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FUNCTIONS WITHOUT EXCEPTIONAL FAMILY OF ELEMENTS AND THE SOLVABILITY OF VARIATIONAL INEQUALITIES ON UNBOUNDED SETS

George Isac — M. Gabriela Cojocaru

Abstract. In this paper we prove an alternative existence theorem for variational inequalities defined on an unbounded set in a Hilbert space. This theorem is based on the concept of exceptional family of elements (EFE) for a mapping and on the concept of $(0, k)$-epi mapping which is similar to the topological degree. We show that when a k-set field is without (EFE) then the variational inequality has a solution. Based on this result we present several classes of mappings without (EFE).

1. Introduction

The theory of *variational inequalities* is now very well developed and the number of papers dedicated to this subject is impressive (see [8], [9], [25], [29]–[33], [36], [37] and many others). The development of this theory has been stimulated by the diversity of applications in Physics, Mechanics, Elasticity, Fluid Mechanics, Engineering and Economics. The solvability of variational inequalities has been studied with several methods based, for example, on coercivity conditions, on compactness, on the fixed point theory, on KKM-mappings and on the minimax theory. Recently, using the topological degree, we introduced the concept of exceptional family of elements for a function ([19], [3]). Applying this concept we studied several problems related to complementarity theory ([3], [4], [10]–[21]).

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In his Ph.D. thesis Y. B. Zhao extended the concept of exceptional family of elements to variational inequalities in the Euclidean space (see [30]).

In [29]–[33], [36] and [37], several existence results are presented. In our recent paper [22] we introduced the concept of exceptional family of elements for a completely continuous field in infinite dimensional Hilbert spaces and we applied this concept to the study of solvability of variational inequalities.

In this paper we will extend the main result proved in [21] to k-set fields and we will show that several classes of fields are without exceptional family of elements. This important fact implies the solvability of variational inequalities on unbounded sets. The main result (Theorem 4.2) will be given using a concept of $(0,k)$-*epi mapping* which is a more refined concept than the topological degree. From this point of view our paper can be considered as an interesting application of $(0,k)$-epi mappings to the study of variational inequalities on unbounded sets. This is a deep relation and it must be exploited in other future papers on variational inequalities.

2. Preliminaries

Let $(H,\langle\cdot\,,\cdot\rangle)$ be a Hilbert space and $\Omega\subset H$ a non-empty unbounded closed convex set. Since Ω is closed and convex, then the projection operator onto Ω, denoted by P_Ω is well defined for every $x\in H$. It is well known that for any $x\in H$, $P_\Omega(x)$ is the unique element in Ω such that

$$\|x-P_\Omega(x)\|=\min_{y\in\Omega}\|x-y\|.$$

Given a mapping $f\colon H\to H$ we can consider the following variational inequality defined by f and Ω:

$$\mathrm{VI}(\Omega,f):\begin{cases}\text{find } x_*\in\Omega \text{ such that}\\ \langle x-x_*,f(x_*)\rangle\geq 0,\ \text{ for all } x\in\Omega.\end{cases}$$

It is known ([10], [11]) that the solvability of the problem $\mathrm{VI}(\Omega,f)$ is equivalent to the solvability in H of the following equation

$$x=P_\Omega(x-f(x)). \tag{2.1}$$

If $X\subset H$ is an arbitrary non-empty subset, we denote by ∂X the boundary of X, by $\mathrm{int}(X)$ the interior of X and by $\mathrm{cl}(X)$ the closure of X.

We say that a subset $K\subset H$ is a cone if $\lambda K\subseteq K$ for all $\lambda\in R_+$ and we say that K is a convex cone if

(a) $\lambda K\subseteq K$ for all $\lambda\in R_+$, and
(b) $K+K\subseteq K$.

If $K \subset H$ is a cone, its dual is (by definition)

$$K^* = \{y \in H \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

We can show that K^* is a convex cone.

If $D \subset H$ is a non-empty convex set and $x \in \mathrm{cl}(D)$ then (by definition) the normal cone of D at the point x is

$$N_D(x) = \{\zeta \in H \mid \langle \zeta, y - x \rangle \leq 0 \text{ for all } y \in D\}$$

or $N_D(x) = -[T_D(x)]^*$ where T_D is the tangent cone of D at the point x, i.e.

$$T_D(x) = \mathrm{cl}\bigg(\bigcup_{\lambda > 0} \lambda(D - x) \bigg).$$

The following proposition is a classical known result.

Proposition 2.1. *For each $x \in H$, we have that $y = P_\Omega(x)$ if and only if $x \in y + N_\Omega(y)$.*

3. $(0,k)$-epi mappings

About the solvability of a variational inequality we will prove in this paper an alternative theorem which is valid for a much larger class of mappings as the main result proved in [22].

To prove this new result, we need to introduce a mathematical tool, similar to the topological degree, but simpler and more refined. This is the concept of $(0,k)$-*epi mapping*, which is a generalization obtained by E. V. Tarafdar and H. B. Thompson (see [28]) of the concept of *zero-epi mapping* introduced by M. Furi, M. Martelli and A. Vignoli in [7].

Now we will give only the definition and the most important properties of this concept.

Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces, $\Omega \subset E$ a subset and $f\colon \Omega \to F$ a mapping. Let $A \subset E$ be a non-empty subset. The Kuratowski measure of noncompactness of A is by definition:

$$\alpha(A) = \inf\{\epsilon > 0 \mid A \text{ can be covered by a finite number of sets of diameter less than } \varepsilon\}.$$

The measure of noncompactness can be consider in E or in F and it will be denoted by the same letter α.

It is known that $\alpha(A) = 0$ if and only if A is relatively compact. A continuous mapping $f\colon \Omega \to F$ is said to be a *k-set contraction* if for each bounded subset A of Ω we have $\alpha(f(A)) \leq k\alpha(A)$, where $k \geq 0$. Let $\Omega \subset E$ be a bounded open subset in E and p an element in F.

Definition 3.1 ([7]). A continuous mapping $f:\overline{\Omega} \to F$ is said to be 0-*admissible* (respectively, *p-admissible*) if $0 \notin f(\partial\Omega)$ (respectively, $p \notin f(\partial\Omega)$).

Definition 3.2 ([28]). A 0-admissible mapping $f:\overline{\Omega} \to F$ is said to be $(0,k)$-*epi* if for each k-set contraction $h:\overline{\Omega} \to F$ with $h(x) = 0$ for each $x \in \partial\Omega$, the equation $f(x) = h(x)$ has a solution in Ω. Similarly, a p-admissible mapping $f:\overline{\Omega} \to F$ is said to be (p,k)-*epi* if the mapping $f - p$ defined by $(f-p)(x) = f(x) - p$, for each $x \in \overline{\Omega}$ is $(0,k)$-epi.

If in Definition 3.2 we replace the term *k-set contraction* by *compact mapping* (i.e. $h(\overline{\Omega})$ is relatively compact in F), then we obtain the concept of 0-epi mapping introduced in [7] and studied in several papers ([10]). The concept of $(0,k)$-epi mapping has the following main properties:

(I) (Existence property) If $f:\overline{\Omega} \to F$ is a (p,k)-epi mapping, then the equation $f(x) = p$ has a solution in Ω.

(II) (Normalization property) The inclusion mapping $i:\overline{\Omega} \to E$ is (p,k)-epi for $k \in [0,1[$, if and only if $p \in \Omega$.

(III) (Localization property) If $f:\overline{\Omega} \to F$ is a (p,k)-mappnig and $f^{-1}(0)$ is contained in an open set $\Omega_1 \subset \Omega$, then f restricted to Ω_1 is also $(0,k)$-epi.

(IV) (Homotopy property) Let $f:\overline{\Omega} \to F$ be $(0,k)$-epi and $h:[0,1] \times \overline{\Omega} \to F$ be a β-set contraction with $0 \le \beta \le k < 1$ such that $h(0,x) = 0$ for all $x \in \overline{\Omega}$. If $f(x) + h(t,x) \ne 0$ for all $x \in \partial\Omega$ and for all $t \in [0,1]$, then $f(\,\cdot\,) + h(1,\,\cdot\,):\overline{\Omega} \to F$ is a $(0, k-\beta)$-epi mapping.

(V) (Boundary dependence property) Let $f:\overline{\Omega} \to F$ be $(0,k)$-epi and $g:\overline{\Omega} \to F$ be a β-set contraction with $0 \le \beta \le k < 1$ and $g(x) = 0$ for each $x \in \partial\Omega$, then $f + g:\overline{\Omega} \to F$ is a $(0, k-\beta)$-epi mapping.

4. Exceptional family of elements and the solvability of variational inequalities on unbounded sets

Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be an arbitrary Hilbert space, $\Omega \subset H$ a non-empty unbounded closed convex set and $f:H \to H$ a mapping. We say that f is a *k-set field* if f has a representation of the form $f(x) = x - T(x)$, where $T:H \to H$ is a k-set contraction with $0 \le k < 1$. When $k = 0$, we have that f is a *completely continuous field*.

Definition 4.1 ([22]). We say that $\{x_r\}_{r>0} \subset H$ is an *exceptional family of elements for the mapping* $f(x) = x - T(x)$ defined on H with respect to the subset Ω if the following conditions are satisfied:

(a) $\|x_r\| \to \infty$ as $r \to \infty$.

(b) For any $r > 0$ there exists a real number $\mu_r > 1$ such that $\mu_r x_r \in \Omega$ and $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$, where $N_\Omega(\mu_r x_r)$ is the normal cone of Ω at the point $\mu_r x_r$.

The importance of Definition 4.1 is given by the following result.

Theorem 4.2. *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and $f\colon H \to H$ a k-set field (with the representation $f(x) = x - T(x)$). Then the problem* $\mathrm{VI}(\Omega, f)$ *has at least one of the following two properties:*

(a) $\mathrm{VI}(\Omega, f)$ *has a solution,*
(b) *the k-set field f has an exceptional family of elements with respect to Ω.*

Proof. We associate to the problem $\mathrm{VI}(\Omega, f)$ the mapping $\Phi\colon H \to H$ defined by

$$\Phi(x) = x - P_\Omega[x - f(x)] = x - P_\Omega(T(x))$$

for any $x \in H$. It is a classical result that the problem $\mathrm{VI}(\Omega, f)$ has a solution if and only if the equation $\Phi(x) = 0$ has a solution. We use the following notations: $S_r = \{x \in H \mid \|x\| = r\}$ and $B_r = \{x \in H \mid \|x\| < r\}$, for any $r > 0$.

Remark that the identity mapping $\mathrm{id}(x) = x$ is a $(0, k)$-epi mapping on any set B_r with $k \in [0, 1[$ and we consider the mapping $h\colon [0, 1] \times \overline{B_r} \to H$ defined by:

$$h(t, x) = t(x - P_\Omega[x - f(x)] - x) = t(-P_\Omega[x - f(x)]).$$

The mapping h is a k-set contraction such that $h(t, x) = 0$ for all $x \in \overline{B_r}$. We have only the following two situations:

(a) There exists $r > 0$ such that $x + t(-P_\Omega[x - f(x)]) \neq 0$ for all $x \in S_r$ and all $t \in [0, 1]$.

In this case applying the homotopy property for $(0, k)$-epi mappings for h and id, we have that $x + t(-P_\Omega[x - f(x)]) = 0$ has a solution in B_r, that is there exists $x_* \in B_r$ such that $x_* = P_\Omega[x_* - f(x_*)]$, which implies that x_* is a solution to the problem $\mathrm{VI}(\Omega, f)$.

(b) For every $r > 0$ there exist $x_r \in S_r$ and $t_r \in [0, 1]$ such that

$$x_r + t_r(-P_\Omega[x_r - f(x_r)]) = 0.$$

If $t_r = 0$, we have that $x_r = 0$, which is impossible since $x_r \in S_r$. If $t_r = 1$ then $x_r - P_\Omega[x_r - f(x_r)] = 0$ which is equivalent to say that $VI(\Omega, f)$ has a solution.

Hence we can say that either the problem $\mathrm{VI}(\Omega, f)$ has a solution or for any $r > 0$ there exist $x_r \in S_r$ and $t_r \in]0, 1[$ such that $x_r = t_r P_\Omega[T(x_r)]$.

By Proposition 2.1 we have that

$$T(x_r) \in \frac{1}{t_r} x_r + N_\Omega\biggl(\frac{1}{t_r} x_r\biggr).$$

If we denote by $\mu_r = 1/t_r$ for all $r > 0$, then we have:

(a) $\|x_r\| = r$ and $\mu_r > 1$ for all $r > 0$,
(b) $\mu_r x_r \in \Omega$ for all $r > 0$,
(c) $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$ for all $r > 0$,

and since $\|x_r\| \to \infty$ as $r \to \infty$, we obtain that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to Ω and the proof is complete. $\square$

If for the k-set field $f(x) = x - T(x)$, $k = 0$, then the mapping f is a completely continuous mapping. In this case, we can prove Theorem 4.2 applying the *Leray–Schauder alternative*. About this classical result the reader is referred to [1], [2] and [6].

Applying the variant of Leray–Schauder alternative proved with the transversality theory in [6, Theorem 5.1] we obtain the following result.

If Ω is such that $0 \in \Omega$, then in this case (supposing that f is a completely continuous filed), the exceptional family of elements $\{x_r\}_{r>0}$, obtained in the proof of Theorem 4.2 can be selected such that for each $r > 0$, $x_r \in \Omega$. Indeed, since $0 \in \Omega$ we apply [6, Theorem 5.1] taking $C = \Omega$ and $U_r = \{x \in \Omega \mid \|x\| < r\}$. The set U_r is open in Ω and its boundary ∂U_r with respect to Ω is the set $\{x \in \Omega \mid \|x\| = r\}$. Obviously, we have for each $r > 0$, $x_r \in \partial U_r$, that is $x \in \Omega$ and $\|x_r\| = r$. A consequence of Theorem 4.2 is the following result.

Theorem 4.3. *Let $(H, \langle \cdot , \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and $f: H \to H$, $f(x) = x - T(x)$ a k-set field on H. If f is without exceptional family of elements with respect to Ω, then the problem* $\mathrm{VI}(\Omega, f)$ *has a solution.*

5. k-set fields without exceptional family of elements

In this section we will present several classes of k-set fields without exceptional family of elements with respect to an unbounded closed convex set. Here f can be supposed a k-set field.

Definition 5.1. We say that a mapping $f: H \to H$ satisfies condition (θ, Ω) with respect to an unbounded closed convex set $\Omega \subset H$ if there exists $\rho > 0$ such that for each couple (x, α) with $\|x\| > \rho$, $\alpha \geq 1$ and $\alpha x \in \Omega$, there exists $y \in \Omega$ such that $\|y\| < \alpha\|x\|$ and $\langle f(x), \alpha x - y \rangle \geq 0$.

Remark 5.2. If Ω is a closed convex cone, then in this case condition (θ, Ω) is equivalent to the following condition:

$$(\theta) : \begin{cases} \text{there exists } \rho > 0 \text{ such that for each } x \in \Omega \text{ with } \|x\| > \rho, \\ \text{there exists } y \in \Omega \text{ such that } \|y\| < \|x\| \text{ and } \langle f(x), x - y \rangle \geq 0. \end{cases}$$

Indeed, if Ω is a closed convex cone and f satisfies condition (θ), then there exists $\rho > 0$ such that for each $x \in \Omega$ with $\|x\| > \rho$, there exists $y \in \Omega$ such that

$\|y\| < \|x\|$ and $\langle f(x), x - y\rangle \geq 0$. In this case $\{x \in H \mid \text{there exists } \alpha \geq 1,\ \alpha x \in \Omega\} = \Omega$. For all $\alpha \geq 1$, $\alpha x \in \Omega$, $\|\alpha y\| < \alpha\|x\|$, $y \in \Omega$ and $\langle f(x), \alpha x - \alpha y\rangle \geq 0$, which means that f satisfies condition (θ, Ω). Conversely, if f satisfies condition (θ, Ω), then obviously f satisfies condition θ with respect to Ω.

We introduced condition (θ) in [14], [20] in Complementarity Theory. Therefore condition (θ, Ω) is an adaptation of condition (θ) for an arbitrary unbounded closed convex set $\Omega \subset H$. The importance of condition (θ, Ω) is given by the following result.

THEOREM 5.3. *Let $(H, \langle,\rangle)$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and $f\colon H \to H$ a k-set field. If f satisfies condition (θ, Ω) with respect to Ω then f is without exceptional family of elements and the problem* $\mathrm{VI}(\Omega, f)$ *has a solution.*

PROOF. Suppose that f has an exceptional family of elements $\{x_r\}_{r>0}$ with respect to Ω. Hence $\{x_r\}_{r>0}$ satisfies Definition 4.1. For each $r > 0$ we have that $\mu_r x_r \in \Omega$ where $\mu_r > 1$ and applying condition (θ, Ω), there exists y_r such that $\|y_r\| < \|\mu_r x_r\|$ and $\langle f(x_r), \mu_r x_r - y_r\rangle \geq 0$, for each $r > 0$ such that $\|x_r\| \geq \rho$. Therefore for $r > 0$ such that $\|x_r\| \geq \rho$ we have $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$, i.e. $\zeta_r = T(x_r) - \mu_r x_r$ satisfies the condition $\langle \zeta_r, y - \mu_r x_r\rangle \leq 0$, for all $y \in \Omega$, and

$$\begin{aligned} 0 &\leq \langle f(x_r), \mu_r x_r - y_r\rangle = \langle x_r - T(x_r), \mu_r x_r - y_r\rangle \\ &= \langle x_r - \mu_r x_r - \zeta_r, \mu_r x_r - y_r\rangle = \langle (1-\mu_r)x_r - \zeta_r, \mu_r x_r - y_r\rangle \\ &= (1-\mu_r)\langle x_r, \mu_r x_r - y_r\rangle + \langle \zeta_r, y_r - \mu_r x_r\rangle \\ &\leq (1-\mu_r)[\mu_r\|x_r\|^2 - \langle x_r, y_r\rangle] < 0, \end{aligned}$$

since $1 - \mu_r < 0$ and

$$\mu_r\|x_r\|^2 - \langle x_r, y_r\rangle \geq \mu_r\|x_r\|^2 - \|x_r\| \cdot \|y_r\| = \|x_r\|[\mu_r\|x_r\| - \|y_r\|] > 0.$$

We have a contradiction which implies that f is without exceptional family of elements. □

Now we give some examples of functions which satisfy condition (θ, Ω). We suppose that $\Omega \subset H$ is an unbounded, closed and convex set.

DEFINITION 5.4. A mapping $f\colon H \to H$ is said to be *ρ-copositive on* Ω if there exists $\rho > 0$ such that for all $x \in \Omega$, with $\|x\| > \rho$ we have $\langle x, f(x)\rangle \geq 0$.

PROPOSITION 5.5. *If $f\colon H \to H$ is ρ-copositive on Ω and there exists $x_* \in \Omega$ such that $\|x_*\| < \rho$ and $\langle x_*, f(x)\rangle \leq 0$ for all $x \in H$ with $\alpha x \in \Omega$ for $\alpha > 1$ and $\|x\| > \rho$, then f satisfies condition (θ, Ω).*

PROOF. Indeed, if $x \in H$ is such that $\|x\| > \rho$ and $\alpha x \in \Omega$ for $\alpha \geq 1$ then we have $\langle x, f(x)\rangle \geq 0$ which implies $\langle \alpha x, f(x)\rangle \geq 0$. Since $\|x_*\| < \rho \leq \|\alpha x\|$ and $\langle \alpha x - x_*, f(x)\rangle \geq 0$, we have that f satisfies condition (θ, Ω) with respect to Ω. □

Corollary 5.6. *If $f: H \to H$ is ρ-copositive on Ω and $0 \in \Omega$, then f satisfies condition (θ, Ω).*

Definition 5.7. We say that $f: H \to H$ satisfies condition (K) with respect to Ω if there exists a bounded set $D \subset \Omega$ such that for all couples (x, α) where $x \in H, \alpha \geq 1$ and $\alpha x \in \Omega \setminus D$ there exists $y \in D$ such that $\langle f(x), \alpha x - y \rangle \geq 0$.

Remark 5.8. Condition (K) is a Karamardian type condition.

Proclaim 5.9. *If $f: H \to H$ satisfies condition* (K) *with respect to Ω, then f satisfies condition (θ, Ω).*

Proof. Let $D \subset \Omega$ be the set defined by condition (K). Since D is bounded, there exists $\rho > 0$ such that $D \subset \{x \in \Omega \mid \|x\| \leq \rho\}$. For each couple (x, α) where $x \in H$, $\alpha \geq 1$ and $\alpha x \in \Omega$ we have $\|\alpha x\| \geq \|x\| > \rho$, which implies $\alpha x \in \Omega \setminus D$ and there exists $y \in D$ such that $\langle f(x), \alpha x - y \rangle \geq 0$. Because $\|y\| \leq \rho < \alpha\|x\|$ we have that f satisfies condition (θ, Ω) on Ω. $\square$

The following condition is inspired by a similar condition introduced by X. P. Ding and K. K. Tan in [5].

Definition 5.10. A mapping $f: H \to H$ satisfies condition (DT) with respect to Ω if there exists two non-empty bounded subsets $D_0, D_* \subset \Omega$ such that for each couple (x, α) where $x \in H$, $\alpha \geq 1$ and $\alpha x \in \Omega \setminus D_*$ there is an $y \in \operatorname{co}(D_0 \cup \{\alpha x\})$ verifying $\langle \alpha x - y, f(x) \rangle \geq 0$.

Proposition 5.11. *If $f: H \to H$ satisfies condition* (DT) *with respect to Ω, then f satisfies condition (θ, Ω).*

Proof. Since D_0 and D_* are bounded there exists $\rho > 0$ such that $D_0, D_* \subset \{x \in \Omega \mid \|x\| \leq \rho\}$. If $x \in H$ is such that $\|x\| > \rho$ and $\alpha x \in \Omega$ for some $\alpha \geq 1$, then we have that $\alpha x \in \Omega \setminus D_*$ and by condition (DT) there exists $y \in \operatorname{co}(D_0 \cup \{\alpha x\})$ verifying $\langle \alpha x - y, f(x) \rangle \geq 0$. We have

$$y = \lambda d_0 + (1 - \lambda)(\alpha x), \quad \text{with } \lambda \in [0, 1] \text{ and } d_0 \in D_0,$$

which implies

$$y = \lambda\|d_0\| + (1 - \lambda)\|\alpha x\| < \lambda\|\alpha x\| + (1 - \lambda)\|\alpha x\| = \alpha\|x\|,$$

since $d_0 \leq \rho < \|x\|$. Therefore f satisfies condition (θ, Ω). $\square$

Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and $f, g: H \to H$ two mappings. The following notion is a variant of a notion introduced in [18] for complementarity theory, i.e. when Ω is a closed pointed convex cone.

Definition 5.12. We say that $f: H \to H$ is *asymptotically strongly g-demimonotone with respect to* Ω if there exist a function $\phi: R_+ \to R_+$, an element $u \in \Omega$ and a real number $\rho > 0$ such that:

(a) $\lim_{t\to\infty} \phi(t) = \infty$, and
(b) for each couple (x, α) where $x \in H$, $\|x\| > \rho$, $\alpha \geq 1$ and $\alpha x \in \Omega$ we have $\langle \alpha x - u, f(x) - g(u)\rangle \geq \|\alpha x - u\| \phi(\|\alpha x - u\|)$.

Proposition 5.13. *If $f: H \to H$ is asymptotically strongly g-demimonotone with respect to Ω then f satisfies condition (θ, Ω).*

Proof. Assume f satisfies condition (θ, Ω) with respect to Ω. For each couple (x, α) where $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$ we have $\|u\| < \alpha\|x\|$ and

$$\langle \alpha x - u, f(x) - g(u)\rangle \geq \|\alpha x - u\| \phi(\|\alpha x - u\|),$$

which implies

$$\langle \alpha x - u, f(x)\rangle \geq \langle \alpha x - u, g(u)\rangle + \|\alpha x - u\| \phi(\|\alpha x - u\|).$$

Since $\alpha\|x\| > \|u\|$ we have $\|\alpha x - u\| > 0$ and

$$\langle \alpha x - u, f(x)\rangle \geq \|\alpha x - u\| \left[\left\langle \frac{\alpha x - u}{\|\alpha x - u\|}, g(u)\right\rangle + \phi(\|\alpha x - u\|)\right].$$

Since $S_1 = \{x \in H \mid \|x\| = 1\}$ is bounded and considering for u fixed, $g(u)$ as a continuous linear functional on H, we deduce that there exists $\gamma \in R$ such that

$$\left\langle \frac{\alpha x - u}{\|\alpha x - u\|}, g(u)\right\rangle \geq \gamma,$$

for each couple (x, α) where $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$. Since Ω is unbounded there exist couples (x, α) such that $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$ and $\|\alpha x - u\| \to \infty$ as $\|x\| \to \infty$. Because $\lim_{t\to\infty} \phi(t) = \infty$ we have that there exists ρ_* such that for all couples (x, α) with $\alpha \geq 1$, $\alpha x \in \Omega$, $\|x\| > \max\{\rho, \|u\|\}$ and $\|\alpha x - u\| > \rho_*$ we have $\phi(\|\alpha x - u\|) \geq -\gamma$ that is $\langle \alpha x - u, f(x)\rangle \geq 0$.

If for any couple (x, α) with $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$ we take $y = u$ we have that f satisfies condition (θ, Ω) (since $\alpha\|x\| \geq \|x\| > \max\{\rho_* + \|u\|, \rho\}$ and $\|\alpha x - u\| > \rho$). □

Definition 5.14. We say that $f: H \to H$ is *scalarly increasing to infinity on* Ω if for each $y \in \Omega$ there exists a real number $\rho(y) > 0$ such that for all couples (x, α), $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| \geq \rho(y)$ we have

$$\langle \alpha x - y, f(x)\rangle \geq 0.$$

Theorem 5.15. *If the mapping $f: H \to H$ is scalarly increasing to infinity on Ω (supposed to be an unbounded closed and convex set) then f satisfies condition (θ, Ω).*

Proof. Since f is scalarly increasing to infinity then for each $y \in \Omega$ there exists a real number $\rho(y) > 0$ such that for all couples (x, α), $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| \geq \rho(y)$ we have $\langle \alpha x - y, f(x)\rangle \geq 0$.

Fix y_0 arbitrarily in Ω with $\|y_0\| > 0$. This is possible since Ω is unbounded. Then there exists a real number $\rho_0 := \rho(y_0) > 0$ such that for all couples (x, α), $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| \geq \rho_0$ we have

$$\langle \alpha x - y_0, f(x)\rangle \geq 0. \tag{5.1}$$

If we put $\rho_* = \rho_0 + \|y_0\|$, certainly we have that (5.1) is satisfied for each couple (x, α), with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| \geq \rho_* \geq \rho_0$. Obviously for such a couple we have $\alpha\|x\| \geq \|x\| > \|y_0\|$, which implies that condition (θ, Ω) is satisfied for f with respect to Ω. □

We denote by $\mathrm{conh}(\Omega)$ the conical hull of Ω, i.e.

$$\mathrm{conh}(\Omega) = \bigcup_{\lambda \geq 0} \lambda\Omega.$$

Definition 5.16. We say that $T: H \to H$ is *monotonically decreasing on rays with respect to* $\mathrm{conh}(\Omega)$ if for every $\alpha \geq 1$ and every $x \in \mathrm{conh}(\Omega)$ we have

$$\langle x, T(x)\rangle \geq \langle x, T(\alpha x)\rangle.$$

Theorem 5.17. *If the mapping $T: H \to H$ is bounded, monotonically decreasing on rays with respect to* $\mathrm{conh}(\Omega)$ *and $0 \in \Omega$ then the mapping $f(x) = x - T(x)$ is without exceptional family of elements with respect to Ω.*

Proof. Suppose that f has an exceptional family of elements $\{x_r\}_{r>0}$. For every x_r with $\|x_r\| \geq 1$ we take $\alpha = \|x_r\|$ and

$$x = \frac{x_r}{\|x_r\|} = \frac{\mu_r x_r}{\|\mu_r x_r\|} \in \mathrm{conh}(\Omega).$$

Because T is monotonically decreasing on rays with respect to $\mathrm{conh}(\Omega)$ we have

$$\langle x, T(x)\rangle \geq \langle x, T(\alpha x)\rangle$$

or

$$\langle x_r, T(x) - T(x_r)\rangle \geq 0, \tag{5.2}$$

for any $r > 0$ with $\|x_r\| \geq 1$. We know that $T(x_r) - \mu_r x_r = \zeta_r \in N_\Omega(\mu_r x_r)$, which implies that $T(x_r) = \mu_r x_r + \zeta_r$ where $\langle \zeta_r, y - \mu_r x_r\rangle \leq 0$ for all $y \in \Omega$. From (5.2) we have

$$0 \leq \langle x_r, T(x) - (\mu_r x_r + \zeta_r)\rangle = \langle x_r, T(x)\rangle - \langle x_r, \mu_r x_r + \zeta_r\rangle.$$

Therefore

$$\langle x_r, \mu_r x_r + \zeta_r\rangle \leq \langle x_r, T(x)\rangle \tag{5.3}$$

Since $0 \in \Omega$ we have $\langle x_r, \zeta_r\rangle \geq 0$, and since T is bounded there exists $M > 0$ such that $\|T(x)\| \leq M$. Considering (5.3) and the fact that $\mu_r > 1$ we obtain that

$$\|x_r\|^2 = \langle x_r, x_r\rangle \leq \langle x_r, T(x)\rangle \leq \|x_r\| M,$$

and consequently $\|x_r\| \leq M$, for all $r > 0$ such that $\|x_r\| \geq 1$, which is impossible because $\|x_r\| \to \infty$ as $r \to \infty$. Therefore f is without exceptional family of elements with respect to Ω. □

In the next definition we adapt for an arbitrary unbounded closed convex set the condition Isac–Gowda considered for convex cones by Y. B. Zhao (see [31]).

Definition 5.18. We say that $f\colon H \to H$ satisfies condition (IG) with respect to Ω if there exists a real number $p > 0$ such that the mapping $\Phi(x) = \|x\|^{p-1}x - f(x)$ is monotonically decreasing on rays with respect to $\operatorname{conh}(\Omega)$.

We have the following result.

Theorem 5.19. *Let $T\colon H \to H$ be a bounded mapping. If $\Omega \subset H$ is an unbounded closed convex subset such that $0 \in \Omega$ and $f(x) = x - T(x)$ satisfies condition* (IG) *with respect to Ω then f is without exceptional family of elements with respect to Ω.*

Proof. Assume f has an exceptional family of elements $\{x_r\}_{r>0}$ with respect to Ω. Because f satisfies condition (IG) we have $\langle x, \Phi(x) - \Phi(\alpha x)\rangle \geq 0$, for all $\alpha \geq 1$ and all $x \in \operatorname{conh}(\Omega)$. For every $r > 0$ such that $\|x_r\| \geq 1$ we take $\alpha = \|x_r\|$ and because $x = x_r/\|x_r\| = \mu_r x_r/\|\mu_r x_r\| \in \operatorname{conh}(\Omega)$ we have

$$\langle x, \Phi(x) - \Phi(x_r)\rangle \geq 0$$

or

$$\langle x_r, \Phi(x) - \|x_r\|^{p-1}x_r + f(x_r)\rangle \geq 0. \tag{5.4}$$

We know that $f(x_r) = x - T(x_r)$ and $T(x_r) = \mu_r x_r + \zeta_r$, where $\langle \zeta_r, y - \mu_r x_r\rangle \leq 0$ for all $y \in \Omega$. From (5.4) we have

$$\langle x_r, \Phi(x) - \|x_r\|^{p-1}x_r + x_r - (\mu_r x_r + \zeta_r)\rangle \geq 0. \tag{5.5}$$

Since $0 \in \Omega$ we deduce that $\langle x_r, \zeta_r\rangle \geq 0$, and from (5.5) we obtain

$$\langle x_r, \Phi(x)\rangle - \|x_r\|^{p+1} + \|x_r\|^2 - \mu_r\|x_r\|^2 \geq 0$$

or

$$\langle x_r, \Phi(x)\rangle - \|x_r\|^{p+1} \geq (\mu_r - 1)\|x_r\|^2 \geq 0,$$

which implies

$$\langle x_r, \Phi(x)\rangle \geq \|x_r\|^{p+1}. \tag{5.6}$$

Because T is bounded we have that Φ is bounded. Therefore there exists $M > 0$ such that $\|\Phi(x_r/\|x_r\|)\| \leq M$ and from (5.6) we obtain that $\|x_r\|^p \leq M$ for all $r > 0$ such that $\|x_r\| \geq 1$, which is impossible since $\lim_{r\to\infty}\|x_r\| = \infty$. This contradiction implies that f is without exceptional families of elements with respect to Ω. □

Now we consider a variant of condition (θ, Ω). Suppose $\Omega \subset H$ to be unbounded closed and convex.

Definition 5.20. We say that $f\colon H \to H$ satisfies condition $(\theta, \Omega)_S$ with respect to Ω if for any family of couples $\{(x_r, \alpha_r)\}_{r>0}$ such that $\alpha_r \geq 1$, $x_r \in H$, $\alpha_r x_r \in \Omega$ and $\|x_r\| \to \infty$ there exists $y_* \in \Omega$ such that

$$\langle \alpha_r x_r - y_*, f(x_r)\rangle \geq 0$$

for some $r > 0$ such that $\alpha_r\|x_r\| > \|y_*\|$.

We have the following result.

Theorem 5.21. *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space, $\Omega \subset H$ an unbounded closed convex set and $f\colon H \to H$ a k-set field with the representation $f(x) = x - T(x)$. If f satisfies condition $(\theta, \Omega)_S$ with respect to Ω then f is without exceptional family of elements and the problem* $\mathrm{VI}(\Omega, f)$ *has a solution.*

Proof. Suppose that f has an exceptional family of elements $\{x_r\}_{r>0}$ with respect to Ω. Hence $\{x_r\}_{r>0}$ satisfies Definition 4.1. Consider the family of couples $\{(x_r, \mu_r)\}_{r>0}$ as obtained by Definition 4.1. We have $\mu_r > 1$ for any $r > 0$ and $\|x_r\| \to \infty$ as $r \to \infty$. By condition $(\theta, \Omega)_S$ there exists $y_* \in \Omega$ such that

$$\langle f(x_r), \mu_r x_r - y_*\rangle \geq 0,$$

for each $r > 0$ such that $\|y_*\| < \|x_r\| \leq \|\mu_r x_r\|$. Now by the same computation as in the proof of Theorem 5.3 we obtain a contradiction which implies that f is without exceptional families of elements. □

Remark 5.22. Our condition $(\theta, \Omega)_S$ is more general than the condition used in [32, Theorem 3.1] because in condition $(\theta, \Omega)_S$ the element y_* is dependent of the family $\{(x_r, \alpha_r)\}_{r>0}$ while in [32, Theorem 3.1] the element y_* is independent of the family $\{x_r\}_{r>0}$.

The following condition is a variant of a condition used by Harker and Pang in Euclidean spaces in [8].

Definition 5.23. We say that $f\colon H \to H$ satisfies condition (HP) with respect to Ω (supposed to be unbounded closed and convex) if there exists a vector $x_* \in \Omega$ such that the set

$$\Omega_I(x_*) = \{x \in H \mid \text{there exists } \alpha \geq 1,\ \alpha x \in \Omega \text{ and } \langle f(x),\ \alpha x - x_* \rangle < 0\}$$

is bounded or empty.

Remark 5.24. The set considered by Harker and Pang is

$$\Omega(x_*) = \{x \in \Omega \mid \langle f(x), x - x_* \rangle < 0\}.$$

Obviously we have $\Omega(x_*) \subseteq \Omega_I(x_*)$ and when $\Omega_I(x_*)$ is bounded or empty we have that $\Omega(x_*)$ has the same property, i.e. Harker and Pang condition is satisfied.

We obtain the following result.

Theorem 5.25. *If the mapping $f\colon H \to H$ satisfies condition* (HP) *with respect to Ω, then f satisfies condition $(\theta, \Omega)_S$.*

Proof. Let $\{(x_r, \mu_r)\}_{r>0}$ be a family of couples such that for each $r > 0$, $\alpha_r \geq 1$, $x_r \in H$, $\alpha_r x_r \in \Omega$ and $\|x_r\| \to \infty$ as $r \to \infty$. If there exists a vector $x_* \in \Omega$ such that the set $\Omega_I(x_*)$ is bounded (or empty), then for $r > 0$ sufficiently large and such that $\|x_r\| \geq \|x_*\|$, we have $x_r \notin \Omega_I(x_*)$. Because $\alpha_r \geq 1$ and $\alpha_r x_r \in \Omega$ we must have $\langle f(x_r), \alpha_r x_r - x_* \rangle \geq 0$. Obviously condition $(\theta, \Omega)_S$ is satisfied for f with respect to Ω. □

A consequence of Theorem 5.25 is the following result.

Proposition 5.26. *Let $f\colon H \to H$ be a k-set field and $\Omega \subset H$ an unbounded closed convex set. If f has an exceptional family of elements with respect to Ω, then for any point $x_* \in \Omega$, the set $\Omega_I(x_*)$ must be non-empty and unbounded.*

Proof. This result is a consequence of Theorems 5.25 and 5.21. □

Y. B. Zhao and J. Y. Han introduced the notion of "p-order coercivity" in the Euclidean space with respect to a set defined by inequalities and equalities (see [32]). Now we will consider the notion of p-order coercivity in an arbitrary Hilbert space H and with respect to an arbitrary unbounded closed convex set

$\Omega \subset H$ such that $0 \in \Omega$. Moreover, we will put the notion of p-order coercivity in relation with our condition $(\theta,\Omega)_S$ and with the notion of scalar asymptotic derivative. We note also that our notion of *exceptional family of elements* is more general than the similar notion used in [32]. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ an unbounded closed convex set.

Definition 5.27. We say that $f\colon H \to H$ is *p-order coercive with respect to* Ω if there exists a real number $p \in\,]-\infty,1[$ such that

$$\lim_{\substack{\|x\|\to\infty\\ x\in\Omega}} \frac{\langle f(x),x\rangle}{\|x\|^p} = \infty.$$

Remark 5.28. When $p = 1$ we have the classical notion of coercivity used by many authors in the theory of variational inequalities.

Any coercive mapping is p-coercive but the converse is not true (see [32]).

Proposition 5.29. *Let $f\colon H \to H$ be a mapping and $\Omega \subset H$ an unbounded closed convex set such that $0 \in \Omega$. If f is p-coercive then f satisfies condition $(\theta,\Omega)_S$.*

Proof. Suppose that f is p-coercive. If $0 \le p < 1$ then

$$\lim_{\substack{\|x\|\to\infty\\ x\in\Omega}} \frac{\langle f(x),x\rangle}{\|x\|^p} = \infty,$$

implies that

$$\lim_{\substack{\|x\|\to\infty\\ x\in\Omega}} \langle f(x),x\rangle = \infty,$$

and hence condition $(\theta,\Omega)_S$ is satisfied, taking in Definition 5.20, y_*=0. If $-\infty < p < 0$, then for any family of couples $\{(x_r,\mu_r)\}_{r>0}$, $\alpha_r \ge 1$, $x_r \in H$, $\alpha_r x_r \in \Omega$ and $\|x_r\| \to \infty$ as $r \to \infty$, we have (using Definition 5.27), that $\langle f(x_r),x_r\rangle \ge 0$ for $r > 0$ sufficiently large. Obviously for such $r > 0$ we have $\langle f(x_r),x_r\rangle \ge 0$ and if we take in Definition 5.20 y_*=0 we obtain that f satisfies condition $(\theta,\Omega)_S$.□

Definition 5.30. Let $f\colon H \to H$ be a mapping and $\Omega \subset H$ an unbounded closed convex set. We say that $T\colon H \to H$ is a *p-scalar asymptotic derivative of f with respect to* Ω if there exists a real number $p \in\,]-\infty,1[$ such that

$$\lim_{\substack{\|x\|\to\infty\\ x\in\Omega}} \frac{\langle f(x)-T(x),x\rangle}{\|x\|^p} = 0.$$

Theorem 5.31. *Let $f: H \to H$ be a mapping and $\Omega \subset H$ an unbounded closed convex set such that $0 \in \Omega$. If f has a p-scalar asymptotic derivative $T: H \to H$ and T is p-coercive with respect to Ω then f satisfies condition $(\theta, \Omega)_S$.*

Proof. The theorem is a consequence of Proposition 5.29 and of the following relation:

$$\lim_{\substack{\|x\|\to\infty \\ x\in\Omega}} \frac{\langle f(x), x\rangle}{\|x\|^p} = \lim_{\substack{\|x\|\to\infty \\ x\in\Omega}} \frac{\langle f(x) - T(x), x\rangle}{\|x\|^p} + \lim_{\substack{\|x\|\to\infty \\ x\in\Omega}} \frac{\langle T(x), x\rangle}{\|x\|^p} = \infty. \qquad \square$$

Comments. In this paper we presented a topological method applicable to the study of solvability of variational inequalities in an arbitrary Hilbert space and with respect to an unbounded closed convex set. This method is based on the concept of $(0,k)$-epi mapping and on the notion of *exceptional family of elements*. In conclusion it is interesting to find other classes of mappings without exceptional families of elements since this property implies the solvability of variational inequalities.

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GEORGE ISAC
Department of Mathematics
Royal Military College of Canada
P.O. Box 17000, STN Forces
Kingston, Ontario K7K 7B4, CANADA

E-mail address: isac-g@rmc.ca

M. GABRIELA COJOCARU
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario, K7L 3N6, CANADA

E-mail address: monica@mast.queensu.ca