chi}{}'\Bigg|\sum_{n\leqslant x}\alpha(n)\chi(n)\Bigg|.$$

Collecting up terms according to the primitive characters by which the χ are induced, we see that this multiple sum does not exceed

$$V=\sum_{w<d\leqslant x^\delta}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\ \widetilde{\sum_{D_1D_2\equiv 0\,(\mathrm{mod}\,d)}}\frac{1}{\phi(D_1D_2)}\left|\sum_{n\leqslant x}\alpha(n)\chi(n)\right|,$$

where the sum $\widetilde{\sum}$ runs over the characters χ to the various moduli D_1D_2 which are induced by χ_1. Here $\chi(n)=\chi_1(n)$ if $(n,D_1D_2)=1$, and is zero otherwise. A typical innermost sum has the representation(s)

$$\begin{aligned}\sum_{\substack{n\leqslant x\\(n,D_2)=1}}\alpha(n)\chi_1(n)&=\sum_{n\leqslant x}\alpha(n)\chi_1(n)\sum_{m\mid(n,D_2)}\mu(m)\\&=\sum_{m\mid D_2}\mu(m)\sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\,m)}}\alpha(n)\chi_1(n).\end{aligned}$$

If $m|D_2$ and $m>1$, then $m>w$. Thus

$$\begin{aligned}V\leqslant&\sum_{w<d\leqslant x^\delta}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\ \widetilde{\sum_{D_1D_2\equiv 0\,(\mathrm{mod}\,d)}}\frac{1}{\phi(D_1D_2)}\left|\sum_{n\leqslant x}\alpha(n)\chi_1(n)\right|\\&+\sum_{w<d\leqslant x^\delta}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\ \widetilde{\sum_{D_1D_2\equiv 0\,(\mathrm{mod}\,d)}}\frac{1}{\phi(D_1D_2)}\sum_{\substack{m\mid D_2\\ m>w}}\mu^2(m)\left|\sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\,m)}}\alpha(n)\chi_1(n)\right|.\end{aligned}$$

I call these two multiple sums V_1 and V_2, respectively.

The innermost sum in V_1 no longer depends upon any induced characters χ. Since $\phi(n)\gg n(\log l)^{-1}$ uniformly for all integers $n\leqslant x$,

$$V_1\ll l\log l\sum_{w<d\leqslant x^\delta}\frac{1}{d}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\left|\sum_{n\leqslant x}\alpha(n)\chi_1(n)\right|.$$

This sum may be decomposed into $O(\log l)$ pieces in each of which the moduli d are confined to a range $y<d\leqslant y^{1+\varepsilon}$. Applications of Lemma 2 now yield the first two of the upper bound terms in the statement of Lemma 3.

The sum V_2 is estimated by changing the order of summation, bringing the summation over m to the outside. Any modulus D_1D_2 counted in $\widetilde{\sum}$ will then be a multiple of d and m, and therefore of $[d,m]$. Hence

$$\begin{aligned}V_2&\ll l\log l\sum_{w<m\leqslant x^\delta}\mu^2(m)\sum_{d\leqslant x^\delta}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\frac{1}{[d,m]}\left|\sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\,m)}}\alpha(n)\chi_1(n)\right|\\&\ll l\log l\sum_{w<m\leqslant x^\delta}\frac{\mu^2(m)}{m}\sum_{h\mid m}h\sum_{\substack{d\leqslant x^\delta\\(d,m)=h}}\frac{1}{d}\sum_{\chi_1\,(\mathrm{mod}\,d)}^{*}\left|\sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\,m)}}\alpha(n)\chi_1(n)\right|.\end{aligned}$$

For a typical pair m, h let F denote the inner (triple) sum over d, χ_1 and n.

Applying the Cauchy–Schwarz inequality several times

$$F^2 \leqslant \sum_{\substack{d\leqslant x^\delta\\(d,m)=h}} \frac{1}{d} \sum_{\substack{d\leqslant x^\delta\\(d,m)=h}} \sideset{}{^*}\sum_{\chi_1 \,(\mathrm{mod}\, d)} \Bigg| \sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\, m)}} \alpha(n)\chi_1(n)\Bigg|^2 .$$

The factor sum involving the reciprocals of d is $\ll h^{-1}l$. The remaining multiple sum is estimated by Lemma 1, using the representation

$$\sum_{\substack{n\leqslant x\\ n\equiv 0\,(\mathrm{mod}\, m)}} \alpha(n)\chi_1(n) = \chi_1(m) \sum_{t\leqslant x/m} \chi_1(t)\alpha(tm).$$

In this way we obtain the bound

$$F^2 \ll \frac{l}{h}\left(\frac{x}{m}+\frac{x^{2\delta}}{h}\right) \sum_{t\leqslant x/m} |\alpha(tm)|^2 \ll x^2 l^3 \left(\frac{1}{hm^2}+\frac{1}{h^2 m}\right).$$

Then

$$V_2 \ll x l^{5/2} \log l \sum_{w<m\leqslant x^\delta} \frac{\mu^2(m)}{m^{3/2}}\, d(m)$$

where the sum is

$$\ll \sum_{2^j>w/2} 2^{-3j/2} \sum_{m\leqslant 2^{j+1}} d(m) \ll \sum_{2^j>w/2} 2^{-j/2} j \ll w^{-1/2}\log w.$$

This completes the estimation of V, and the proof of Lemma 3.

Integrating by parts we obtain from Lemma 3:

LEMMA 4. *Let $A\geqslant 0$, $b=(\log x)^{6A+15}$. Then*

$$\sum_{D_1 D_2\leqslant x^\delta} \max_{(r,D_1D_2)=1} \Bigg| \sum_{\substack{2\leqslant n\leqslant x\\ n\equiv r\,(\mathrm{mod}\, D_1D_2)}} \frac{\alpha(n)}{\log n} - \frac{1}{\phi(D_2)} \sum_{\substack{2\leqslant n\leqslant x,\,(n,D_2)=1\\ n\equiv r\,(\mathrm{mod}\, D_1)}} \frac{\alpha(n)}{\log n}\Bigg|$$

$$\ll x l^{-A}(\log l)^2 + w^{-1} x l^{2A+8}(\log l)^2 + w^{-1/2} x l^{5/2}\log l,$$

where D_1 is confined to those integers whose prime factors do not exceed w, and D_2 to integers whose prime factors exceed w. The implied constant depends at most upon A.

Proof. Since the terms of Lemma 3 involving $\alpha(1)$ contribute $O(x^\delta)$ to the sum, we are reduced to the estimate of

$$\sum_{D_1 D_2\leqslant x^\delta} \max_{(r,D_1D_2)=1} \Bigg| \frac{1}{\log x}\Bigg\{ \sum_{\substack{n\leqslant x\\ n\equiv r\,(\mathrm{mod}\, D_1D_2)}} \alpha(n) - \frac{1}{\phi(D_2)} \sum_{\substack{n\leqslant x\\ n\equiv r\,(\mathrm{mod}\, D_1)}} \alpha(n)\Bigg\}\Bigg|$$

and

$$\sum_{D_1D_2\leqslant x^\delta} \max_{(r,D_1D_2)=1} \int_2^x \left| \sum_{\substack{n\leqslant y\\ n\equiv r \pmod{D_1D_2}}} \alpha(n) - \frac{1}{\phi(D_2)} \sum_{\substack{n\leqslant y\\ n\equiv r \pmod{D_1}}} \alpha(n) \right| \frac{dy}{y(\log y)^2}. \tag{4}$$

The first of these expressions may be estimated directly by Lemma 3. Lemma 3 will also apply to the terms in the second expression which involve the range $(x^{4\delta/(2\delta+1)}, x]$ of the integral, since over that range $D_1D_2\leqslant x^\delta \leqslant y^{(\delta+1/2)/2}$ where $\frac{1}{2}(\delta+\frac{1}{2})<\frac{1}{2}$. Otherwise we argue crudely that

$$\int_2^{x^{2\delta/(2\delta+1)}} \left| \sum_{\substack{n\leqslant y\\ n\equiv r \pmod{D_1D_2}}} \alpha(n) - \frac{1}{\phi(D_2)} \sum_{\substack{n\leqslant y\\ n\equiv r \pmod{D_1}}} \alpha(n) \right| \frac{dy}{y(\log y)^2}$$
$$\ll \int_2^{x^{2\delta/(2\delta+1)}} l^2\left(\frac{y}{D_1D_2}+1\right)\frac{dy}{y(\log y)^2}.$$

The corresponding contribution to the expression at (4) is then

$$\ll l^2 \sum_{D\leqslant x^\delta} \left(\frac{x^{2\delta/(2\delta+1)}}{Dl^2}+1\right) \ll x^{2\delta/(2\delta+1)}.$$

This completes the proof of Lemma 4.

Remarks. We have $\alpha(n)(\log n)^{-1}=g(n)-k_1(n)(\log n)^{-1}-k_2(n)(\log n)^{-1}$. It follows from the definition (3) that each function $k_j(n)(\log n)^{-1}$ is at most one in absolute value, but is not generally multiplicative. Since b is small compared to x, $k_1(n)(\log n)^{-1}$ is comparable to $g(n)(\log x)^{-1}$. The function $k_2(n)(\log n)^{-1}$ is largely supported on the primes, where it is essentially bounded. Whilst we might treat k_1 using an argument by induction, k_2 remains an obstacle to forming a simple general analog of the well-known theorem of Bombieri and Vinogradov concerning primes in arithmetic progression.

Consider now a multiplicative function g which satisfies $|g(n)|\leqslant 1$. Define an exponentially multiplicative function g_1 by $g_1(p)=g(p)$, and the multiplicative function h by convolution: $g=h*g_1$.

LEMMA 5.

$$\sum_{r\leqslant x} r^{-1/2}|h(r)| \ll (\log x)^{3/2}$$

uniformly in g, and $x\geqslant 2$.

Proof. By examining Euler products we see that

$$h(p^k)=\sum_{u+v=k} \frac{(-g(p))^u}{u!} g(p^v),$$

with the understanding that 0-powers are to be replaced by 1. In particular, $h(p)=0$, $h(p^2)=g(p^2)-\frac{1}{2}g(p)^2$, $|h(p^k)|\leqslant e$ for $k\geqslant 3$. The sum to be estimated therefore does not exceed

$$\prod_{p\leqslant x}\left(1+\sum_{k=1}^{\infty}|h(p^k)|p^{-k/2}\right)\ll\exp\left(\sum_{p\leqslant x}\sum_{k=1}^{\infty}|h(p^k)|p^{-k/2}\right)$$
$$\ll\exp\left(\frac{3}{2}\sum_{p\leqslant x}\frac{1}{p}\right)\ll(\log x)^{3/2}.$$

For $B\geqslant 0$ define

$$\beta_1(n)=\sum_{\substack{ump=n\\ u\leqslant l^B,\, p\leqslant b}}h(u)\,\frac{g_1(m)g(p)\log p}{\log mp},\quad \beta_2(n)=\sum_{\substack{urp=n\\ u\leqslant l^B,\, r\leqslant b}}h(u)\,\frac{g_1(r)g(p)\log p}{\log rp},$$

and set $\beta(n)=g(n)-\beta_1(n)-\beta_2(n)$. Clearly

$$|\beta_j(n)|\leqslant\sum_{\substack{u|n\\ u\leqslant l^B}}|h(u)|\ll l^{(B+3)/2}\quad\text{uniformly in } n,j.$$

LEMMA 6. *Let $B\geqslant 0$, $A\geqslant 0$, $b=(\log x)^{6A+15}$, $0<\delta<\frac{1}{2}$. Then, for any multiplicative function which satisfies $|g(n)|\leqslant 1$,*

$$\sum_{D_1D_2\leqslant x^\delta}\max_{(r,D_1D_2)=1}\left|\sum_{\substack{n\leqslant x\\ n\equiv r \ (\mathrm{mod}\ D_1D_2)}}\beta(n)-\frac{1}{\phi(D_2)}\sum_{\substack{n\leqslant x,\,(n,D_2)=1\\ n\equiv r\ (\mathrm{mod}\ D_1)}}\beta(n)\right|$$

$$\ll xl^{-A}(\log l)^2+w^{-1}xl^{2A+8}(\log l)^2+w^{-1/2}xl^{5/2}\log l+xl^{5/2-B/2},$$

where D_1 is confined to those integers whose prime factors do not exceed w, and D_2 to integers whose prime factors exceed w. The implied constant depends at most upon A,B.

Proof. Since $g=h*g_1$

$$\sum_{\substack{n\leqslant x\\ n\equiv r\ (\mathrm{mod}\ D_1D_2)}}\beta(n)=\sum_{u\leqslant l^B}h(u)\sum_{\substack{v\leqslant x/u\\ uv\equiv r\ (\mathrm{mod}\ D_1D_2)}}\alpha(v)(\log v)^{-1}$$
$$+\sum_{l^B<u\leqslant x}h(u)\sum_{\substack{v\leqslant x/u\\ uv\equiv r\ (\mathrm{mod}\ D_1D_2)}}g_1(v).$$

The (multiple) sum which we wish to estimate does not exceed

$$\sum_{D_1D_2\leqslant x^\delta}\sum_{u\leqslant l^B}|h(u)|\max_{(r,D_1D_2)=1}\Bigg|\sum_{\substack{v\leqslant x/u\\ uv\equiv r\ (\mathrm{mod}\ D_1D_2)}}\alpha(v)(\log v)^{-1} -\frac{1}{\phi(D_2)}\sum_{\substack{v\leqslant x/u,\,(v,D_2)=1\\ uv\equiv r\ (\mathrm{mod}\ D_1)}}\alpha(v)(\log v)^{-1}\Bigg|$$

$$+\sum_{D_1D_2\leqslant x^\delta}\sum_{l^B<u\leqslant x}|h(u)|\max_{(r,D_1D_2)=1}\Bigg|\sum_{\substack{v\leqslant x/u\\ uv\equiv r\ (\mathrm{mod}\ D_1D_2)}}g_1(v)-\frac{1}{\phi(D_2)}\sum_{\substack{v\leqslant x/u,\,(v,D_2)=1\\ uv\equiv r\ (\mathrm{mod}\ D_1)}}g_1(v)\Bigg|. \tag{5}$$

The second of these expressions is estimated crudely to be

$$\ll\sum_{D\leqslant x^\delta}\sum_{l^B<u\leqslant x}|h(u)|\left(\frac{x}{u\phi(D)}+1\right)$$
$$\ll xl\sum_{l^B<u\leqslant x}u^{-1}|h(u)|+x^\delta\sum_{l^B<u\leqslant x}|h(u)|,$$

which by Lemma 5 is $\ll xl^{5/2-B/2}$.

Let $\delta'=\frac{1}{2}\left(\delta+\frac{1}{2}\right)$. To estimate the first multiple sum at (5) we change its order of summation. For all sufficiently large x the restriction $u\leqslant l^B$ in the summation conditions ensures that $D_1D_2\leqslant x^\delta<(x/u)^{\delta'}$. Temporarily fixing u we apply Lemma 4. The multiple sum is

$$\ll\sum_{u\leqslant l^B}|h(u)|(u^{-1}xl^{-A}(\log l)^2+w^{-1}u^{-1}xl^{2A+8}(\log l)^2+w^{-1/2}u^{-1}xl^{5/2}\log l).$$

The proof of Lemma 6 is completed by a further appeal to Lemma 5.

The next lemma is specialised to the present situation.

LEMMA 7. *In the notation of Lemma* 6 *set* $B=2A+5$. *Let*

$$(\log x)^{3A+8}\leqslant w\leqslant\exp(\sqrt{\log x}).$$

Let P be a product of primes which do not exceed w. Then

$$\sum_{\substack{D\leqslant x^\delta\\ p|D\Rightarrow p>w}}\Bigg|\sum_{\substack{n\leqslant x,\,(n-a,P)=1\\ n\equiv a\ (\mathrm{mod}\ D)}}\beta(n)-\frac{1}{\phi(D)}\sum_{\substack{n\leqslant x,\,(n-a,P)=1\\ (n,D)=1}}\beta(n)\Bigg|\ll x(\log x)^{1-A}.$$

The following well known result will play an auxiliary rôle in the proof of Lemma 7.

LEMMA 8. *Let $2\leqslant r\leqslant x$. The number of integers, not exceeding x, which are comprised only of primes up to r is $\ll x\exp(-\log x/\log r)+x^{14/15}$.*

Proof of Lemma 8. A proof of a stronger result, using sieve methods and argument from Probabilistic Number Theory, may be found in Lemma 13 of [9]. In fact the weaker bound $\ll x\exp(-\log x/(2\log r))$ will suffice, and this may be rapidly obtained by Rankin's method, as demonstrated by Tenenbaum [16, III.5, théorème 1, p. 396].

Proof of Lemma 7. The condition $(n-a,P)=1$ may be incorporated into the sums involving $\beta(n)$ by means of the Möbius function:

$$\begin{aligned}\sum_{\substack{n\leqslant x,\,(n-a,P)=1\\ n\equiv a\ (\mathrm{mod}\ D)}}\beta(n) &= \sum_{\substack{n\leqslant x\\ n\equiv a\ (\mathrm{mod}\ D)}}\beta(n)\sum_{\substack{m\mid(n-a,P)\\ m\leqslant x+|a|}}\mu(m)\\ &= \sum_{\substack{m\mid P\\ m\leqslant x+|a|}}\mu(m)\sum_{\substack{n\equiv a\ (\mathrm{mod}\ mD)\\ n\leqslant x}}\beta(n).\end{aligned}$$

The multiple sum to be estimated in Lemma 7 is therefore at most

$$\sum_{\substack{m\mid P\\ m\leqslant x+|a|}}|\mu(m)|\sum_{\substack{D\leqslant x^{\delta}\\ p\mid D\Rightarrow p>w}}\Bigg|\sum_{\substack{n\leqslant x\\ n\equiv a\ (\mathrm{mod}\ mD)}}\beta(n)-\frac{1}{\phi(D)}\sum_{\substack{n\leqslant x,\,(n,D)=1\\ n\equiv a\ (\mathrm{mod}\ m)}}\beta(n)\Bigg|$$

The contribution arising from the range $m\leqslant x^{(1/2-\delta)/2}$ is estimated using Lemma 6. The remaining contribution, estimated crudely, is

$$\ll\sum_{\substack{m\mid P\\ x^{(1/2-\delta)/2}<m\leqslant x+|a|}}|\mu(m)|\sum_{\substack{D\leqslant x^{\delta}\\ p\mid D\Rightarrow p>w}}\frac{xl^{A+4}}{m\phi(D)}\ll xl^{A+5}\sum_{\substack{m\mid P\\ x^{(1/2-\delta)/2}<m\leqslant x+|a|}}\frac{|\mu(m)|}{m}.$$

Note that the conditions $n\leqslant x$, $n\equiv a$ (mod mD) can only be simultaneously satisfied if $mD\leqslant x+|a|$. The last sum is covered by disjoint intervals of the form $(U,2U]$, on each of which Lemma 8 is applied:

$$\sum_{\substack{m\mid P\\ U<m\leqslant 2U}}\frac{1}{m}\ll\exp\left(-\frac{\log U}{\log w}\right)+U^{-1/15}\ll\exp(-(\log x)^{1/3}).$$

Since there are $\ll\log x$ such intervals, the proof of Lemma 7 is complete.

Remark. It is possible to introduce an extra uniformity into Lemma 3. In the statement of that lemma the expression maximized over the reduced residue classes $(\mathrm{mod}\,D_1D_2)$ may be replaced by

$$\max_{y\leqslant x}\Bigg|\sum_{\substack{n\leqslant y\\ n\equiv r\ (\mathrm{mod}\ D_1D_2)}}\alpha(n)-\frac{1}{\phi(D_2)}\sum_{\substack{n\leqslant y,\,(n,D_2)=1\\ n\equiv r\ (\mathrm{mod}\ D_1)}}\alpha(n)\Bigg|.$$

To effect this improvement we replace Lemma 1 by a maximal version of the Large Sieve:

$$\sum_{d\leqslant Q}\frac{Dd}{\phi(Dd)}\sum_{\chi\ (\mathrm{mod}\ Dd)}^{*}\max_{y\leqslant N}\left|\sum_{n\leqslant y}a_n\chi(n)\right|^2 \ll (N+DQ^2(\log N)^{2+\varepsilon})\sum_{n\leqslant N}|a_n|^2,$$

valid for each fixed $\varepsilon>0$. A proof of this inequality for $D=1$ is given as Theorem 1 of [8, p. 149]. The general case may be obtained in the same way if Lemma 1 of [8, p. 150] is replaced by Lemma 1 of the present paper.

The extra uniformity gained in Lemma 3 carries forward to give corresponding extra uniformities in Lemmas 4 to 7.

4. Proof of Theorem 1, continuation

Returning to the integral at (2) we introduce the functions $\beta_j(n)$. Thus

$$Q_h \ll \frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leqslant x,\ (n,P_{y,w})=1\\ n\equiv 0\ (\mathrm{mod}\ [d_1,d_2])}}\beta(n+a)\,dt$$

$$+\sum_{j=1}^{2}\frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leqslant x,\ (n,P_{y,w})=1\\ n\equiv 0\ (\mathrm{mod}\ [d_1,d_2])}}\beta_j(n+a)\,dt+O(x^{-1/4}\log x).$$

I denote these integrals by I_0, I_j, $j=1,2$, respectively. With a view to the application of Lemma 7 we write

$$\begin{aligned} I_0 = {} & \frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}}\lambda_{d_1}\lambda_{d_2}\frac{1}{\phi([d_1,d_2])}\sum_{\substack{n\leqslant x,\ (n,P_{y,w})=1\\ (n+a,[d_1,d_2])=1}}\beta(n+a)\,dt \\ & +\frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith}E(t)\,dt, \end{aligned} \tag{6}$$

where

$$E(t)=\sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}}\lambda_{d_1}\lambda_{d_2}\Bigg(\sum_{\substack{n\leqslant x,\ (n,P_{y,w})=1\\ n\equiv 0\ (\mathrm{mod}\ [d_1,d_2])}}\beta(n+a)-\frac{1}{\phi([d_1,d_2])}\sum_{\substack{n\leqslant x,\ (n,P_{y,w})=1\\ (n+a,[d_1,d_2])=1}}\beta(n+a)\Bigg).$$

Assuming that $\lambda_d\ll 1$ uniformly in d (as is usually the case) and noting that for squarefree integers D there are at most $3^{\omega(D)}$ pairs d_1, d_2 for which $[d_1,d_2]=D$, we see that

$$E(t)\ll \sum_{\substack{D\leqslant z^2\\ p|D\Rightarrow p>w}}3^{\omega(D)}\left|\sum_{\substack{n\leqslant x,\ (n-a,P_{y,w})=1\\ n\equiv a\ (\mathrm{mod}\ D)}}\beta(n)-\frac{1}{\phi(D)}\sum_{\substack{n\leqslant x,\ (n-a,P_{y,w})=1\\ (n,D)=1}}\beta(n)\right|+O(z^2l^{A+4}).$$

An application of the Cauchy–Schwarz inequality gives

$$|E(t)|^2$$
$$\ll \sum_{D\leqslant z^2} \frac{3^{2\omega(D)}}{\phi(D)} \sum_{\substack{D\leqslant z^2\\ p|D\Rightarrow p>w}} \phi(D)\Bigg| \sum_{\substack{n\leqslant x,\,(n-a,P_{y,w})=1\\ n\equiv a \pmod D}} \beta(n) - \frac{1}{\phi(D)} \sum_{\substack{n\leqslant x,\,(n-a,P_{y,w})=1\\ (n,D)=1}} \beta(n)\Bigg|^2.$$

The first sum over D is

$$\ll \prod_{p\leqslant z^3}\left(1+\frac{9}{p}\right) \ll (\log x)^9.$$

Since $\beta(n)\ll 1$ on average, assuming that $z\leqslant x^{1/6}$ and that w satisfies the conditions of Lemma 7, the remaining sum over D is

$$\ll x \sum_{\substack{D\leqslant z^2\\ p|D\Rightarrow p>w}} \Bigg| \sum_{\substack{n\leqslant x,\,(n-a,P_{y,w})=1\\ n\equiv a \pmod D}} \beta(n) - \frac{1}{\phi(D)} \sum_{\substack{n\leqslant x,\,(n-a,P_{y,w})=1\\ (n,D)=1}} \beta(n)\Bigg|,$$

which by Lemma 7 is $\ll x(\log x)^{1-A}$. Thus $E(t)\ll x(\log x)^{5-A/2}$, and the second integral at (6) is $\ll(\log x)^{6-A/2}$.

Separating $\beta(n+a)$ into its component parts we reach the estimate

$$I_0 = \frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith} \sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}} \lambda_{d_1}\lambda_{d_2}\frac{1}{\phi([d_1,d_2])} \sum_{\substack{n\leqslant x,\,(n,P_{y,w})=1\\ (n+a,[d_1,d_2])=1}} g(n+a)\,dt$$
$$-\sum_{j=1}^{2} K_j + O((\log x)^{6-A/2}), \tag{7}$$

where

$$K_j = \frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith} \sum_{\substack{d_k\leqslant z\\ d_k|P_{w,z}}} \lambda_{d_1}\lambda_{d_2}\frac{1}{\phi([d_1,d_2])} \sum_{\substack{n\leqslant x,\,(n,P_{y,w})=1\\ (n+a,[d_1,d_2])=1}} \beta_j(n+a)\,dt.$$

My next aim is to show that in the estimation of Q_h the integrals I_j, K_j, $j=1,2$, contribute a negligible amount. The main contribution to Q_h will come from the first of the integrals representing I_0 at (7).

The treatment of I_1 is a paradigm. For ease of notation let $P=P_{y,w}$, $R=P_{w,z}$. Then reforming the square:

$$I_1 = \frac{1}{\pi(x)}\int_{-1}^{1}(1-|t|)e^{-ith} \sum_{\substack{n\leqslant x\\ (n,P)=1}} \beta_1(n+a)\Bigg(\sum_{\substack{d|(n,R)\\ d\leqslant z}} \lambda_d\Bigg)^2 dt$$
$$\leqslant \frac{2}{\pi(x)} \max_{|t|\leqslant 1} \sum_{\substack{n\leqslant x\\ (n,P)=1}} |\beta_1(n+a)|\Bigg(\sum_{\substack{d|(n,R)\\ d\leqslant z}} \lambda_d\Bigg)^2.$$

We note that $\beta_1(n) \ll l^{A+4}$ always, and that if for fixed $\varepsilon > 0$, $n > x^\varepsilon$, then

$$\beta_1(n) \ll l^{-1} \sum_{\substack{ump=n \\ u \leqslant l^B,\, p \leqslant l^{6A+15}}} |h(u)| \log p \ll l^{-1} \sum_{\substack{ump=n \\ u \leqslant l^B,\, p \leqslant l^{6A+15}}} h_0(u) \log p,$$

where $h_0(u)$ is the multiplicative function defined by $h_0(p)=0$, $h_0(p^2)=\frac{3}{2}$, $h_0(p^k)=e$ if $k \geqslant 3$. The implied constants depend upon ε. If we denote this upper bound for $\beta_1(n)$ by $\gamma(n)$, then

$$I_1 \ll \pi(x)^{-1} \sum_{n \leqslant x} \gamma(n) \Bigg(\sum_{\substack{d | (n-a, R) \\ d \leqslant z}} \lambda_d \Bigg)^2 + x^{-1+2\varepsilon}.$$

This is a wasteful step, since the condition $(n-a, P)=1$ has been thrown away. With a little more effort the condition may be retained, but the consequent improvement in the bound for I_1 would not be significant here. Once again unwinding the square, and changing the order of summation,

$$I_1 \ll \pi(x)^{-1} \sum_{\substack{d_j \leqslant z \\ d_j | R}} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{n \leqslant x \\ n \equiv a \pmod{[d_1, d_2]}}} \gamma(n) + x^{-1+2\varepsilon}.$$

Here the innermost sum has the alternative form

$$l^{-1} \sum_{\substack{u \leqslant l^B \\ p \leqslant l^{6A+15}}} h_0(u) \log p \sum_{\substack{m \leqslant x/up \\ mup \equiv a \pmod{[d_1, d_2]}}} 1. \tag{8}$$

If $w > l^{6A+15}$ $(> l^B)$, then $(up, [d_1, d_2])=1$, and this double sum has the estimate

$$l^{-1} \sum_{\substack{u \leqslant l^B \\ p \leqslant l^{6A+15}}} h_0(u) \log p \left(\frac{x}{up[d_1, d_2]} + O(1) \right) = \frac{M}{[d_1, d_2]} + O(l^{7A+18}),$$

with

$$M = \frac{x}{\log x} \sum_{u \leqslant l^B} \frac{h_0(u)}{u} \sum_{p \leqslant l^{6A+15}} \frac{\log p}{p}.$$

In particular $M \ll x l^{-1} \log l$. It follows that

$$I_1 \ll \frac{M}{\pi(x)} \sum_{\substack{d_j \leqslant z \\ d_j | R}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + x^{-2/3}.$$

We now choose the λ_d to minimize the quadratic form, which by its derivation is non-negative definite. An account of a suitable version of Selberg's argument may be found in [2, Chapter 2, pp. 79–89]. For a more elaborate account of sieve methods, including that of Selberg, see Halberstam and Richert [13]. The particular choice of λ_d is not important here save that $|\lambda_d|\leqslant 1$ and that the quadratic form realises its minimum value L^{-1}, where

$$L=\sum_{\substack{m\leqslant z\\ m|R}}\frac{\mu^2(m)}{\phi(m)}.$$

Formally speaking we are following Selberg's classical method of obtaining an upper bound for the number of primes in an interval. The lower bound $L\geqslant\phi(P)P^{-1}\log x$ is vouchsafed by Lemma 3.1, p. 102 of Halberstam and Richert [13], and

$$I_1\ll\frac{\log\log x}{\log x}\prod_{y<p\leqslant w}\left(1-\frac{1}{p}\right)^{-1}+x^{-2/3}\ll\frac{(\log\log x)^2}{\log x}$$

provided we choose w to be a fixed power of $\log x$.

The treatment of I_2 is similar, employing

$$|\beta_2(n)|\ll l^{-1}\sum_{\substack{urp=n\\ u\leqslant l^B,\, r\leqslant l^{6A+15}}}h_0(u)\log p$$

over a range $x^\varepsilon<n\leqslant x$, and $\beta_2(n)\ll l^{A+4}$ otherwise. The appropriate analog of (8) is

$$l^{-1}\sum_{\substack{u\leqslant l^B\\ r\leqslant l^{6A+15}}}h_0(u)\sum_{\substack{p\leqslant x/ur\\ pur\equiv a\ (\mathrm{mod}\ [d_1,d_2])}}\log p.$$

It is not currently possible to deal with individual moduli $[d_1,d_2]$, but we can successfully treat their average. We write

$$\begin{aligned}I_2\ll x^{-1}&\sum_{\substack{u\leqslant l^B\\ r\leqslant l^{6A+15}}}h_0(u)\sum_{\substack{d_j\leqslant z\\ d_j|R}}\lambda_{d_1}\lambda_{d_2}\frac{x}{ur\phi([d_1,d_2])}\\ &+x^{-1}\sum_{\substack{u\leqslant l^B\\ r\leqslant l^{6A+15}}}h_0(u)\sum_{\substack{d_j\leqslant z\\ d_j|R}}\lambda_{d_1}\lambda_{d_2}\Bigg\{\sum_{\substack{p\leqslant x/ur\\ pur\equiv a\ (\mathrm{mod}\ [d_1,d_2])}}\log p-\frac{x}{ur\phi([d_1,d_2])}\Bigg\}\qquad(9)\\ &+O(x^{-1+2\varepsilon}).\end{aligned}$$

Assuming that $z\leqslant x^{1/6}$, the second of these multiple sums may be shown to be $\ll l^{-A}$ in the way that the earlier estimate $E(t)\ll x(\log x)^{2-A/2}$ was obtained. In place of Lemma 7 the validity of

$$\sum_{D\leqslant x^{1/3}}\Bigg|\sum_{\substack{p\leqslant x\\ p\equiv a\ (\mathrm{mod}\ D)}}\log p-\frac{x}{\phi(D)}\Bigg|\ll x(\log x)^{-C}$$

for a certain positive C will suffice, and this readily follows from the well known theorem of Bombieri and Vinogradov [1, théorème 17, p. 57].

In the first of the two multiple sums used to bound I_2, it is convenient to replace $\phi([d_1, d_2])$ by $[d_1, d_2]$. Since for $D=[d_1, d_2]$

$$0 \leqslant \frac{1}{\phi(D)} - \frac{1}{D} \leqslant \frac{1}{\phi(D)} \sum_{\substack{p|D \\ p>w}} \frac{1}{p},$$

this introduces an error of

$$\begin{aligned}
&\ll x^{-1} \sum_{\substack{u \leqslant l^B \\ r \leqslant l^{6A+15}}} h_0(u) \sum_{D \leqslant z^2} 3^{\omega(D)} \frac{x}{ur\phi(D)} \sum_{\substack{p|D \\ p>w}} \frac{1}{p} \\
&\ll \sum_{u \leqslant l^B} \frac{h_0(u)}{u} \sum_{r \leqslant l^{6A+15}} \frac{1}{r} \sum_{p>w} \frac{1}{p\phi(p)} \sum_{d \leqslant z^2} \frac{3^{\omega(d)}}{\phi(d)} \\
&\ll l^3 \log l \sum_{p>w} p^{-2} \ll (\log x)^{-A}.
\end{aligned}$$

With this replacement

$$I_2 \ll \log l \sum_{\substack{d_j \leqslant z \\ d_j | R}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + l^{-A} \ll l^{-1} (\log l)^2,$$

the same bound as for I_1.

It is not difficult to show that the integrals K_1 and K_2 satisfy the stronger bound $K_j \ll l^{-1} \log l$. It suffices to remove the condition $(n, [d_1, d_2])=1$ in the sums over the β_j, and again replace $\phi([d_1, d_2])$ by $[d_1, d_2]$.

Within a permissible error we may effect these same changes in the first of the integrals representing I_0 at (7). We so reach

$$Q_h \ll (\pi(x)L)^{-1} \int_{-1}^{1} (1-|t|) e^{-ith} \sum_{n \leqslant x,\, (n, P_{y,w})=1} g(n+a)\, dt + l^{-1} (\log l)^2. \tag{10}$$

I fix z at $x^{1/6}$, A at 2 and w at a suitably large power of $\log x$. Then $(\pi(x)L)^{-1} \ll x^{-1} \log w$. Further progress depends upon information concerning the distribution of general multiplicative functions on arithmetic progressions with differences of moderate size.

5. Middle moduli

I begin this section by considering how often a multiplicative function can be near to a (generalized) character.

LEMMA 9. *Let $B\geqslant 0$. There is a c so that if $2\leqslant Q\leqslant M$, then those Dirichlet characters to moduli not exceeding Q, which for some real τ, $|\tau|\leqslant Q^B$, satisfy*

$$\sum_{Q<p\leqslant M} p^{-1}(1-\operatorname{Re} g(p)\chi(p)p^{i\tau})<\frac{1}{4}\log\left(\frac{\log M}{\log Q}\right)-c \tag{11}$$

are all induced by the same primitive character.

For the purposes of this statement all principal characters are regarded as induced by the function which is identically one on the positive integers.

LEMMA 10. *Let $0<\beta<1$, $0<\varepsilon<\frac{1}{8}$, $2\leqslant\log M\leqslant Q\leqslant M$. Then*

$$\sum_{\substack{n\leqslant x\\ n\equiv r \ (\mathrm{mod}\ D)}} g(n)=\frac{1}{\phi(D)}\sum_{\substack{n\leqslant x\\ (n,D)=1}} g(n)+O\left(\frac{x}{\phi(D)}\left(\frac{\log Q}{\log x}\right)^{1/8-\varepsilon}\right)$$

uniformly for $M^\beta\leqslant x\leqslant M$, all r prime to D, and all moduli D not exceeding Q save possibly for the multiples of a single modulus $D_0>1$. For each of these exceptional moduli D there is a pair (χ,τ), with χ non-principal (mod D), *$|\tau|\leqslant Q^B$, so that* (11) *holds with $c+1$ in place of c.*

Proofs of Lemmas 9 *and* 10. Except for the final assertion, Lemma 10 is Theorem 1 of [9]. Both the final assertion and Lemma 9 follow readily from the estimates for Dirichlet series given in §2 of that work, and in particular from Lemmas 3 and 4 there. The modulus D_0 in Lemma 10 is the modulus to which the primitive character is Lemma 9 is defined.

Set $M=x$, $Q=\exp((\log\log x)^3)$. Let P be the product of the primes in the interval $(y,w]$.

LEMMA 11. *Either*

$$\sum_{\substack{n\leqslant x\\ (n-a,P)=1}} g(n)=\prod_{y<p\leqslant w}\left(1-\frac{1}{p-1}\right)\sum_{n\leqslant x} g(n)\prod_{\substack{p|n\\ p>y}}\left(\frac{p-1}{p-2}\right)+O(x(\log x)^{-1/10})$$

or there is an extra term

$$\prod_{y<p\leqslant w}\left(1-\frac{1}{p-1}\right)\frac{\mu(D_0)}{D_0}\prod_{p|D_0}\left(1-\frac{2}{p}\right)^{-1}\bar{\chi}_1(a)\sum_{n\leqslant x}\chi(n)g(n)\prod_{\substack{p|n\\ p>y}}\left(\frac{p-1}{p-2}\right)$$

on the right hand side, where $\chi \pmod{D_0}$ *is the primitive character defined via Lemmas* 10 *and* 9. *Moreover,* D_0 *divides* P, *and does not exceed* $\log x$.

Proof. Suppose first that for a given $x \geqslant 2$ in Lemma 10 there are no exceptional moduli relative to g. In terms of the Möbius function, the sum which we wish to estimate has the representation

$$\sum_{n \leqslant x} g(n) \sum_{d \mid (n-a, P)} \mu(d) = \sum_{\substack{d \mid P \\ d \leqslant x+|a|}} \mu(d) \sum_{\substack{n \leqslant x \\ n \equiv a \pmod d}} g(n).$$

In view of Lemma 10, the moduli $d \leqslant Q$ contribute

$$\sum_{\substack{d \mid P \\ d \leqslant Q}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{n \leqslant x \\ (n,d)=1}} g(n) + O\left(\sum_{\substack{d \mid P \\ d \leqslant Q}} \frac{\mu^2(d)}{\phi(d)} \frac{x}{(\log x)^{1/9}} \right). \tag{12}$$

Here the sum in the error term is

$$\ll \prod_{p \leqslant w} \left(1 + \frac{1}{p-1}\right) \ll \log \log x.$$

The moduli in the range $Q < d \leqslant x+|a|$ contribute

$$\ll x \sum_{\substack{d \mid P \\ Q < d \leqslant x+|a|}} \frac{\mu^2(d)}{d} \ll lx \exp\left(-\frac{\log Q}{\log w}\right) \ll x \exp(-(\log \log x)^{3/2}),$$

these last two from an application of Lemma 8. Within an error of the same size the condition $d \leqslant Q$ may be removed from the double sum at (12), and the first estimate of Lemma 11 is established.

Suppose next that inequality (11) of Lemma 9 holds for a character induced by the primitive character $\chi \pmod{D_0}$, with $D_0 \leqslant Q$, $D_0 \mid P$. Each character to a squarefree modulus D can be factorised $\chi_1 \chi_2$ where χ_1 is induced by a character defined $\pmod{D_1}$, χ_2 by a character $\pmod{D_2}$, $D = D_1 D_2$, $D_1 = (D, D_0)$. There is a corresponding representation

$$\sum_{\substack{n \leqslant x \\ n \equiv a \pmod D}} g(n) = \frac{1}{\phi(D_1)} \sum_{\chi_1} \bar{\chi}_1(a) \sum_{\substack{n \leqslant x \\ n \equiv a \pmod{D_2}}} g(n) \chi_1(n).$$

A formality of this type occurs in the proof of Lemma 3. Since D_0, D_2 are mutually prime, $g\chi_1\chi'$ cannot satisfy inequality (11) of Lemma 9 for any non-principal character

χ' (mod D_2). We may therefore apply Lemma 10 to estimate the sum over $g\chi_1$. We so reach

$$\sum_{\substack{n\leqslant x\\(n-a,P)=1}} g(n) = \prod_{\substack{y<p\leqslant w\\(p,D_0)=1}}\left(1-\frac{1}{p-1)}\right) \sum_{\substack{n\leqslant x\\(n-a,D_0)=1}} g(n)\prod_{\substack{p|n,p>y\\(p,D_0)=1}}\left(\frac{p-1}{p-2}\right)$$
$$+O(x(\log x)^{-1/10}).$$

Moreover, again introducing the Möbius function, in order to represent the condition $(n-a, D_0)=1$, we consider

$$\sum_{d|D_0}\mu(d)\sum_{\substack{n\leqslant x\\n\equiv a \ (\mathrm{mod}\ d)}} g(n)\prod_{\substack{p|n,p>y\\(p,D_0)=1}}\left(\frac{p-1}{p-2}\right).$$

Since D_0 is minimal, for moduli $d<D_0$ no exceptional character of the type needed to fulfill (11) of Lemma 9 can occur. For such moduli we may again apply Lemma 10. For $d=D_0$ a term

$$\mu(D_0)\Bigg\{\sum_{\substack{n\leqslant x\\n\equiv a \ (\mathrm{mod}\ D_0)}} g(n)\prod_{\substack{p|n\\p>y}}\left(\frac{p-1}{p-2}\right)-\frac{1}{\phi(D_0)}\sum_{\substack{n\leqslant x\\(n,D_0)=1}} g(n)\prod_{\substack{p|n\\p>y}}\left(\frac{p-1}{p-2}\right)\Bigg\}$$

is separated off. The remaining terms combine to give the first estimate of Lemma 11.

The separated term is not significant unless $D_0\leqslant\log x$. It has the representation

$$\frac{\mu(D_0)}{\phi(D_0)}\sum_{\chi_1 \text{ non-principal}}\bar{\chi}_1(a)\sum_{n\leqslant x}\chi_1(n)g(n)\prod_{\substack{p|n\\p>y}}\left(\frac{p-1}{p-2}\right).$$

For χ_1 distinct from χ, the inequality (11) fails. We may apply a method of Halász, e.g., in the form [4, Lemma 3], to show that an innersum here is then

$$\ll x\exp\left(-\frac{1}{2}\min_{|\tau|\leqslant(\log x)^2}\sum_{p\leqslant x}\frac{1}{p}(1-\mathrm{Re}\, g(p)\chi_1(p))\right)\ll x(\log x)^{-1/9}.$$

The factor $\Pi((p-1)/(p-2))$, $p|n$, $p>y$, may be carried through the argument of [4, Lemma 3]; or represented as a Dirichlet convolution $1*h$, and stripped off, using the fact that $\sum|h(n)|n^{-1/2}$ converges.

Lemma 11 is established.

6. Proof of Theorem 1, end

In his treatment of additive functions on the natural numbers, Ruzsa (loc. cit.) has a division into effectively two cases: *There is a real* λ, $|\lambda|\leqslant(\log x)^2$ *so that*

$$U=\sum_{p} p^{-1}\min(1,|f(p)-\lambda\log p|^2)<\tfrac{1}{100}\log\log x, \tag{13}$$

or there is not. In fact I have changed to $\frac{1}{100}$ the factor $\frac{1}{10}$ in his account, and noted that his argument needs only $|\lambda|\leqslant(\log x)^2$, rather than λ unrestricted. These changes do not affect his argument beyond changing the value of certain implied absolute constants.

Assume that $U<\frac{1}{100}\log\log x$ for some λ, $|\lambda|\leqslant(\log x)^2$. If $g(n)=\exp(itf(n))$, $|t|\leqslant 1$, then $|1-g(p)p^{-i\lambda t}|=|1-\exp(it\{f(p)-\lambda\log p\})|\leqslant\min(2,|f(p)-\lambda\log p|)$. Then by an application of the Cauchy–Schwarz inequality

$$\sum_{p\leqslant x} p^{-1}(1-\operatorname{Re} g(p)p^{-i\lambda t})\leqslant\sum_{p\leqslant x} p^{-1}|1-g(p)p^{-i\lambda t}|\leqslant\left(4U\sum_{p\leqslant x}\frac{1}{p}\right)^{1/2},$$

which does not exceed $\left(\frac{1}{5}+o(1)\right)\log\log x$ as $x\to\infty$. Bearing in mind the small size of Q, we see that

$$\sum_{Q<p\leqslant x} p^{-1}(1-\operatorname{Re} g(p)p^{-i\lambda t})<\frac{1}{4}\log\left(\frac{\log x}{\log Q}\right)-c$$

for all sufficiently large x, and that $|\lambda|\leqslant(\log x)^2<Q$. The inequality (11) of Lemma 9 is valid (with $B=1$), and the asymptotic estimate of Lemma 10 is available for all moduli D up to Q without exception.

Consider the integral

$$\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{n\leqslant x\\ (n,P)=1}} g(n+a)\,dt$$

appearing in the upper bound for Q_h given at (10), where $P=P_{y,w}$. I set $y=3|a|$. For each g (parametrized by t) we may apply the first estimate of Lemma 11, to obtain

$$Q_h\ll x^{-1}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{n\leqslant x} g(n)\prod_{\substack{p\mid n\\ p>3|a|}}\left(\frac{p-1}{p-2}\right)dt+O((\log x)^{-1/12}). \tag{14}$$

We then come to an integral which apart from the factor $\prod((p-1)/(p-2))$ over $p|n$, $p>3|a|$, is of the type considered by Ruzsa in his study of the concentration function of additive functions on the natural integers, [15, §3, pp. 218–219]. The extra factor may either be carried through his entire analysis, or included at a later stage by means of a convolution argument. A convolution argument of a related type already appears in §10, pp. 230–231 of Ruzsa's paper. In this way we obtain the estimate of Theorem 1; a bound of the same quality as Ruzsa's. I shall reduce myself to this case.

We may therefore suppose that (13) fails for every λ, $|\lambda|\leqslant(\log x)^2$. For real u, a Dirichlet character χ (mod D), real t, define

$$m(u,\chi,t)=\sum_{\substack{p\leqslant x\\(p,D)=1}}\frac{1}{p}(1-\operatorname{Re}e^{itf(p)}\chi(p)p^{-iu}).$$

Denote by $M(t)$ the minimum of $m(\tau,\chi,t)$ taken over all characters to moduli not exceeding $\log x$, and real τ, $|\tau|\leqslant\log x$. From Lemma 11

$$\sum_{\substack{n\leqslant x\\(n-a,P)=1}} g(n)\ll x\frac{\phi(P)}{P}\exp(-cM(t)),\quad |t|\leqslant 1,$$

for a positive absolute constant c. The value of c may need to be depressed in order to absorb various error terms involving a negative power of $\log x$. From what we established in §4 we see that

$$Q_h\ll\int_{-1}^{1}\exp(-cM(t))\,dt. \tag{15}$$

For each positive integer k, let E_k denote the set of (real) t in the interval $|t|\leqslant 1$ for which $M(t)\leqslant k$, and let $|E_k|$ denote its Lebesgue measure. I shall elaborate ideas from Ruzsa (loc. cit.) to show that $|E_k|\ll k^{1/2}(\log\log x)^{-1/2}$ uniformly in k. As a consequence

$$Q_h\ll\sum_{k=0}^{\infty}e^{-ck}|E_{k+1}|\ll(\log\log x)^{-1/2}\ll W(x)^{-1/2}$$

will follow.

According to a theorem of Raikov, cf. [2, Lemma 1.6], if E is a measurable subset of $[-1,1]$, symmetric with respect to the origin, and r is a positive integer, then those reals in $[-1,1]$ which are representable as a sum of r members of E have measure at least $\min(2,r|E|)$. Assuming that E_k has a positive measure, we shall have $t=t_1+\ldots+t_r$ with $r=1+[2|E_k|^{-1}]$. Corresponding to each t_j will be a character ψ_j, to a modulus D_j not exceeding $\log x$, and a real u_j, $|u_j|\leqslant\log x$, for which $m(u_j,\psi_j,t_j)\leqslant k$.

An application of the Cauchy–Schwarz inequality shows that the inequality

$$1-\operatorname{Re}z_1\ldots z_r\leqslant r\sum_{j=1}^{r}(1-\operatorname{Re}z_j)$$

holds for all unimodular complex z_j. It continues to hold if any z_j is replaced by zero. Accordingly, there is a character $\chi_t = \psi_1 \ldots \psi_r$ viewed $(\bmod [D_1, \ldots, D_r])$, and a real $u(t) = u_1 + \ldots + u_r$, for which $m(u(t), \chi_t, t) \leqslant r^2 k$. The modulus of χ_t is at most $(\log x)^r$, moreover, $|u(t)| \leqslant r \log x$.

Let t_1, t_2 and $t_1 + t_2$ lie in $[-1, 1]$. A simpler version of this last step shows that $m(v, \chi, t) \leqslant 9r^2 k$, with $v = u(t_1 + t_2) - u(t_1) - u(t_2)$, $\chi = \chi_{t_1+t_2} \bar{\chi}_{t_1} \bar{\chi}_{t_2}$. Let Δ be a defining modulus for χ. We have arrived at

$$\sum_{\substack{p \leqslant x \\ (p, \Delta) = 1}} \frac{1}{p}(1 - \operatorname{Re} \chi(p) p^{-iv}) \leqslant 9r^2 k.$$

If χ is non-principal, then a consideration of Euler products shows the sum in this bound to be

$$\log \left| \frac{L(\sigma, \chi_0)}{L(\sigma + iv, \chi)} \right| + O(1),$$

where $\sigma = 1 + (\log x)^{-1}$, χ_0 is the principal character $(\bmod \Delta)$.

Let $0 < \gamma < 1$. We may assume $9r^2 k \leqslant \gamma \log\log x$, otherwise the asserted bound for $|E_k|$ is immediate. The standard bounds

$$L(\sigma + iv, \chi) \ll \log \Delta(|v| + 2), \quad |L(\sigma, \chi_0)| \gg \zeta(\sigma)\phi(\Delta)\Delta^{-1}$$

together with $r \ll (\log\log x)^{1/2}$ show that $r^2 k \gg \log\log x$. We again have the asserted bound on $|E_k|$.

We can therefore assume $\chi_{t_1+t_2} \bar{\chi}_{t_1} \bar{\chi}_{t_2}$ to be principal for every permitted pair t_1, t_2. Let H be the product of the primes up to $\log x$, $h = \phi(H)$. Let X_t be the character $(\bmod H)$ induced by χ_t. Then $X_{t_1+t_2} = X_{t_1} X_{t_2}$ for all permitted t_j. In particular, $X_t = (X_{t/h})^h = 1$. Every X_t is principal. As a consequence

$$\sum_{\substack{p \leqslant x \\ (p, H) = 1}} \frac{1}{p}(1 - \operatorname{Re} e^{itf(p)} p^{-iu(t)}) \leqslant r^2 k \leqslant \tfrac{1}{9}\gamma \log\log x.$$

Since

$$\sum_{p | H} \frac{1}{p} \leqslant \log\log\log H + O(1) \leqslant \log\log\log x + O(1),$$

with γ in place of $\frac{1}{9}\gamma$ we may omit the requirement that $(p, H) = 1$.

Non-principal Dirichlet characters and their associated L-series play no part in Rusza's treatment of additive functions (loc. cit.) but the bounds involving $u(t)$, without the condition $(p,H)=1$, are of exactly the type studied by Ruzsa. We may now directly follow his argument to reach

$$\sum_{p\leqslant x}\frac{1}{p}(1-\operatorname{Re}e^{itf(p)}p^{-i\lambda t})\ll\gamma\log\log x,\quad \lambda=u(1).$$

For all sufficiently large x the implied constant depends at most upon a. Moreover, $|\lambda|\leqslant r\log x<(\log x)^2$. With a suitable initial choice of γ, integration over the interval $-1\leqslant t\leqslant 1$ shows (13) to be satisfied by λ, contradicting our temporary assumption. The bound on $|E_k|$ is indeed satisfied, and our proof of Theorem 1 complete.

With a little more effort we can show that within an error of $O((\log\log x)^{-1/2})$ over the interval $(0,x]$, the concentration function of f on the shifted primes is dominated by the concentration function of $f-\lambda\log$ on the natural numbers, for some λ, $|\lambda|\leqslant(\log x)^2$.

7. Proof of Theorem 2, beginning

The proof begins with

$$S_h\ll\frac{\log N}{N}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{n<N\\(n,PR)=1}}g(N-n)\,dt+\frac{z\log N}{N},$$

where R denotes the product of the primes in the range $w<p\leqslant z$ which do not divide N, and P denotes the product of the primes in the range $y<p\leqslant w$. The first part of the proof of Theorem 2 follows that of Theorem 1 until we reach

$$S_h\ll\frac{\log w}{N}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{n<N\\(n,P)=1}}g(N-n)\,dt+\frac{(\log\log N)^2}{\log N},$$

which is the analog of (10). In the notation used for the estimation of I_1, the relevant quadratic form becomes

$$\sum_{\substack{d_k\leqslant z\\d_k|R,\,(d_k,N)=1}}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}.$$

The condition $(d_k,N)=1$ may be retained, so that the analog of L has lower bound

$$\frac{\phi(P)}{P}\prod_{\substack{w<p\leqslant z\\(p,N)=1}}\left(1-\frac{1}{p}\right)\log z,$$

or removed before the quadratic form is minimized, the removal costing

$$\ll \sum_{p|N} \sum_{\substack{d_1 \leqslant z \\ d_1 \equiv 0 \ (\mathrm{mod}\ p)}} \sum_{d_2 \leqslant z} \frac{1}{[d_1, d_2]} \ll \sum_{p|N} \sum_{\substack{D \leqslant z^2 \\ D \equiv 0 \ (\mathrm{mod}\ p)}} \frac{3^{\omega(D)}}{D}$$

$$\ll \sum_{p|N} \frac{1}{p} \prod_{q \leqslant z} \left(1+\frac{3}{q}\right) \ll \frac{(\log z)^3 \log N}{w} \ll \frac{1}{\log z},$$

provided $w \geqslant (\log N)^5$. The extra condition $(d_k, N)=1$ has negligible effect.

In the section on middle moduli the condition $n \leqslant x$ is replaced by $n<N$; $(n-a, P)=1$ by $(N-n, P)=1$. The latter is equivalent to $(n, N_1)=1$, where N_1 is the product of the prime divisors of N in the interval $(y, w]$, and $(N-n, PN_1^{-1})=1$. The first of the sums to be estimated in the analog of Lemma 11 is therefore

$$\sum_{\substack{n<N \\ (n,N_1)=1}} g(n) \sum_{d|(N-n, PN_1^{-1})} \mu(d) = \sum_{\substack{d|PN_1^{-1} \\ d<N}} \mu(d) \sum_{\substack{n<N,\,(n,N_1)=1 \\ n \equiv N \ (\mathrm{mod}\ d)}} g(n). \tag{16}$$

Here $(N, d)=1$ for each d. Let $Q=\exp((\log\log N)^3)$. We apply Lemma 10 to the multiplicative function which coincides with g except on the powers of the primes which divide N_1, where it is zero. Assuming that there are no exceptional moduli, the contribution arising from the $d \leqslant Q$ will be

$$\sum_{\substack{d|PN_1^{-1} \\ d \leqslant Q}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{n<N \\ (n,N_1 d)=1}} g(n) + O\Bigg(\sum_{\substack{d|PN_1^{-1} \\ d \leqslant Q}} \frac{\mu^2(d) N}{\phi(d)} \left(\frac{\log Q}{\log N}\right)^{1/8-\varepsilon} \Bigg) \tag{17}$$

for any fixed $\varepsilon>0$. Here the error term is

$$\ll N \left(\frac{(\log\log N)^3}{\log N} \right)^{1/8-\varepsilon} \prod_{p \leqslant w} \left(1+\frac{1}{p-1}\right) \ll N(\log N)^{-1/9}.$$

Applying Lemma 8 we can show that the contribution to the double sums at (16) which arises from the moduli $Q<d<N$ is $\ll N \exp(-(\log\log N)^{3/2})$. Within the same expense the condition $d \leqslant Q$ may be removed in the double sum at (17). In this way we obtain

$$\sum_{\substack{n<N \\ (N-n,P)=1}} g(n) = \prod_{\substack{y<p \leqslant w \\ (p,N)=1}} \left(1-\frac{1}{p-1}\right) \sum_{\substack{n<N \\ (n,N)=1}} g(n) \prod_{\substack{p|n \\ p>y}} \left(\frac{p-1}{p-2}\right) + O(N(\log N)^{-1/9}).$$

If there are exceptional moduli in the relevant version of Lemma 10, we introduce an auxiliary term, exactly as for the second estimate of Lemma 11. Fixing y at 3 we reach an analog of (13):

$$S_h \ll \phi(N)^{-1}\int_{-1}^{1}(1-|t|)e^{-ith}\sum_{\substack{n<N\\(n,N)=1}} g(n)\prod_{\substack{p|n\\p>3}}\left(\frac{p-1}{p-2}\right)dt+(\log N)^{-1/10},$$

with attendant modifications corresponding to any auxiliary terms.

It is tempting to declare that we may again follow our earlier treatment, and this is possible except in one regard. The analog of Halász' bound for the mean value of multiplicative functions is not immediately available. There are many variant proofs of the original result of Halász, but without modification they appear only to deliver

$$\sum_{\substack{n<N\\(n,N)=1}} g(n)\ll N\left(\frac{\phi(N)}{N}\right)^{c}\exp\left(-c\min_{|u|\leqslant l^{d}}\sum_{\substack{p\leqslant N\\(p,N)=1}}\frac{1}{p}(1-\operatorname{Re} g(p)p^{iu})\right)$$

with (absolute) constants $d>0$, and c in the interval $(0,1)$. If we could choose $c=1$ we would be done, but this is not currently possible even in the formulation (15) without the condition $(n,N)=1$. The following result is therefore of independent interest.

8.

LEMMA 12. *Let γ,δ be positive numbers. Then there is a further positive number c, depending at most upon γ, so that*

$$\sum_{\substack{n\leqslant x\\(n,N)=1}} g(n)\ll \frac{\phi(N)}{N}x\left\{\exp\left(-\min_{|u|\leqslant T}\sum_{\substack{p\leqslant x\\(p,N)=1}}\frac{1}{p}(1-\operatorname{Re} g(p)p^{-iu})\right)+\frac{1}{T}\right\}^{c}$$

uniformly in g, $T\geqslant 2$ and all real $x\geqslant 2$, integral $N\geqslant 1$ which satisfy

$$\left(\frac{N}{\phi(N)}\right)^{\gamma}\sum_{p|N}\frac{\log p}{p}\leqslant \delta\log x.$$

In particular we may set $x=N$ for all sufficiently large N.

Remark. It is not difficult to show that

$$\left(\frac{N}{\phi(N)}\right)^{\gamma}\sum_{p|N}\frac{\log p}{p}\lll(\log\log N)^{\gamma+1}.$$

Proof of Lemma 12. I do not give a detailed proof, only the key step. We largely follow the method of Halász [11], as exhibited in [2, Chapter 6]. Let $1<\beta\leqslant 2$, $s=\sigma+i\tau$. An important ingredient of Halász' proof is an inequality of the type

$$\int_{-1}^{1}\left|\sum_{n=1}^{\infty} g(n)n^{-s}\right|^{\beta} d\tau \ll (\sigma-1)^{\beta-1}, \quad \sigma>1,$$

which he derives from Plancherel's formula. He considers only the case $\beta=\frac{3}{2}$ but his argument works generally. In place of his inequality I use

$$\int_{-1}^{1}\left|\sum_{\substack{n=1\\(n,N)=1}}^{\infty} g(n)n^{-s}\right|^{\beta} d\tau \ll \left(\frac{\phi(N)}{N}\right)^{\beta}(\sigma-1)^{\beta-1}, \quad \sigma>1, \tag{18}$$

valid whenever

$$\left(\frac{N}{\phi(N)}\right)^{\frac{\beta-1}{\beta+1}} \sum_{p\mid N} \frac{\log p}{p} \leqslant \frac{c_0}{\sigma-1}, \tag{19}$$

the implied constant of (18) depending upon the (arbitrary positive) constant c_0 appearing in (19). It is the proof of (18) which I give here.

To this end we may employ either the inequality

$$\int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty} a_n n^{-s}\right|^{2} \frac{d\tau}{|s|^2} \leqslant \int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty} b_n n^{-s}\right|^{2} \frac{d\tau}{|s|^2}$$

valid whenever $|a_n|\leqslant b_n$ for all n and the series $\sum b_n n^{-\sigma}$ converges, or a finite version

$$\int_{-T}^{T}\left|\sum_{n=1}^{\infty} a_n n^{-s}\right|^{2} d\tau \leqslant 3\int_{-T}^{T}\left|\sum_{n=1}^{\infty} b_n n^{-s}\right|^{2} d\tau, \quad T\geqslant 0. \tag{20}$$

The former may be obtained by applying Plancherel's formula in the style of Halász (cf. [2, Chapter 6, pp. 228–229]), the latter is due to Montgomery, given in Tenenbaum [16, Lemma (6.1), pp. 373–374]. In either case we begin with an appeal to the representation

$$\sum_{n=1}^{\infty} g(n)n^{-s} = (1+h)G_1(s)\exp\left(\sum_{p\geqslant 3} g(p)p^{-s}\right) \tag{21}$$

where

$$h=\sum_{k=1}^{\infty} g(2^{ks})2^{-ks},$$

and $G_1(s)$ analytic in the half-plane $\sigma>\frac{1}{2}$, bounded by $e^{-5}\leqslant|G_1(s)|\leqslant e^5$ in the half-plane $\sigma\geqslant 1$. This is Lemma (6.6) of [2, pp. 230–231], established by means of Euler products.

It holds for all complex multiplicative functions of modulus at most 1. In particular, if $G(s)$ denotes the sum function of the series $\sum g(n)n^{-s}$ taken over the natural numbers prime to N, then in the half-plane $\sigma>1$,

$$|G(s)|^{\beta/2} \ll \left|\exp\Big(\sum_{\substack{p\geqslant 3\\(p,N)=1}} \frac{\beta g(p)}{2}\frac{1}{p^s}\Big)\right|.$$

We can expand the exponential as a Dirichlet series

$$\prod_{\substack{p\geqslant 3\\(p,N)=1}} \left(1+\frac{\beta g(p)}{2}\frac{1}{p^s}+\left(\frac{\beta g(p)}{2}\right)^2\frac{1}{2!}\frac{1}{p^{2s}}+\dots\right),$$

and it is clear that a typical coefficient of n^{-s}, which I shall denote by d_n, does not exceed in absolute value the expression k_n obtained from it by replacing every $g(p)$ by 1. It is also convenient to note that from a further application of (21)

$$\left|\sum_{n=1}^{\infty} k_n n^{-s}\right| = \left|\exp\Big(\sum_{\substack{p\geqslant 3\\(p,N)=1}} \frac{\beta}{2p^s}\Big)\right| \ll \left|\prod_{p|N}\left(1-\frac{1}{p^s}\right)\zeta(s)\right|^{\beta/2}$$

in the half-plane $\sigma>1$, where $\zeta(s)$ denotes the standard Riemann Zeta function.

After these preliminary remarks we may argue that

$$\int_{-1}^{1}|G(s)|^{\beta}d\tau \ll \int_{-1}^{1}\left|\sum_{n=1}^{\infty}d_n n^{-s}\right|^2 \ll \int_{-1}^{1}\left|\sum_{n=1}^{\infty}k_n n^{-s}\right|^2 d\tau$$
$$\ll \int_{-1}^{1}\left|\prod_{p|N}(1-p^{-s})\zeta(s)\right|^{\beta} d\tau,$$

the second step by (20) with $T=1$. We are reduced to the particular case that g is identically one on the integers prime to N.

Since $\zeta(s)$ is analytic in the disc $|s-1|<2$ except for a simple pole at $s=1$, this last integral is

$$\ll \int_{-1}^{1}\left|\prod_{p|N}(1-p^{-s})\right|^{\beta}\frac{d\tau}{|s-1|^{\beta}},$$

which after the change of variable $\tau=(\sigma-1)t$, with $w=\sigma+it(\sigma-1)$, becomes

$$\ll (\sigma-1)^{1-\beta}\int_{-(\sigma-1)^{-1}}^{(\sigma-1)^{-1}}\left|\prod_{p|N}(1-p^{-w})\right|^{\beta}\frac{dt}{(1+t^2)^{\beta/2}}. \tag{22}$$

Let $M \geqslant 1$. The contribution to this integral which arises from the range $M < |t| \leqslant (\sigma - 1)^{-1}$ is estimated crudely, to be

$$\ll \prod_{p|N}\left(1+\frac{1}{p}\right)\int_M^\infty \frac{dt}{t^\beta} \ll \frac{NM^{1-\beta}}{\phi(N)}.$$

Over the range $|t| \leqslant M$,

$$\frac{N}{\phi(N)}\prod_{p|N}\left(1-\frac{1}{p^w}\right) = \exp\left(\sum_{p|N}\left(\frac{1}{p}-\frac{1}{p^w}\right)+O(1)\right)$$
$$\ll \exp\left((\sigma-1)M\sum_{p|N}\frac{\log p}{p}\right).$$

The corresponding contribution towards the integral at (22) is

$$\ll \left(\frac{\phi(N)}{N}\right)^\beta (\sigma-1)^{1-\beta}\exp\left((\sigma-1)M\sum_{p|N}\frac{\log p}{p}\right)\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^{\beta/2}}.$$

If $N > 1$, then choose M to satisfy $(\sigma-1)M\sum_{p|N}p^{-1}\log p = 1$. Otherwise set $M=1$. Provided the condition (19) is satisfied,

$$\frac{NM^{1-\beta}}{\phi(N)} = \frac{N}{\phi(N)}\left((\sigma-1)\sum_{p|N}\frac{\log p}{p}\right)^{\beta-1} \ll \left(\frac{\phi(N)}{N}\right)^\beta,$$

and the inequality (18) is established.

Using (18) together with the estimate

$$G(s)\zeta(\sigma)^{-1} \ll \phi(N)N^{-1}\exp\left(-\sum_{\substack{p\geqslant 3\\(p,N)=1}} p^{-\sigma}(1-\operatorname{Re} g(p)p^{-i\tau})\right)$$

which may be derived without difficulty from (21), the proof of Lemma 12 can be completed along the lines of Lemma (6.10) in [2, Chapter 6].

9. Proof of Theorem 2, end

There are no new difficulties and the proof of Theorem 2 may be completed along the lines of the proof of Theorem 1.

10.

The study of lower bounds for the concentration function of additive functions on sequences of arithmetic interest is worthy for itself. To illustrate that Theorems 1 and 2 are in certain senses best possible, I furnish two examples. L will denote $\log x$.

The first example concerns functions of the form 'logarithm + small'.

Let h be an additive function which vanishes on the primes in the interval $(x^{1/2}, x]$ and satisfies $h(p^k)=h(p)$ with $|h(p)|\leqslant \frac{1}{2}$ when $k\geqslant 1$ and the prime p does not exceed $x^{1/2}$. Define $A=\sum h(p)p^{-1}$, $B=(\sum |h(p)|^2p^{-1})^{1/2}\geqslant 0$, the sums taken over the primes up to $x^{1/2}$. The variant of the Turán–Kubilius inequality given as Lemma 4.11 in [2], shows that for a certain absolute constant c_1

$$\sum_{p+1\leqslant x} |h(p+1)-A|^2 \leqslant c_1B^2\pi(x), \quad x\geqslant 2.$$

For at least half of the primes p up to x, $|h(p+1)-A|\leqslant (2c_1)^{1/2}B$.

The function

$$U(\lambda)=\lambda^2+\sum_{p\leqslant x} p^{-1}\min(1, |h(p)-\lambda\log p|^2)$$

is continuous in λ. With λ_0 a value which minimizes U, define the additive function $f=h+\lambda_0\log$.

The number of primes up to $\frac{1}{5}x$ is asymptotically $(1+o(1))x(5\log x)^{-1}$ as $x\to\infty$, and so does not exceed $\frac{1}{4}\pi(x)$ for all large enough x. For the remaining primes up to x, $|\lambda_0(\log(p+1)-\log x)|\leqslant 5|\lambda_0|$. Altogether there are at least $\frac{1}{4}\pi(x)$ primes p not exceeding x for which

$$|f(p+1)-A-\lambda_0\log x|\leqslant (2c_1)^{1/2}B+5|\lambda_0|.$$

Covering the interval which has centre $A+\lambda_0\log x$, and length $2((2c_1)^{1/2}B+5(\lambda_0|)$, by adjacent intervals of length 1, we see that the concentration function of f is at least $c_2W(x)^{-1/2}$ for a certain positive absolute constant c_2.

The lower bound in this example loses its precision if the restriction $|h(p)|\leqslant\frac{1}{2}$ is removed. I give a further example, of the form 'logarithm + large and lattice valued'.

Let J be the set of primes in the interval $(\exp(\sqrt{\log x}), \exp(L(\log L)^{-2})]$, and define $E(x)=\sum p^{-1}$, p in J. Define an additive function w by $w(q^k)=k$ if the prime q belongs to J, $w(q^k)=0$ otherwise. Let N be a positive integer exceeding 1.

The frequency of those primes p, not exceeding x, for which $Nw(p+1)$ has the integral value t, is zero unless t is a multiple of N, when it is the frequency of the p for which $w(p+1)=tN^{-1}$. The maximal frequency for which $Nw(p+1)$ assumes a value is the maximal frequency for which $w(p+1)$ assumes a value.

Let $c_3>0$. Let h be an additive function which is zero on the primes p not belonging to J. The methods of Probabilistic Number Theory as expounded in Chapter 3 of [2] furnish a representation

$$\nu_x(p;\, h(p+1)\in F)=P_1\Big(\sum_{q\in J} X_{1q}\in F\Big)+O((\log x)^{-c_3})$$

uniform in sets F, the implied constant depending at most upon c_3. Here $\nu_x(p;\,...)$ denotes the frequency amongst the primes p not exceeding $x-1$ for which $h(p+1)$ belongs to F. The independent random variables X_{1q}, one for each prime q in J, are distributed according to

$$X_{1q}=\begin{cases}1 & \text{with probability } 1/(q-1),\\ 0 & \text{with probability } 1-1/(q-1).\end{cases}$$

There is a similar representation

$$\nu_x(n;\, h(n)\in F)=P_2\Big(\sum_{q\in J} X_{2q}\in F\Big)+O((\log x)^{-c_3}),$$

where the frequency is among the positive integers not exceeding x, and the independent random variables X_{2q} are distributed according to

$$X_{2q}=\begin{cases}1 & \text{with probability } 1/q,\\ 0 & \text{with probability } 1-1/q.\end{cases}$$

The relevant particular references are Lemma 37 and Lemma 3.4 of [2], respectively.

Although the distribution functions $P_j(\sum X_{jq}\leqslant z)$ can be individually treated, it suffices in the present circumstances to step between them. We may then estimate the frequence $\nu_x(p;\, h(p+1)\leqslant z)$ using results already in the literature.

If $F_j(z)$ has characteristic function $\phi_j(t)$, $j=1,2$, then the concentration function $Q_j(h)=\sup(F_j(z+h)-F_j(z))$ taken over all real z, satisfies

$$Q_2(1)\ll Q_1(1)+\int_{-1}^{1}|t^{-1}(\phi_1(t)-\phi_2(t))|\,dt.$$

This is almost immediate from Lemma 1.47 of [2]. With $F_j(z)=P_j(\sum X_{jq}\leqslant z)$, an easy computation shows that

$$\phi_1(t)=\prod_{q\in J}\Big(1+\frac{1}{q-1}(e^{it}-1)\Big).$$

There is a similar representation for $\phi_2(t)$, the weight $(q-1)^{-1}$ being replaced by q^{-1}. In view of the inequality

$$\Big|\prod_{j=1}^{m} a_j-\prod_{j=1}^{m} b_j\Big|\leqslant\sum_{j=1}^{m}|a_j-b_j|,$$

valid for all a_j, b_j in the complex unit disc,

$$|\phi_1(t)-\phi_2(t)| \leqslant \sum_{q\in J} \frac{1}{q(q-1)}|e^{it}-1| \ll |t|e^{-\sqrt{L}}.$$

Hence

$$Q_2(1) \ll Q_1(1)+\exp\left(-\tfrac{1}{2}\sqrt{L}\right).$$

There is a similar inequality with the rôles of the $Q_j(1)$ interchanged.

A theorem of Halász, an exposition may be found in [2, Theorem 21.3], shows that

$$x^{-1} \sum_{\substack{x/2<n\leqslant x\\ w(n)=[E(x)]}} 1 = \frac{1}{\sqrt{2\pi E(x)}}(1+O(E(x)^{-1/2})).$$

The concentration function of Nw on the shifted primes is therefore $\gg E(x)^{-1/2}$.

Let $N=[(\log x)^{1/2}]$. Choose λ_0 to minimize

$$\lambda^2+\sum_{p\leqslant x} p^{-1}\min(1, |N-\lambda\log p|^2)$$

and define $f=Nw+\lambda_0\log$. Then for a certain β,

$$\nu_x\left(n; n>\tfrac{1}{2}x, |f-\lambda_0\log n-\beta|\leqslant \tfrac{1}{2}\right) \gg E(x)^{-1/2}.$$

The numbers λ_0 are bounded uniformly in x. Indeed, if $\lambda_0\neq 0$, then there must be a prime q in J for which $|N-\lambda_0\log q|<1$; otherwise $\lambda_0^2+\sum p^{-1}\min(1, |N-\lambda_0\log p|^2)$ is decreased by replacing λ_0 with ($\lambda=$) 0. Assuming $|\lambda_0\log q-N|\leqslant 1$, our restrictions upon q and N ensure that $\lambda_0\ll 1$.

For any (positive) value of λ, $|N-\lambda\log p|\leqslant 1$ if and only if $e^{(N-1)/\lambda}<p<e^{(N+1)/\lambda}$. Either $\lambda<N$ and

$$\sum_{e^{(N-1)/\lambda}<p\leqslant e^{(N+1)/\lambda}} \frac{1}{p} = \log\left(\frac{N+1}{N-1}\right)+O\left(\frac{\lambda}{N}\right) \ll 1+\frac{\lambda}{N} \ll 1,$$

or $\lambda\geqslant N$ and this sum over the reciprocals of primes is trivially bounded. Thus

$$\sum_{p\leqslant x} p^{-1}\min(1, |Nw(p)-\lambda\log p|^2) = E(x)+O(1)$$

for all λ. It follows that $W(x)=E(x)+O(1)$.

Since $\lambda_0\log(n/x)\ll 1$ uniformly for integers n in the interval $\left(\tfrac{1}{2}x, x\right]$, and since $\lambda_0\log n-\beta=\lambda_0\log(n/x)-(\beta-\lambda_0\log x)$, the concentration function of f is $\gg W(x)^{-1/2}$.

In this example both $f(p)$ and $f(p)-\lambda_0 \log p$ are often large. If $|\lambda_0| \leqslant L^{-1/4}$, then $f(p)=N+O(L^{3/8})=(1+o(1))L^{1/2}$ uniformly for $\exp(\sqrt{L}) < p \leqslant \exp(L^{5/8})$. If $|\lambda_0| > L^{-1/4}$, then for $\exp(L^{7/8}) < p \leqslant \exp(L(\log L)^{-2})$, $|f(p)| \geqslant |\lambda_0| \log p - N \geqslant (1+o(1))L^{5/8}$. In particular,

$$\sum_{p \in J} p^{-1} f(p)^2 > \log x,$$

$$\sum_{p \in J} p^{-1} |f(p)-\lambda_0 \log p|^2 > \log x,$$

for all sufficiently large x. The argument given for the first example would not here yield a sufficiently strong lower bound.

It will have been observed that the second example does not lie particularly near to the surface.

We cannot directly tackle a general frequency $\nu_x(p;\, f(p+1) \leqslant z)$ using the representation by $P_1(\sum X_{1q} \leqslant z)$, since we begin with no control over the values of f on primes in the interval $(\exp(L(\log L)^{-2}), x]$.

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