Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 27, 2006, 133–147

THE 8π-PROBLEM FOR RADIALLY SYMMETRIC SOLUTIONS OF A CHEMOTAXIS MODEL IN A DISC

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Abstract. We study the properties and the large time asymptotics of radially symmetric solutions of a chemotaxis system in a disc of $\mathbb{R}^2$ when the parameter is either critical and equal to 8π or subcritical.

1. Introduction

We investigate properties and large time asymptotics of radially symmetric solutions to a parabolic-elliptic model of chemotaxis (the simplified Keller–Segel system [15]) in a disc of $\mathbb{R}^2$. Denoting by $u = u(x,t) \geq 0$ the density of microorganisms (e.g. amoebae), and by $\varphi = \varphi(x,t)$ the concentration of a chemoattractant secreted by themselves, the simplified Keller–Segel system we study herein reads

$$u_t = \nabla \cdot (\nabla u + u\nabla\varphi), \tag{1.1}$$

$$\varphi = E_2 * u, \tag{1.2}$$

with the space variable x and the time variable t ranging in $B(0,R) \equiv \{x \in \mathbb{R}^2 : |x| < R\}$, $R > 0$, and $(0,\infty)$, respectively. Here $E_2(z) = (1/(2\pi)) \log |z|$

2000 *Mathematics Subject Classification*. 35Q99, 35K60, 35B40.
Key words and phrases. Chemotaxis system, critical mass, blow up in infinite time, convergence to steady states.

denotes the fundamental solution of the Laplacian in $\mathbb{R}^2$, so that (1.2) leads to the Poisson equation $\Delta\varphi = u$. The system is supplemented with the no flux boundary condition

$$\frac{\partial u}{\partial \nu} + u\frac{\partial \varphi}{\partial \nu} = 0, \tag{1.3}$$

where ν denotes the outward unit normal vector field to the boundary of $B(0,R)$, and with an initial condition

$$u(x,0) = u_0(x). \tag{1.4}$$

Let us first recall some known results about the system (1.1)–(1.4) considered more generally for x in a bounded domain $\Omega \subset \mathbb{R}^2$. First, the nonnegativity of the initial datum u_0 is preserved by the system. Moreover, owing to the boundary condition (1.3), the total mass of $u(t)$ equal to the L^1-norm $|u(t)|_1$ is conserved, that is, $|u(t)|_1 = \widehat{M} \equiv |u_0|_1$ for $t \geq 0$. It is actually well known that the properties of the solution (u,φ) to (1.1)–(1.4) strongly depend on the parameter $\widehat{M}$. Indeed, if $\widehat{M} > 8\pi$, then solutions of (1.1)–(1.4) blow up in a finite time $T = T(u_0)$, that is,

$$\lim_{t\nearrow T} \|u(t)\|_{H^1} = \lim_{t\nearrow T} |u(t)|_p = \lim_{t\nearrow T} \int_\Omega u(x,t)\log u(x,t)\,dx = \infty$$

for each $p > 1$, cf. [14], [13], [8], [2], [10], [19]. This phenomenon can be accompanied by a concentration of mass at the origin if $\Omega = B(0,R)$. On the other hand, global solutions do exist if $\widehat{M} \in [0,8\pi)$ [8], cf. [11] for the case of the whole plane $\mathbb{R}^2$.

In this paper, we discuss the radially symmetric densities $u(x,t) = u(|x|,t)$ in the disc $B(0,R) \subset \mathbb{R}^2$ (we refer to the companion paper [7] for a discussion on similar issues in the whole plane $\mathbb{R}^2$, cf. [17] for an alternative approach mainly for the supercritical case in the plane). In this situation the nonlocal parabolic-elliptic problem (1.1)–(1.2) can be reformulated as a single nonlinear parabolic equation with singular coefficients for the cumulative mass distribution $Q(r,t)$ defined by

$$Q(r,t) \equiv \int_{B(0,r)} u(x,t)\,dx, \quad r \in [0,R],$$

which reads

$$Q_t = Q_{rr} - \frac{1}{r}\,Q_r + \frac{1}{2\pi r}\,Q\,Q_r, \tag{1.5}$$

supplemented with the boundary conditions

$$Q(0,t) = 0, \quad Q(R,t) = \widehat{M}. \tag{1.6}$$

Here $\widehat{M}$ still denotes the total mass $|u_0|_1$ of the initial datum u_0. Such a formulation is also available in any space dimension, see [6, (6)–(7)]. The initial condition $Q(r,0) = Q_0(r)$, $r \in [0,R]$, is a positive nondecreasing function and satisfies the obvious compatibility conditions $Q_0(0) = 0$ and $Q_0(R) = \widehat{M}$.

It is worth mentioning at this point that the formulation (1.5) allows us to consider some initial data for the density u which could be either unbounded or singular (such as measures). Such initial data would correspond to unbounded derivatives $Q_{0,r}$ or even discontinuous Q_0. Other approaches allowing to consider measures as initial data have been developed in [18], [10], [3]–[5]. We also remark that our problem is equivalent to the problem of self-gravitating particles studied in, e.g. [21], [8], [6], [2], [3], [10].

The scaling properties of (1.5) permit us to assume, without loss of generality, that $R = 1$. Indeed, together with $Q(r,t)$, the function $Q(Rr, R^2t)$ is a solution of (1.5)–(1.6) with the same $\widehat{M}$. Observe that (R times) larger domain implies (R^2 times) slower evolution. Next, the problem (1.5)–(1.6) can be transformed, using a new independent variable $s = r^2$ (cf. [6, (12)]). Performing the transformation $M(r^2,t) \equiv Q(r,t)$, we end up with

$$M_t = 4sM_{ss} + \frac{1}{\pi}MM_s \tag{1.7}$$

together with the boundary

$$M(0,t) = 0, \quad M(1,t) = \widehat{M} \tag{1.8}$$

and initial conditions

$$M(s,0) = M_0(s). \tag{1.9}$$

The remainder of the paper is devoted to the study of the properties of the solutions M to (1.7)–(1.9) when $\widehat{M} \in [0,8\pi]$. We first recall that, in the radially symmetric case with $\widehat{M} > 8\pi$, the occurrence of the blow up phenomenon for u results in a concentration of mass at the origin. In terms of M, this means that $M(0+,t)$ becomes positive after some time T and the boundary condition at $s = 0$ is no longer fulfilled. In other words, the degeneracy of the elliptic operator $4sM_{ss}$ at $s = 0$ does not allow the diffusion to compensate the growth induced by the convection term MM_s/π. On the one hand, we will show that, in the critical case $\widehat{M} = 8\pi$, the blow up in the disc does not take place in finite time but occurs in infinite time, i.e. the whole mass concentrates at $s = 0$ in infinite time. We also obtain some temporal decay estimates on $|M(t) - 8\pi|_1$ for large times. Let us point out here that the situation is completely different in the case of the whole plane, see [17], [7]. On the other hand, if $\widehat{M} \in [0,8\pi)$, we show the exponential convergence of $M(t)$ towards the unique stationary solution to (1.7)–(1.8).

The plan of the paper is the following: Section 2 deals with the existence and regularity of solutions issues, while Section 3 is devoted to uniqueness and stability questions. Large time behaviour results are established in Section 4.

Notation. In the sequel $|\cdot|_p$ will denote the $L^p(\Omega)$ norms, $\|\cdot\|_{H^k}$ will be used for the Sobolev space $H^k(\Omega)$ norm, and $\|\cdot\|_{\mathcal{C}^\varepsilon}$ — for the Hölder space $\mathcal{C}^\varepsilon$ norm. The letter C will denote inessential constants which may vary from line to line.

2. Existence and regularity of solutions

In this section we study the problem (1.7)–(1.9) on $(0,1)\times(0,\infty)$ rewritten as

$$
\begin{aligned}
&(2.1) && M_t = 4sM_{ss} + \frac{1}{\pi} M M_s, && (s,t) \in (0,1)\times(0,\infty),\\
&(2.2) && M(0,t) = \widehat{M} - M(1,t) = 0, && t \in (0,\infty),\\
&(2.3) && M(s,0) = M_0(s), && s \in (0,1),
\end{aligned}
$$

where the initial condition

$$
(2.4) \qquad M_0 \in \mathcal{C}([0,1]), \quad M_0(0) = 0 \quad \text{and} \quad M_0(1) = \widehat{M},
$$

is a nondecreasing function.

We first establish the well-posedness of (2.1)–(2.3) whenever $\widehat{M} \in [0, 8\pi]$.

Theorem 2.1. *Consider $\widehat{M} \in [0,8\pi]$ and a function M_0 satisfying* (2.4). *There exists a unique function $M \in \mathcal{C}([0,\infty); L^2(0,1)) \cap \mathcal{C}^{2,1}_{s,t}((0,1)\times(0,\infty))$ such that*

$$
\begin{aligned}
&(2.5) && 0 \le M(s,t) \le \widehat{M}, \quad M_s(s,t) \ge 0 && \textit{for } (s,t) \in (0,1)\times(0,\infty),\\
&(2.6) && M^*(t) \equiv \inf_{s\in(0,1)} M(s,t) = 0 && \textit{a.e. in } (0,\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
&(2.7) && M_t = 4sM_{ss} + \frac{1}{\pi} M M_s, && (s,t) \in (0,1)\times(0,\infty),\\
&(2.8) && M(1,t) = \widehat{M}, && t \in (0,\infty),\\
&(2.9) && M(s,0) = M_0(s), && s \in (0,1).
\end{aligned}
$$

The proof of the existence part of Theorem 2.1 relies on the analysis of a regularized problem for $\widehat{M} \in [0,8\pi]$. More precisely, for $\varepsilon \in (0,1)$, we consider $M_{0,\varepsilon} \in H^1(0,1)$ satisfying (2.4) and $|M_{0,\varepsilon} - M_0|_\infty \le \varepsilon$. We then denote by M_ε

the unique classical solution to the uniformly parabolic problem

$$
\begin{aligned}
&(2.10) \qquad M_{\varepsilon,t} = 4(s+\varepsilon)M_{\varepsilon,ss} + \frac{1}{\pi}M_\varepsilon M_{\varepsilon,s}, && (s,t)\in(0,1)\times(0,\infty),\\
&(2.11) \qquad M_\varepsilon(0,t) = \widehat{M} - M_\varepsilon(1,t) = 0, && t\in(0,\infty),\\
&(2.12) \qquad M_\varepsilon(s,0) = M_{0,\varepsilon}(s), && s\in(0,1),
\end{aligned}
$$

see, e.g. [1, Sections 14, 15]. In particular,

$$M_\varepsilon \in \mathcal{C}([0,1]\times[0,\infty)) \cap \mathcal{C}^{2,1}_{s,t}((0,1)\times(0,\infty)),$$

and we infer from (2.4), (2.10), (2.11) and the comparison principle that

$$0 \le M_\varepsilon(s,t) \le \widehat{M} \quad \text{and} \quad M_{\varepsilon,s}(s,t) \ge 0 \quad \text{for } (s,t)\in[0,1]\times(0,\infty). \tag{2.13}$$

We next observe that, if $\delta \in (0,1)$, we have $s+\varepsilon \ge \delta$ for $s\in[\delta,1]$, which, together with (2.10) and (2.13), allows us to apply classical parabolic regularity results [16, Theorem VI.10.1] to deduce that

$$\|M_\varepsilon\|_{\mathcal{C}^{2+\alpha,1+\alpha}_{s,t}([\delta,1]\times[\tau,T])} \le C(\alpha,\delta,\tau,T) \tag{2.14}$$

for each $T>0$, $\tau\in(0,T)$ and $\alpha\in(0,1)$, where $C(\alpha,\delta,\tau,T)$ is a positive constant depending on α, δ, τ and T but independent of $\varepsilon\in(0,1)$.

Next we turn to the behaviour of M_ε for small s where the equation (2.1) is no longer uniformly parabolic and establish the following key estimate.

Lemma 2.2. *For each $T\in(0,\infty)$, there is a constant $C_1(T)>0$ such that*

$$0 \le \int_0^T \int_0^1 \frac{M_\varepsilon(s,t)(8\pi - M_\varepsilon(s,t))}{s+\varepsilon}\,ds\,dt \le C_1(T) \tag{2.15}$$

for every $\varepsilon\in(0,1)$.

Proof. We multiply (2.10) by $-\log(s+\varepsilon)$ and integrate over $(0,1)$ to obtain

$$
\begin{aligned}
-\frac{d}{dt}\int_0^1 M_\varepsilon \log(s+\varepsilon)\,ds &= -4(1+\varepsilon)\log(1+\varepsilon)M_{\varepsilon,s}(1,t)\\
&\quad + 4\varepsilon\log(\varepsilon)M_{\varepsilon,s}(0,t) + 4\int_0^1 (1+\log(s+\varepsilon))M_{\varepsilon,s}\,ds\\
&\quad - \frac{\log(1+\varepsilon)}{2\pi}\, M_\varepsilon(1,t)^2 + \frac{1}{2\pi}\int_0^1 \frac{M_\varepsilon^2}{s+\varepsilon}\,ds\\
&\le 4(1+\log(1+\varepsilon))M_\varepsilon(1,t)\\
&\quad - 4\int_0^1 \frac{M_\varepsilon}{s+\varepsilon}\,ds + \frac{1}{2\pi}\int_0^1 \frac{M_\varepsilon^2}{s+\varepsilon}\,ds\\
&\le 32\pi(1+\log(1+\varepsilon)) - \frac{1}{2\pi}\int_0^1 \frac{M_\varepsilon(8\pi - M_\varepsilon)}{s+\varepsilon}\,ds.
\end{aligned}
$$

Observing that the integrand in the last term of the right-hand side of the above inequality is nonnegative by (2.13), we integrate over $(0,T)$, and use (2.4) and (2.13) to conclude that (2.15) holds true with some $C_1(T) = CT + C(\widehat{M})$. $\square$

As a final step towards the proof of Theorem 2.1, we study the behaviour of M_ε for small times.

Lemma 2.3. *For each $T > 0$, there is a constant $C_2(T) > 0$ such that*

$$\int_0^T \int_0^1 (s+\varepsilon)|M_{\varepsilon,s}(s,t)|^2 \, ds \, dt + \int_0^T \|M_{\varepsilon,t}(t)\|_{H^{-1}}^2 \, dt \le C_2(T) \tag{2.16}$$

for every $\varepsilon \in (0,1)$.

Proof. For $\varepsilon \in (0,1)$ and $(s,t) \in (0,1)\times(0,\infty)$, we put $N_\varepsilon(s,t) \equiv M_\varepsilon(s,t) - \widehat{M}\, s$ and notice that $N_\varepsilon(0,t) = N_\varepsilon(1,t) = 0$ by (2.2). We multiply (2.1) by N_ε and integrate over $(0,1)$. Using (2.13) we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|N_\varepsilon|_2^2 = &-4 \int_0^1 (s+\varepsilon)|N_{\varepsilon,s}|^2 \, ds \\
&-4 \int_0^1 N_\varepsilon \, N_{\varepsilon,s} \, ds - \frac{\widehat{M}^3}{6\pi} + \frac{\widehat{M}}{2\pi} \int_0^1 M_\varepsilon^2 \, ds \\
\le &-4 \int_0^1 (s+\varepsilon)|N_{\varepsilon,s}|^2 \, ds + C,
\end{aligned}$$

whence the first assertion in (2.16).

Consider next any $\varphi \in H_0^1(0,1)$. We multiply (2.1) by φ, integrate over $(0,1)$, and infer from (2.13) that

$$\begin{aligned}
\left| \int_0^1 M_{\varepsilon,t}\varphi \, ds \right| &\le 4\left| \int_0^1 (s+\varepsilon)\varphi_s \, M_{\varepsilon,s} \, ds \right| + 4\left| \int_0^1 \varphi M_{\varepsilon,s} \, ds \right| + \frac{1}{2\pi}\left| \int_0^1 M_\varepsilon^2 \varphi_s \, ds \right| \\
&\le C|\varphi_s|_2 \left\{ 1 + \left(\int_0^1 (s+\varepsilon)^2 |M_{\varepsilon,s}|^2 \, ds \right)^{1/2} \right\} + 4\left| \int_0^1 \varphi_s \, M_\varepsilon \, ds \right| \\
&\le C|\varphi_s|_2 \left\{ 1 + \left(\int_0^1 (s+\varepsilon) |M_{\varepsilon,s}|^2 \, ds \right)^{1/2} \right\}.
\end{aligned}$$

The second assertion in (2.16) then follows from the previous inequality and the first assertion in (2.16). $\square$

Proof of Theorem 2.1. By (2.14) and the Arzelà–Ascoli theorem, there exists a subsequence of (M_ε) (not relabeled) and a function

$$M \in \mathcal{C}_{s,t}^{2,1}((0,1] \times (0,\infty))$$

such that

$$M_\varepsilon \to M \quad \text{in } \mathcal{C}([\delta,1] \times [\tau,T]) \cap \mathcal{C}_{s,t}^{2,1}([\delta,1] \times [\tau,T]) \tag{2.17}$$

for each $\delta \in (0,1)$, $T>0$ and $\tau \in (0,T)$. It readily follows from (2.10), (2.11) and (2.17) that

$$\begin{aligned} M_t &= 4sM_{ss} + \frac{1}{\pi}\ M\ M_s, && (s,t) \in (0,1)\times(0,\infty),\\ M(1,t) &= \widehat{M}, && t \in (0,\infty), \end{aligned}$$

and that

$$0 \le M(s,t) \le \widehat{M} \quad \text{and} \quad M_s(s,t) \ge 0 \quad \text{for } (s,t) \in (0,1]\times(0,\infty). \tag{2.18}$$

We are thus left with identifying the initial datum and the boundary condition at $s=0$.

First, let $T>0$. Lemma 2.3 and the Arzelà-Ascoli theorem warrant that we may assume that (M_ε) converges towards M in $\mathcal{C}([0,T];H^{-1}(0,1))$, and thus $M(\cdot\,,0) = M_0$ in $H^{-1}(0,1)$ by (2.12). In addition, Lemma 2.3 and a weak compactness argument ensure that $M_s \in L^2(0,T;H^1(\delta,1))$ for each $\delta \in (0,1)$. Consequently, $M \in \mathcal{C}([0,T];L^2(\delta,1))$ and $M(\cdot\,,0) = M_0$ in $L^2(\delta,1)$ for each $\delta \in (0,1)$. But, recalling (2.13), we actually conclude that $M \in \mathcal{C}([0,T];L^2(0,1))$ with $M(\cdot\,,0)=M_0$.

We next infer from (2.15), (2.17), (2.18) and the Fatou lemma that for each $T>0$

$$0 \le \int_0^T \int_0^1 \frac{M(s,t)(8\pi - M(s,t))}{s}\, ds\, dt \le C_1(T). \tag{2.19}$$

Now, for $t>0$, we put

$$M^*(t) \equiv \lim_{s\to 0} M(s,t) = \inf_{s\in(0,1)} M(s,t) \in [0,\widehat{M}], \tag{2.20}$$

which is well defined by (2.18) and claim that

$$M^*(t) \in \{0, 8\pi\} \quad \text{for a.e. } t \in (0,\infty). \tag{2.21}$$

Indeed, fix $T>0$. If $t \in (0,T)$ is such that $M^*(t) < 8\pi$, there is $s(t) \in (0,1)$ such that $M(s,t) \le (M^*(t)+8\pi)/2$ for $s \in (0,s(t)]$. We then infer from (2.19) that, for each $\vartheta \in (0,1)$,

$$\begin{aligned} C_1(T) &\ge \int_0^T \mathbf{1}_{\{M^*<8\pi\}}(t) \int_{\vartheta s(t)}^{s(t)} \frac{M(s,t)\ (8\pi - M(s,t))}{s}\, ds\, dt\\ &\ge \int_0^T \mathbf{1}_{\{M^*<8\pi\}}(t) \int_{\vartheta s(t)}^{s(t)} \frac{M^*(t)\ (8\pi - M^*(t))}{2s}\, ds\, dt\\ &\ge \frac{|\log(\vartheta)|}{2} \int_0^T \mathbf{1}_{\{M^*<8\pi\}}(t)\ M^*(t)(8\pi - M^*(t))\, dt. \end{aligned}$$

Letting $\vartheta \to 0$ yields $\mathbf{1}_{\{M^*<8\pi\}}(t) M^*(t)(8\pi - M^*(t)) = 0$ for a.e. $t \in (0,T)$, whence the claim (2.21).

Now, either $\widehat{M} < 8\pi$ and (2.21) readily implies that $M^*(t) = 0$ for a.e. $t \in (0,\infty)$. Or $\widehat{M} = 8\pi$ and, if $t_0 > 0$ is such that $M^*(t_0) = 8\pi$, it follows from the monotonicity of M and (2.18) that $M(s,t_0) = 8\pi$ for $s \in (0,1)$. Then, $M_s(1,t_0) = 0$, which contradicts the strong maximum principle. Therefore, $M^*(t) = 0$ for a.e. $t \in (0,\infty)$ and the proof of the existence statement in Theorem 2.1 is complete. As for the uniqueness, it is a straightforward consequence of Theorem 3.1 below. □

Note that, moreover, we have the following continuity property for M.

Proposition 2.4. *Let $t_0 \in (0,\infty)$ be such that $M^*(t_0) = 0$. Then M is continuous at $(0,t_0)$.*

Proof. Consider any $\delta \in (0,1)$. Since $M^*(t_0) = 0$, there is $s_0 \in (0,1)$ such that $M(s_0,t_0) \le \delta/2$. As $s_0 > 0$, the continuity of $t \longmapsto M(s_0,t)$ ensures that there is $\alpha \in (0,1)$ such that $M(s_0,t) \le \delta$ for $t \in (t_0-\alpha, t_0+\alpha)$. Then, if $s \in (0,s_0)$ and $t \in (t_0-\alpha,t_0+\alpha)$, the monotonicity of M with respect to the variable s implies that $M(s,t) \le M(s_0,t) \le \delta$, whence the claimed continuity. □

Note that the property $M^*(t) = 0$ a.e. is intimately connected with the behaviour of the derivative $M_s(s,t)$ near $s = 0$. Namely, the solution in Theorem 2.1 satisfies for each $T > 0$ the property

$$\lim_{s\to 0} \int_0^T s\ M_s(s,t)\,dt = 0. \tag{2.22}$$

Proof of (2.22). Once we have the existence of the solution, we may multiply (2.5) by $-\log\sigma$ and integrate over $\sigma \in (s,1)$ with $s \in (0,1/2)$. We have

$$\begin{aligned}
\frac{d}{dt}\int_s^1 |\log\sigma| M(\sigma,t)\,d\sigma &= -\frac{d}{dt}\int_s^1 \log\sigma\ M(\sigma,t)\,d\sigma \\
&= -\,[4\sigma\log\sigma M_s(\sigma,t)]_s^1 + 4\int_s^1 (1+\log\sigma) M_s(\sigma,t)\,d\sigma \\
&\quad - \frac{1}{2\pi}[\log\sigma\ M^2(\sigma,t)]_s^1 + \int_s^1 \frac{M^2(\sigma,t)}{2\pi\sigma}\,d\sigma \\
&= 4s\log s M_s(s,t) + 4[(1+\log\sigma)M(\sigma,t)]_s^1 \\
&\quad - 4\int_s^1 \frac{M(\sigma,t)}{\sigma}\,d\sigma + \frac{\log s}{2\pi} M^2(s,t) + \int_s^1 \frac{M^2(\sigma,t)}{2\pi\sigma}\,d\sigma \\
&\le -\,4s|\log s|\ M_s(s,t) + 4\,\widehat{M} - 4(1+\log s)M(s,t) \\
&\quad + \frac{\log s}{2\pi} M^2(s,t) + \int_s^1 \frac{M(\sigma,t)(M(\sigma,t)-8\pi)}{2\pi\sigma}\,d\sigma \\
&\le -\,4s|\log s| M_s(s,t) + 4\widehat{M} + \frac{\log s}{2\pi} M(s,t)\ (M(s,t)-8\pi),
\end{aligned}$$

where we have used the fact that $0 \le M(s,t) \le \widehat{M} \le 8\pi$. Integrating with respect to time over $(0,T)$ and using the nonnegativity and monotonicity of M, we obtain

$$\begin{aligned} 0 &\le 4s|\log s| \int_0^T M_s(s,t)\,dt \\ &\le \int_s^1 |\log\sigma|\ M(\sigma,0)\,d\sigma + 4T\widehat{M} + \frac{|\log s|}{2\pi}\int_0^T M(s,t)(8\pi - M(s,t))\,dt \\ &\le (1+4T)\widehat{M} + \frac{|\log s|}{2\pi}\int_0^T M(s,t)(8\pi - M(s,t))\,dt, \end{aligned}$$

whence

$$0 \le 4s\int_0^T M_s(s,t)\,dt \le \frac{(1+4\,T)\widehat{M}}{|\log s|} + \int_0^T \frac{M(s,t)(8\pi - M(s,t))}{2\pi}\,dt.$$

Since $M(s,t) \to 0$ as $s \to 0$ for almost every $t \in (0,T)$ and satisfies (2.18), the Lebesgue dominated convergence theorem implies that the second term of the right-hand side of the above inequality converges to zero as $s \to 0$. We may then let $s \to 0$ in the previous inequality and conclude that

$$\lim_{s\to 0}\int_0^T s\ M_s(s,t)\,dt = 0,$$

whence (2.22). □

Finally, there is a class of initial data for which $M^*(t) = 0$ holds true for every $t \in (0,\infty)$.

Proposition 2.5. *If there is $\delta \in (0,1)$ such that $M_0(s) \le (8\pi s)/\delta$ for $s \in (0,1)$, then $M^*(t) = 0$ for each $t \ge 0$.*

Observe that if the derivative of M_0 is finite: $M_{0,s}(0) < \infty$, then the condition on M_0 is satisfied with a suitable $\delta > 0$.

Proof. We denote by $\widetilde{M}$ the solution to (2.1)–(2.3) with the initial datum $\widetilde{M}(s,0) = 8\pi s$, $s \in (0,1)$. Observing that

$$4s\widetilde{M}_{ss}(s,0) + \frac{1}{\pi}\widetilde{M}(s,0)\widetilde{M}_s(s,0) \ge 0$$

for $s \in (0,1)$, the maximum principle applied to $\widetilde{M}_t$ ensures that $\widetilde{M}_t(s,t) \ge 0$ for $(s,t) \in (0,1)\times(0,\infty)$. Therefore, if $t_2 > 0$ and $t_1 \in (0,t_2)$, we have $\widetilde{M}(s,t_2) \ge \widetilde{M}(s,t_1)$ for $s \in (0,1)$ and thus

$$t \mapsto \int_0^1 \widetilde{M}(s,t)\,ds \quad \text{is a nondecreasing function of time.}$$

Since $\widetilde{M}^*(t) = 0$ for a.e. $t \in (0,\infty)$, we conclude that $\widetilde{M}^*(t) = 0$ for each $t \in (0,\infty)$.

Now, owing to the homogeneity properties of (2.1), the function $\widetilde{M}_\delta$ given by $\widetilde{M}_\delta(s,t) = \widetilde{M}(s/\delta, t/\delta)$ is the solution to (2.1)–(2.3) in $(0,\delta)\times(0,\infty)$ (instead of $(0,1)\times(0,\infty)$) with the initial datum $s \longmapsto 8\pi s/\delta$ and M is clearly a subsolution to (2.1)–(2.3) in $(0,\delta)\times(0,\infty)$. Since $M_0 \le \widetilde{M}_\delta(\,\cdot\,,0)$, the comparison principle entails that $M(s,t) \le \widetilde{M}_\delta(s,t)$ for $(s,t)\in(0,\delta)\times(0,\infty)$. Therefore

$$M^*(t) \le \inf_{s\in(0,\delta)} \widetilde{M}_\delta(s,t) = \widetilde{M}^*(t/\delta) = 0$$

for every $t \ge 0$, and the proof of Proposition 2.5 is complete. $\square$

Remark 2.6. Using the methods above, similar existence and regularity results can be obtained for the problem considered in [10, Theorem 1(i)]. Namely, the equation (2.1) with the boundary conditions $M(0,t) = m^* \in (0,4\pi)$, $M(1,t) = \widehat{M} \le 8\pi - m^*$, and suitable initial conditions, has global solutions satisfying similar properties as those in Theorem 2.1.

3. Uniqueness and stability of solutions

Here we investigate the uniqueness of solutions to (2.1)–(2.3) in $(0,1)\times(0,\infty)$ for arbitrary initial data satisfying (2.4). Since (2.1) is a convection-diffusion equation, we anticipate that it may enjoy some contraction property with respect to some L^1-norm. We actually show the following L^1-stability property for solutions.

Theorem 3.1. *If M_j, $j = 1, 2$, are two solutions to* (2.1)–(2.3) (*as in Theorem* 2.1) *with initial data $M_1(0)$ and $M_2(0)$ satisfying* (2.4) *with the same $\widehat{M}$, $\widehat{M}\in[0,8\pi]$, then $t\mapsto|\varrho(M_1(t)-M_2(t))|_1$ is a nonincreasing function of time for each nonnegative, nonincreasing and concave weight $\varrho\in W^{2,\infty}(0,1)$. Furthermore, if $\widehat{M}\in[0,8\pi)$,*

$$|M_1(t)-M_2(t)|_1 \le 2|M_1(0)-M_2(0)|_1 e^{-(4-(\widehat{M}/2\pi))t}. \tag{3.1}$$

Proof. Consider the difference $N = M_1 - M_2$ which satisfies the equation

$$N_t = \frac{\partial}{\partial s}\left(4sN_s + \frac{1}{2\pi}N(M_1+M_2-8\pi)\right) \tag{3.2}$$

with $N(0,t) = N(1,t) = 0$ for a.e. $t\in(0,\infty)$. For $\delta\in(0,1)$ and $r\in\mathbb{R}$, we put

$$\Phi_\delta(r) \equiv \begin{cases} \dfrac{1}{\delta}\left(|r| - \dfrac{\delta}{2}\right)_+^2 & \text{if } |r|\in[0,\delta], \\ |r| - \dfrac{3}{4}\delta & \text{if } |r|\in(\delta,\infty), \end{cases}$$

which is a convex approximation of $r\mapsto|r|$. Indeed, $r\mapsto\Phi_\delta(r)$ and $r\mapsto r\,\Phi'_\delta(r)$ converge uniformly to the absolute value $|r|$ over $\mathbb{R}$, and $r\mapsto r\Phi''_\delta(r)$ is bounded and converges a.e. to zero as $\delta \to 0$. We multiply (3.2) by $\varrho\Phi_\delta'(N)$ and integrate over $(0,1)$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \varrho(s)\Phi_\delta(N)\,ds\\
&\quad= 4s\varrho(s)N_s\Phi_\delta'(N)|_0^1 + \frac{1}{2\pi}\varrho(s)\Phi_\delta'(N)N(M_1+M_2-8\pi)|_0^1\\
&\qquad-\int_0^1 4s\varrho(s)\Phi_\delta''(N)N_s^2\,ds - \int_0^1 4s\varrho'(s)\Phi_\delta'(N)N_s\,ds\\
&\qquad-\frac{1}{2\pi}\int_0^1 \varrho(s)\Phi_\delta''(N)N_sN(M_1+M_2-8\pi)\,ds\\
&\qquad-\frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi_\delta'(N)N(M_1+M_2-8\pi)\,ds\\
&\quad\le -\frac{1}{2\pi}\int_0^1 \varrho(s)\Phi_\delta''(N)NN_s(M_1+M_2-8\pi)\,ds\\
&\qquad-\frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi_\delta'(N)N(M_1+M_2-16\pi)\,ds\\
&\qquad+4\int_0^1 s\varrho''(s)\Phi_\delta(N)\,ds + 4\int_0^1 \varrho'(s)(\Phi_\delta(N)-N\Phi_\delta'(N))\,ds.
\end{aligned}$$

On the one hand, N_s belongs to $L^\infty(0,\infty;L^1(0,1))$, M_1, M_2 and N are bounded, and $r \mapsto r\ \Phi_\delta''(r)$ is bounded and converges a.e. towards zero as $\delta \to 0$. The Lebesgue dominated convergence theorem ensures that the first term of the right-hand side of the above inequality converges to zero as $\delta \to 0$. On the other hand, both $r \mapsto \Phi_\delta(r)$ and $r \mapsto r\,\Phi_\delta'(r)$ converge uniformly towards $r \mapsto |r|$ on $\mathbb{R}$. Thanks to the boundedness of M_1, M_2 and N, we can pass to the limit as $\delta \to 0$ in the other terms of the above inequality, and end up with

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \varrho(s)|N|\,ds\\
&\qquad\le -\frac{1}{2\pi}\int_0^1 \varrho'(s)|N|(M_1+M_2-16\pi)\,ds + \ 4\int_0^1 s\varrho''(s)|N|\,ds.
\end{aligned}\tag{3.3}$$

Since $M_1+M_2 \le 2\widehat{M} \le 16\pi$ and ϱ' and ϱ'' are both nonpositive, the right-hand side of (3.3)) is nonpositive, from which the first assertion of Theorem 3.1 follows.

We now turn to the decay rate (3.1) and assume that $\widehat{M} \in [0,8\pi)$. We take $\varrho(s) = 2-s$ in (3.3). Since $M_1+M_2 \le 2\widehat{M} < 16\pi$, we infer from (3.3) that

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \le \frac{1}{2\pi}\int_0^1 |N|(2\widehat{M}-16\pi)\,ds \le \frac{\widehat{M}-8\pi}{2\pi}\int_0^1 (2-s)|N|\,ds\,,$$

whence

$$\int_0^1 (2-s)|N(t)|\,ds \le \int_0^1 (2-s)|N(0)|\,ds\ \ e^{-(4-(\widehat{M}/2\pi))t}\,,$$

from which (3.1) readily follows. □

The exponential decay rate does not hold true for $\widehat{M} = 8\pi$ but the following weaker assertion is available.

Proposition 3.2. *Let M be the solution to* (2.1)–(2.3) (*as in Theorem* 2.1) *with the initial datum $M(0)$ satisfying* (2.4) *with $\widehat{M} = 8\pi$. Then, for $t \geq 1$, we have*

$$|M(t) - 8\pi|_1 \leq 8\pi/t. \tag{3.4}$$

Proof. For $(s,t) \in (0,1) \times (0,\infty)$, we put $N(s,t) = M - 8\pi$, $\varrho(s) = 2 - s$ and proceed as in the proof of Theorem 3.1. We notice that N solves

$$N_t = \frac{\partial}{\partial s}\left(4sN_s + \frac{1}{2\pi}NM\right) \tag{3.5}$$

with $N(0,t) = -8\pi$ and $N(1,t) = 0$ for a.e. $t \in (0,\infty)$. Keeping the notations from the proof of Theorem 3.1, we multiply (3.5) by $\varrho\Phi_\delta'(N)$ and integrate over $(0,1)$ to obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 \varrho(s)\Phi_\delta(N)\,ds &= 4s\varrho(s)N_s\Phi_\delta'(N)|_0^1 + \frac{1}{2\pi}\varrho(s)\Phi_\delta'(N)NM|_0^1\\ &\quad - \int_0^1 4s\varrho(s)\Phi_\delta''(N)N_s^2\,ds - \int_0^1 4s\varrho'(s)\Phi_\delta'(N)N_s\,ds\\ &\quad - \frac{1}{2\pi}\int_0^1 \varrho(s)\Phi_\delta''(N)N_sNM\,ds - \frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi_\delta'(N)NM\,ds.\end{aligned}$$

Since Φ_δ' vanishes on a neighbourhood of 0 and $M^*(t) = 0$, the boundary terms vanish and

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \varrho(s)\Phi_\delta(N)\,ds \leq -\frac{1}{2\pi}\int_0^1 \varrho(s)\Phi_\delta''(N)NN_sM\,ds\\ &- \frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi_\delta'(N)NM\,ds + 4\int_0^1 s\varrho''(s)\Phi_\delta(N)\,ds + 4\int_0^1 \varrho'(s)\Phi_\delta(N)\,ds.\end{aligned}$$

We then proceed as in the proof of (3.3) to pass to the limit as $\delta \to 0$ and end up with

$$\frac{d}{dt}\int_0^1 \varrho(s)|N|\,ds \leq \frac{1}{2\pi}\int_0^1 \varrho'(s)(8\pi - M)|N|\,ds,$$

i.e.

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \leq -\frac{1}{2\pi}\int_0^1 |N|^2\,ds.$$

We infer from the Cauchy–Schwarz inequality that

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \leq -\frac{1}{2\pi}\left(\int_0^1 |N|\,ds\right)^2 \leq -\frac{1}{8\pi}\left(\int_0^1 (2-s)|N|\,ds\right)^2,$$

whence

$$|M(t) - 8\pi|_1 \leq \int_0^1 (2-s)|N(t)|\, ds \leq \frac{8\pi}{t + 4\pi|8\pi - M_0|_1^{-1}}.$$

The assertion of Proposition 3.2 then readily follows. □

4. Asymptotics

The large time behaviour of solutions to (2.1)–(2.3) when $\widehat{M} \in [0, 8\pi]$ is a straightforward consequence of Theorem 3.1 and Proposition 3.2.

We first consider the case $\widehat{M} < 8\pi$ and recall that (2.1)–(2.2) has a single stationary solution

$$M_b(s) = 8\pi \frac{s}{s+b}, \quad s \in (0,1), \text{ with } b = \frac{8\pi}{\widehat{M}} - 1 > 0. \tag{4.1}$$

Let M be the solution to (2.1)–(2.3) (as in Theorem 2.1) with the initial datum M_0 satisfying (2.4) with $\widehat{M} \in [0, 8\pi)$. Owing to Theorem 3.1, we have

$$|M(t) - M_b|_1 \leq 2|M_0 - M_b|_1 \; e^{-(4-(\widehat{M}/2\pi))t},$$

and M decays towards M_b exponentially fast in L^1. The convergence holds also in $L^p(0,1)$ for each $p \in (1, \infty)$ as a consequence of the boundedness of M and M_b, and the Hölder inequality. As a further remark, let us recall that there are initial data M_0 satisfying (2.4) with $\widehat{M} \in [0, 8\pi)$ such that the solution M to (2.1)–(2.3) (as in Theorem 2.1 enjoys the additional property $\sup_{t \geq 0} |M_s|_\infty < \infty$ [9], [12]. In that particular case, it follows from the exponential decay in $L^1(0,1)$ and the bound in $W^{1,\infty}(0,1)$ by interpolation inequalities that $|M(t) - M_b|_\infty$ decays exponentially to zero as $t \to \infty$.

If $\widehat{M} = 8\pi$, we infer from Proposition 3.2 that $M(t) \to 8\pi$ when $t \to \infty$ in $L^1(0,1)$. An alternative proof of this fact can be given by a comparison argument. Indeed, if we take the initial conditions $M_{0,\delta} = \min(M_0, 8\pi - \delta)$, $\delta \in (0,1)$, in (2.4), then the corresponding solutions of (2.1)–(2.3) converge to M_{b_δ} with $b_\delta \to 0$ as $\delta \to 0$. A diagonal argument then shows that $M(s,t) \to 8\pi$ a.e. on $(0,1)$ as $t \to \infty$. However, the approach used in Proposition 3.2 provides an additional algebraic decay rate of the distance in $L^1(0,1)$ between $M(t)$ and 8π. Seemingly, this decay estimate is far from being optimal since formal computations performed in [20, Section 4.2] seem to indicate a temporal decay as $e^{-Ct^{1/2}}$ for large times.

For a restricted class of initial data, the pointwise convergence of $M(t)$ to 8π can be proved with the help of suitable subsolutions. In fact,

$$\underline{M}(s,t) = 8\pi \, \frac{1 + a/(t+T)}{s + a/(t+T)} \, s \tag{4.2}$$

is a subsolution of the equation (1.7) for each $a \geq 1/8$ and $T > 0$. Thus, if $M_{0,s}(0) > 8\pi$ (so that $M_0(s) > 8\pi s$ in a neighbourhood of 0), then one can find a and T such that $\underline{M}$ is a subsolution of the initial-boundary value problem (1.7)–(1.9). Since $\lim_{t\to\infty} \underline{M}(s,t) = 8\pi$ for each $s > 0$, we obtain asymptotics of M: $\lim_{t\to\infty} M(s,t) = 8\pi$ for each $s > 0$. Observe, however, that $|\underline{M}(t) - 8\pi|_1$ behaves as $\log{(t+T)}/(t+T)$ for large times and provides a weaker decay estimate.

Acknowledgements. The preparation of this paper was partially supported by the KBN grant 2/P03A/002/24, the POLONIUM project ÉGIDE–KBN 2003 –05643SE, and by the EU network HYKE under the contract HPRN–CT–2002–00282. We thank Nikos Kavallaris and Benoît Perthame for interesting remarks.

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Manuscript received March 31, 2005

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