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PACKING AND HAUSDORFF MEASURES OF CANTOR SETS ASSOCIATED WITH SERIES

Abstract

We study a generalization of Morán's sum sets, obtaining information about the h-Hausdorff and h-packing measures of these sets and certain of their subsets.

1 Introduction

In [10], Morán introduced the notion of a sum set,

$$C_a = \left\{ \sum_{i=1}^{\infty} \varepsilon_i a_i : \varepsilon_i = 0, 1 \right\};$$

the set of all possible subsums of the series $\sum a_n$, where $a = (a_n)$ is a sequence of vectors in $\mathbb{R}^p$ with summable norms. The classical Cantor middle-third set is

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one example with $a_i = 3^{-i}2$. Assuming a suitable separation condition, Morán [11] related the h-Hausdorff measure of C_a to the quantities $R_n = \sum_{i>n} \|a_i\|$.

In this paper, we generalize Morán's sum set notion to permit a greater diversity in the geometry; see (1) for the definition of the generalization. For example, our generalization includes Cantor-like sets in $\mathbb{R}$ which have the property that the Cantor intervals of a given level, but not necessarily the gaps, are all of the same length. Moreover, unlike Morán's sets, our generalized sum sets can have Hausdorff dimension greater than one.

We obtain the analogue of Morán's results on h-Hausdorff measures for these generalized sum sets and prove dual results for h-packing measures. We show that for any of these sum sets, there is a doubling dimension function h for which the sum set has both finite and positive h-Hausdorff and h-packing measure. We give formulas for the Hausdorff and packing dimensions and show that given any α less than the Hausdorff dimension (or β less than the packing dimension), there is a sum subset that has Hausdorff dimension α (or packing dimension β). In fact, there is even a sum subset with both Hausdorff dimension α and packing dimension β, provided α/β is dominated by the ratio of the Hausdorff dimension to the packing dimension of the original set. Furthermore, if the Hausdorff and/or packing measure is finite and positive (in the corresponding dimension), then we can choose this sum subset to have finite and positive Hausdorff and/or packing measure.

2 Preliminaries

Let $s_n > 0$ with $\sum_n s_n < \infty$. Fix $N \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let the *nth digit set* $\mathcal{D}^n = \{0 = d_1^n, d_2^n, \ldots, d_N^n\} \subset \mathbb{R}^p$ be given.

We define $C_{s,\mathcal{D}}$ by

$$C_{s,\mathcal{D}} = \left\{ \sum_{i=1}^{\infty} s_i b_i : b_i \in \mathcal{D}^i \right\}; \tag{1}$$

the set of all possible sums with choices drawn from $\mathcal{D}^n$ and scaled by s_n. Morán's sum set is the special case when $s_i = \|a_i\|$, $N = 2$ and $\mathcal{D}^i = \{0, a_i/\|a_i\|\}$. This generalized sum set is the main object of study in this paper.

For each n define

$$\kappa_n = \max\{\|d_i^n - d_j^n\| : 0 \le i, j \le N, i \ne j\}$$

and

$$\tau_n = \min\{\|d_i^n - d_j^n\| : 0 \le i, j \le N, i \ne j\}.$$

In Morán's case, $\kappa_n = \tau_n = 1$. We assume that $\kappa := \sup_n \kappa_n < \infty$, as well as $\tau := \inf \tau_n > 0$; the intent is that the sequence s_n controls the decay rate, not the (possibly varying) geometry of the digit sets $\mathcal{D}^n$. In addition, we assume the rapid decay condition

$$\sup_n \frac{\kappa R_n}{\tau s_n} = M < 1, \tag{2}$$

where $R_n = \sum_{i>n} s_i$. This is the analogue of Morán's separation condition. The quantity R_n is very important for describing the geometry of $C_{s,\mathcal{D}}$.

In certain situations where we have precise information about the geometry of $\mathcal{D}^n$, it is possible to assume something weaker than (2) and still have a suitable separation property to allow for dimensions to be calculated; see Example 8.

Example 1. *1. A very simple example is the classical Cantor set with $s_n = 2 \cdot 3^{-n}$ and $\mathcal{D} = \{0, 1\}$.*

2. Consider a finite set $\mathcal{D} \subset \mathbb{R}^p$, a real number $r < d/(2D)$ (where $d = \min \tilde{\mathcal{D}}$, $D = \max \tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}} = \{\|d-d'\| : d, d' \in \mathcal{D}, d \neq d'\}$), an orthogonal matrix $O \in \mathbb{R}^{p\times p}$ and the contractions $S_d(x) = rO(x+d)$. The attractor of this IFS is $C_{s,D}$ with $s_n = r^n$ and $\mathcal{D}^n = O^n\mathcal{D}$.

We now examine some basic properties of $C_{s,\mathcal{D}}$. First, we argue that $C_{s,\mathcal{D}}$ is a compact and perfect set. To do this, let $\Xi = \{1, 2, \ldots, N\}^{\mathbb{N}}$ with the product topology induced by the discrete topology on each factor. Further, for $n \in \mathbb{N}$, let $\Xi^n = \{1, \ldots, N\}^n$. We note that Ξ is a totally disconnected, perfect metric space. Define the function $\Phi : \Xi \to \mathbb{R}^p$ by

$$\Phi(\sigma) = \sum_i s_i d^i_{\sigma_i}.$$

Then the range of Φ is $C_{s,\mathcal{D}}$. Since Ξ is compact and perfect, we need only show that Φ is continuous and injective to show that $C_{s,\mathcal{D}}$ is compact and perfect. Let $\Phi_n : \Xi \to \mathbb{R}^p$ be defined by $\Phi_n(\sigma) = \sum_{i\leq n} s_i d^i_{\sigma_i}$. Then Φ_n is constant on each of the sets $\Xi_\alpha = \{\sigma \in \Xi : \sigma_i = \alpha_i, 1 \leq i \leq n\}$ for any fixed $\alpha \in \Xi^n$. This means that each Φ_n is continuous. Furthermore, $\|\Phi_n(\sigma) - \Phi(\sigma)\| \leq \kappa R_n$. Thus, $\Phi_n \to \Phi$ uniformly on Ξ, and so Φ is also continuous. Thus, $C_{s,\mathcal{D}}$ is compact.

If n is the first place where σ and σ' disagree, then

$$\begin{aligned}\|\Phi(\sigma)-\Phi(\sigma')\| &= \Big\|\sum_i s_i(d^i_{\sigma(i)}-d^i_{\sigma'(i)})\Big\| \\ &\geq \|s_n(d^n_{\sigma(n)}-d^n_{\sigma'(n)})\| - \Big\|\sum_{i>n} s_i(d^i_{\sigma(i)}-d^i_{\sigma'(i)})\Big\| \\ &\geq s_n\tau-\kappa R_n>0. \end{aligned} \tag{3}$$

This means that Φ is injective and is thus a homeomorphism, so that $C_{s,\mathcal{D}}$ is also totally disconnected and perfect.

For a given $n\in\mathbb{N}$ and $\sigma\in\Xi^n$, we define

$$x_\sigma=\sum_{i\leq n} s_i d^i_{\sigma_i}$$

and

$$C_{\sigma,n}=x_\sigma+\Big\{\sum_{i>n} s_i b_i : b_i\in\mathcal{D}^i\Big\}.$$

Our condition (2) ensures the non-overlapping of the sets $C_{\sigma,n}$.

Using this notation, we see two very important facts. First, $C_{\sigma,n}=x_\sigma-x_\alpha+C_{\alpha,n}$ for any $\sigma,\alpha\in\Xi^n$. That is, for a fixed n the collection of $C_{\sigma,n}$ are all translates of each other. Secondly, we can decompose $C_{s,\mathcal{D}}$ into N^n copies of $C_{\sigma,n}$ as

$$C_{s,\mathcal{D}}=\bigcup_{\sigma\in\Xi^n} C_{\sigma,n}=\{x_\sigma:\sigma\in\Xi^n\}+C_{1,n},$$

where by $1\in\Xi^n$ we mean the element all of whose terms are equal to 1.

An elementary estimate gives that

$$|C_{\sigma,n}|\leq\Big\|\sum_{i>n} s_i b_i-\sum_{i>n} s_i b_i'\Big\|\leq\kappa\sum_{i>n} s_i=\kappa R_n, \tag{4}$$

where $|C|$ means the diameter of the set C.

3 Hausdorff and packing measures

We first recall some facts about Hausdorff and packing measures; see [12, 9]. For us, a *dimension function* is a continuous, non-decreasing function $h:[0,\infty)\to[0,\infty)$ with $h(0)=0$. It is said to be *doubling* if there is some constant $c>0$ such that $h(2x)\leq c\,h(x)$ for all $x>0$.

For two dimension functions f and g, we say that $f \prec g$ if

$$\lim_{t \to 0+} g(t)/f(t) = 0.$$

For each $\delta > 0$, a δ-covering of a set E is a countable collection $\{B_i\}$ of subsets of $\mathbb{R}^p$ with diameters dominated by δ, that is $|B_i| \leq \delta$, and for which $E \subseteq \bigcup_i B_i$. We define

$$\mathcal{H}^h_\delta(E) = \inf \Big\{ \sum_i h(|B_i|) : \{B_i\} \text{ is a } \delta\text{-covering of } E \Big\}$$

and the *Hausdorff h-measure* as

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E).$$

Notice that in the definition of $\mathcal{H}^h_\delta$, it is sufficient to consider coverings by balls.

Now we turn to the h-packing measure $\mathcal{P}^h$. A δ-packing of a set E is a disjoint family of open balls $\{B(x_i, r_i)\}$ with $x_i \in E$ and $r_i \leq \delta$. The h-packing pre-measure is given by

$$\mathcal{P}^h_0(E) = \lim_{\delta \to 0} \mathcal{P}^h_\delta(E),$$

where

$$\mathcal{P}^h_\delta = \sup \Big\{ \sum_i h(|B_i|) : \{B_i\} \text{ is a } \delta\text{-packing of } E \Big\}.$$

Unfortunately, $\mathcal{P}^h_0$ is not a measure, as it is in general not countably additive. Thus, we need one more step to construct the packing measure $\mathcal{P}^h$, namely

$$\mathcal{P}^h(E) = \inf \Big\{ \sum_i \mathcal{P}^h_0(E_i) : E \subset \bigcup_i E_i \Big\}.$$

The next theorem gives estimates for the Hausdorff and packing measures of $C_{s,\mathcal{D}}$. The first two claims about the Hausdorff measure of $C_{s,\mathcal{D}}$ are given in [11] for the special case of $\mathcal{D}^n$ containing two digits.

Theorem 2. *Suppose that h is a doubling dimension function.*

1. *If* $\liminf N^n h(\kappa R_n) = \alpha$, *then* $\mathcal{H}^h(C_{s,\mathcal{D}}) \leq \alpha$.
2. *If* $\liminf N^n h(\kappa R_n) = \alpha > 0$, *then* $\mathcal{H}^h(C_{s,\mathcal{D}}) > 0$.
3. *If* $\limsup N^n h(\kappa R_n) = \alpha < \infty$, *then* $\mathcal{P}^h(C_{s,\mathcal{D}}) \leq N\alpha$.

4. If $\limsup N^n h(\kappa R_n) = \alpha > 0$*, then* $\mathcal{P}^h(C_{s,\mathcal{D}}) > 0$*.*

Remark 3. *If* h *is a doubling dimension function and* $0 < \liminf N^n h(R_n) < \infty$*, then* $0 < \mathcal{H}^h(C_{s,\mathcal{D}}) < \infty$*, and so* $C_{s,\mathcal{D}}$ *is an* h*-Hausdorff set. Similarly, if* $\limsup N^n h(R_n)$ *is positive and finite, then* $C_{s,\mathcal{D}}$ *is an* h*-packing set. Finally, if* $\liminf N^n h(R_n)$ *is positive and* $\limsup N^n h(R_n)$ *is finite, then* $C_{s,\mathcal{D}}$ *is both an* h*-Hausdorff and an* h*-packing set.*

Proof. The first item is trivial by considering the covering $C_{I,n}$ for all $I \in \{0, 1, \ldots, N\}^n$, which consists of N^n sets each of diameter at most κR_n.

To prove the rest of the statements, we will use the fact that there is a Borel measure μ supported on $C_{s,\mathcal{D}}$ for which $\mu(C_{\sigma,n}) = N^{-n}$ for each σ and n. This measure is often called the *natural probability measure*.

2. Let $\beta < \alpha$ so that we have $N^n h(\kappa R_n) > \beta$ for all large n. Now choose $x \in C_{s,\mathcal{D}}$ and $\delta > 0$, and let n be such that $\kappa R_n < \delta \leq \kappa R_{n-1}$. By a simple modification of Lemma 2 in [10], there is a $q \in \mathbb{N}$ such that the number of $C_{\sigma,n}$ which intersect $B(x,\delta)$ is less than q, independent of B and δ. (This is where the condition (2) is used.) But then we have

$$\mu(B(x,\delta)) \leq q\mu(C_{\sigma,n}) = q\, N^{-n} < \frac{q}{\beta} h(\kappa R_n) < \frac{qh(\delta)}{\beta}.$$

By the mass distribution principle, see [5], we have $\mathcal{H}^h(C_{s,\mathcal{D}}) \geq \alpha/q$.

3. Let $\beta > \alpha$ so that we have $N^n h(\kappa R_n) < \beta$ for all large n. Now choose $x \in C_{s,\mathcal{D}}$ and $\delta > 0$, and let n be such that $\kappa R_n < \delta \leq \kappa R_{n-1}$. We know that $x \in C_{\sigma,n}$ for some σ, and since $|C_{\sigma,n}| \leq \kappa R_n < \delta$, we have that $C_{\sigma,n} \subseteq B(x, \kappa R_n) \subseteq B(x,\delta)$. But then

$$\mu(B(x,\delta)) \geq \mu(C_{\sigma,n}) = N^{-n} = \frac{N^{-(n-1)}}{N} > \frac{h(\kappa R_{n-1})}{N\beta} \geq \frac{h(\delta)}{N\beta},$$

since h is a non-decreasing function. But then we have that

$$\liminf \mu(B(x,\delta))/h(\delta) \geq (N\alpha)^{-1},$$

and so $\mathcal{P}^h(C_{s,\mathcal{D}}) \leq N\alpha$ by Theorem 3.16 in [4].

4. Let $0 < \beta < \alpha$. Then there are n_j so that $N^{n_j} h(\kappa R_{n_j}) > \beta$ for all j. Let $x \in C_{s,\mathcal{D}}$ be given. For any j, we have $x \in C_{\sigma_j, n_j}$ for some σ_j. By the same simple modification of Lemma 2 in [10], there is a $q \in \mathbb{N}$ such that for any $\delta > 0$ and any ball B of radius δ, if $m \in \mathbb{N}$ is the smallest value with $\kappa R_m < \delta$, then the number of $C_{I,m}$ which intersect B is less than q, independent of B and δ. Let $\delta = \kappa R_{n_j - 1}$, so that $\kappa R_{n_j} < \delta = \kappa R_{n_j-1}$. Then

$$\mu(B(x, \kappa R_{n_j})) \leq \mu(B(x,\delta)) \leq q\,\mu(C_{\sigma_j,n_j}) = q\, N^{-n_j} < q\, h(\kappa R_{n_j})/\beta,$$

and thus, $\liminf \mu(B(x,\delta))/h(\delta) \leq q/\alpha$. By Theorem 3.16 in [4], it follows that $\mathcal{P}^h(C_{s,\mathcal{D}}) \geq c\alpha/q$, where c is the doubling constant for h. □

Remark 4. *Since $\kappa s_{n+1} < \kappa R_n \leq M\tau s_n < \kappa s_n$, for any doubling dimension function h, we could instead relate the two quantities, $\liminf N^n h(s_n)$ and $\limsup N^n h(s_n)$, to the h-Hausdorff and h-packing measure of $C_{s,\mathcal{D}}$.*

Theorem 5. *For any sequence s_n and collections of digits $\mathcal{D}^n$ which satisfy (2), there is a doubling dimension function h for which $C_{s,\mathcal{D}}$ is simultaneously an h-Hausdorff set and an h-packing set.*

PROOF. Following the pattern in [3, Section 5], we define the function $h : [0,\kappa R_0] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = 1/f^{-1}(x)$, where $f(x)$ is given by

$$f(x) = \kappa R_n + \frac{\kappa R_{n+1} - \kappa R_n}{N^{n+1} - N^n}(x - N^n), \quad x \in [N^n, N^{n+1}).$$

Clearly, h is non-decreasing and continuous, so we only need to show that h is doubling. For $x > 0$, let $n, m \in \mathbb{N}$ be such that $\kappa R_{m+1} < x \leq \kappa R_m < \kappa R_n \leq 2x < \kappa R_{n-1}$. Then $\kappa R_i \leq \tau M s_i \leq \kappa \frac{\tau M}{\kappa} R_{i-1}$ for all i. Letting $\theta = \tau M/\kappa < 1$,

$$\theta^{n-m} \leq \frac{\kappa R_n}{\kappa R_m} < \frac{2x}{x} = 2,$$

and so we have $m - n \leq -\ln(2)/\ln(\theta)$. As $f(N^j) = \kappa R_j$,

$$\frac{h(2x)}{h(x)} = \frac{f^{-1}(x)}{f^{-1}(2x)} \leq \frac{N^{m+1}}{N^{n-1}} \leq N^{2-\ln(2)/\ln(\theta)},$$

and so h is doubling.

Since $N^n h(\kappa R_n) = 1$ for all n, we have that $C_{s,\mathcal{D}}$ is an h-Hausdorff set and an h-packing set for this dimension function h, as desired. □

The next theorem is a simple consequence of some known results. However, it shows that the set of dimensional subsets of $C_{s,\mathcal{D}}$ is an initial segment in the partially ordered set of all doubling dimension functions.

Theorem 6. *Let f, h be doubling dimension functions and assume $f \prec h$.*

1. *If $0 < \mathcal{H}^h(C_{s,\mathcal{D}}) < \infty$, then for any $t > 0$, there is a compact and perfect subset $E \subset C_{s,\mathcal{D}}$ such that $\mathcal{H}^f(E) = t$.*

2. *If $0 < \mathcal{P}^h(C_{s,\mathcal{D}}) < \infty$, then for any $t > 0$, there is a compact and perfect subset $E \subset C_{s,\mathcal{D}}$ such that $\mathcal{P}^f(E) = t$.*

Proof. 1. From Theorem 40 in [12], we have that $\mathcal{H}^f(C_{s,\mathcal{D}}) = \infty$. Then by Theorem 2 in [8], there is some closed subset $E' \subset C_{s,\mathcal{D}}$ for which $\mathcal{H}^f(E') = t$. As E' is a closed subset of a perfect set, it is the union of a perfect set E and a countable set. So, $\mathcal{H}^f(E) = \mathcal{H}^f(E') = t$ and E is a perfect subset of $C_{s,\mathcal{D}}$.

2. By the same argument as Theorem 40 in [12], but adapted to packing measures, we have that $\mathcal{P}^f(C_{s,\mathcal{D}}) = \infty$. Now, if we obtain a closed subset $E' \subset C_{s,\mathcal{D}}$ for which $\mathcal{P}^f(E') = t$, then we find a perfect subset in a similar way to the case 1 before. In [7], Joyce and Preiss proved that if a set has infinite h-packing measure (for any given $h \in \mathcal{D}$), then the set contains a compact subset with finite h-packing measure. With a simple modification of their proof (in particular, their Lemma 6), we obtain a set of finite packing measure greater than t. By Lyapunov's convexity theorem, there is a subset whose h-packing measure is exactly t [13, Theorem 5.5]. $\square$

We now specialize to the "usual" dimension functions $h_s(x) = x^s$ and let $\dim_H$ and $\dim_P$ denote the "usual" Hausdorff and packing dimension. In analogy with the case of a "cut-out" Cantor subset of $\mathbb{R}$, see [1, 3, 6], we have the following proposition.

Proposition 7. *We have that*

$$\dim_H(C_{s,\mathcal{D}}) = \liminf \frac{-n\ln(N)}{\ln(s_n)} \quad \text{and} \quad \dim_P(C_{s,\mathcal{D}}) = \limsup \frac{-n\ln(N)}{\ln(s_n)}.$$

Proof. First, we note that

$$\liminf \frac{-n\ln(N)}{\ln(\kappa R_n)} = \liminf \frac{-n\ln(N)}{\ln(R_n)} = \liminf \frac{-n\ln(N)}{\ln(s_n)},$$

with a similar equality for the limit superior.

If $\beta > \alpha := \liminf \frac{-n\ln(N)}{\ln(\kappa R_n)}$, then there is a subsequence (n_j) such that $N^{n_j}(\kappa R_{n_j})^\beta < 1$. Thus, $\liminf N^n(\kappa R_n)^\beta < 1$ and so $\dim_H(C_{s,\mathcal{D}}) \le \alpha$ by Theorem 2.

Conversely, if $\gamma < \alpha$, then for large n we have $N^n(R_n)^\gamma > 1$. Thus, $\liminf N^n(R_n)^\gamma > 1$ and so $\dim_H(C_{s,\mathcal{D}}) \ge \alpha$ by Theorem 2.

The proof for packing dimension is similar. $\square$

Example 8. *For any $\alpha \in [0,p)$, it is possible to construct a sum set $C_{s,\mathcal{D}} \subset \mathbb{R}^p$ with $\dim_H(C_{s,\mathcal{D}}) = \alpha$. The simplest way of doing this is to choose $\mathcal{D}^n = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_p) : \epsilon_i \in \{0,1\}\}$, the set of all corners of a p-dimensional unit cube, and set $s_n = \lambda^n$ where $\lambda = 2^{-p/\alpha}$. This will generate a self-similar set $C_{s,\mathcal{D}}$ that is a product of classical Cantor sets. The problem is that condition*

(2) *requires that* $\lambda < 1/(1+\sqrt{p})$, *which does not allow the full range of dimensions (and, in fact, gets worse as p increases). However, from the simple geometry of this example, we can see that the sets* $C_{\sigma,n}$ *are non-overlapping, provided* $s_n > R_n$. *Under this (weaker) assumption,* $C_{s,\mathcal{D}}$ *is a self-similar set satisfying the open set condition, and hence its dimensions are as stated in the previous proposition. This separation condition allows for any* $\lambda \in [0, 1/2)$.

In the case of the "usual" dimension functions h_s, Theorem 6 has a stronger form in that, not only is there a Cantor subset with the correct dimension, but this subset corresponds to all the subsums of a subsequence of (s_n).

Theorem 9. *Suppose that* $\dim_H(C_{s,\mathcal{D}}) = A$ *and* $dim_P(C_{s,\mathcal{D}}) = B$. *Then for any* $0 \le \alpha \le A$ *and* $0 \le \beta \le B$, *with* $\alpha/A \le \beta/B$, *there is a subsequence* (t_n) *of* (s_n) *such that* $\dim_H(C_{t,\mathcal{D}}) = \alpha$ *and* $\dim_P(C_{t,\mathcal{D}}) = \beta$.

PROOF. We will assume $0 < \alpha < A$, $0 < \beta < B$ and leave the details of the endpoint cases for the reader. Choose n_i and m_i to be disjoint sequences of indices such that

$$\lim_i \frac{-n_i \ln(N)}{\ln(s_{n_i})} = A \quad \text{and} \quad \lim_i \frac{-m_i \ln(N)}{\ln(s_{m_i})} = B.$$

If necessary, we take subsequences in order to assure that $n_1 \ge 100$, $m_i \ge 2^{n_i}$, and $n_{i+1} \ge 2^{m_i}$. To obtain the new sequence t_k, we remove terms from s_n in segments, each in a "uniform" manner with some density $\xi \in (0,1)$. To explain, suppose the segment is the set of indices $\{q, q+1, \ldots, \ell\} \subset \mathbb{N}$. Then to uniformly remove terms with density ξ from this segment, we remove all the terms of the form $q + \lfloor i/\xi \rfloor$ for $i = 0, \ldots, \lfloor \xi(\ell - q) - 1 \rfloor$ (to make sure we do not remove ℓ). Note that removing with density ξ is the same as retaining with density $1 - \xi$.

From the set of indices $\{1, 2, \ldots, n_1\}$, we remove terms in a "uniform" way with density $1 - \frac{\alpha}{A}$. Then from the set of indices $\{n_1 + 1, \ldots, m_1\}$, we remove terms in a "uniform" way with density $1 - \frac{\beta}{B}$. We continue alternating, removing terms with density $1 - \frac{\alpha}{A}$ from $\{m_i + 1, \ldots, n_{i+1}\}$ and with density $1 - \frac{\beta}{B}$ from $\{n_i + 1, \ldots, m_i\}$. Call the resulting sequence t_ℓ, where we have $t_\ell = s_n$ with $\ell = n\Theta(n)$, where $\Theta : \mathbb{N} \to [\alpha/A, \beta/B]$ is a measure of the "local scaling" of the index. From the construction, we have $\Theta(n_j) \approx \alpha/A$, $\Theta(m_j) \approx \beta/B$, Θ is increasing on $\{n_i + 1, \ldots, m_i\}$ and decreasing on $\{m_i + 1, \ldots, n_{i+1}\}$. Further,

$$\frac{-\ell \ln(N)}{\ln(t_\ell)} = \theta(n) \frac{-n \ln(N)}{\ln(s_n)}.$$

From here it is straightforward to show that $\liminf \frac{-\ell \ln(N)}{\ln(t_\ell)} = \alpha$ and also that $\limsup \frac{-\ell \ln(N)}{\ln(t_\ell)} = \beta$, as desired. The condition $\alpha/A \le \beta/B$ is used to check the new liminf and limsup. Since the original sequence satisfies condition (2), it is easy to see that any subsequence will as well. □

Of course, this construction does not guarantee that $C_{t,\mathcal{D}}$ will satisfy $0 < \mathcal{H}^{\alpha}(C_{t,\mathcal{D}}) < \infty$ even if it has the proper dimension. Comparing Theorem 6 with Theorem 9, we trade the ability to specify the $\mathcal{H}^t$-measure of the subset with the ability to ensure that the subset is of a particularly nice form, in Theorem 9 being the full set of subsums of some subsequence. However, if we assume a bit more on $C_{t,\mathcal{D}}$, then we can obtain a substantially stronger result.

Theorem 10.

1. *Suppose that $0 < \mathcal{H}^A(C_{s,\mathcal{D}}) < \infty$. Then for any $0 \le a \le A$, there is a subsequence (t_n) of (s_n) such that $0 < \mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$.*

2. *Suppose that $0 < \mathcal{P}^B(C_{s,\mathcal{D}}) < \infty$. Then for any $0 \le b \le B$, there is a subsequence (t_n) of (s_n) such that $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$.*

3. *Suppose that $0 < \mathcal{H}^A(C_{s,\mathcal{D}}) < \infty$ and $0 < \mathcal{P}^B(C_{s,\mathcal{D}}) < \infty$. Then for any $0 \le a \le A$ and $0 \le b \le B$ with $a/A \le b/B$, there is a subsequence (t_n) of (s_n) such that $0 < \mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$ and $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$.*

Proof. We prove the third statement as it is the most involved. The other two are similar. As in Theorem 9, we work with s_n rather than R_n.

Let m_i, n_i be natural numbers such that $\lim N^{m_j} s_{m_j}^B = \limsup N^n s_n^B = S$ and $\lim N^{n_j} s_{n_j}^A = \liminf N^n s_n^A = I$. In addition, we assume that $n_j < m_j < n_{j+1} < m_{j+1}$, $m_j/n_j \to \infty$, and $n_{j+1}/m_j \to \infty$. The two cases $a/A = b/B$ and $a/A < b/B$ require different techniques and so we do them separately.

Case 1: $a/A = b/B$

If $a/A = 1$, then there is nothing to prove. We define our subsequence (t_n) by defining the indexing function $\pi : \mathbb{N} \to \mathbb{N}$ such that $t_n = s_{\pi(n)}$. Define $\hat{\pi} : \mathbb{N} \to \mathbb{N}$ by $\hat{\pi}(i) = \lfloor (A/a)i \rfloor$. If $m_j, n_j \in \hat{\pi}(\mathbb{N})$ for all j, then we let $\pi = \hat{\pi}$. Otherwise, suppose that $m_j \notin \hat{\pi}(\mathbb{N})$. Then $i := \lfloor m_j a/A \rfloor < m_j a/A$, so we define $\pi(i) = m_j$. We do the same procedure for any $n_j \notin \hat{\pi}(\mathbb{N})$. Since $A/a > 1$, we know that $\hat{\pi}$ is injective. If we assume that $|n_j - m_k| > 2A/a$ for all j and k, then π is also guaranteed to be injective. Since $[k, k+A/a+1] \cap \hat{\pi}(\mathbb{N})$ is nonempty for any k, we know that $-1 \le \pi(i) - (A/a)i \le A/a + 1$ or, more useful for us,

$$-1 - \frac{a}{A} + \pi(i)\left(\frac{a}{A}\right) \le i \le \frac{a}{A} + \pi(i)\left(\frac{a}{A}\right).$$

This means that

$$\begin{aligned} N^i t_i^a &= N^i s_{\pi(i)}^a \geq N^{-1-a/A} N^{\pi(i)(a/A)} s_{\pi(i)}^{(a/A)A} = N^{-1-a/A} \left(N^{\pi(i)} s_{\pi(i)}^A \right)^{a/A} \\ &\geq N^{-1-a/A} (I-\epsilon)^{a/A} > 0 \end{aligned}$$

for large enough i. Thus, $\liminf N^i t_i^a > 0$ and so $\mathcal{H}^a(C_{t,\mathcal{D}}) > 0$. By construction, there is a sequence $q_j \in \mathbb{N}$ such that $\pi(q_j) = n_j$, and so

$$\begin{aligned} N^{q_j} t_{q_j}^a &\leq N^{a/A} N^{\pi(q_j)(a/A)} s_{n_j}^{(a/A)A} = N^{a/A} \left(N^{n_j} s_{n_j}^A \right)^{a/A} \\ &\leq N^{a/A} (I+\epsilon)^{a/A} < \infty. \end{aligned}$$

Thus, $\mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$ as well. The proof that $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$ is similar.
Case 2: $a/A < b/B$

Let

$$\gamma_0 = \frac{\frac{B}{b} - 1}{\frac{A}{a} - 1}$$

and then choose $\delta > 0$ such that $\gamma := \gamma_0 + \delta < 1$. Define

$$n'_j = \left\lfloor \frac{n_j}{\gamma} \right\rfloor \quad \text{and} \quad m'_j = \lfloor \gamma m_j \rfloor,$$

and notice that $n'_j > n_j$ and $m'_j < m_j$. For notational ease, let

$$\begin{aligned} P_j &= m_j - \left\lceil (1 - \frac{b}{B}) m_j \right\rceil, & P'_j &= m'_j - \left\lceil (1 - \frac{b}{B}) m'_j \right\rceil, \\ Q_j &= n_j - \left\lfloor (1 - \frac{a}{A}) n_j \right\rfloor, \quad \text{and} & Q'_j &= n'_j - \left\lfloor (1 - \frac{a}{A}) n'_j \right\rfloor. \end{aligned}$$

Further, let

$$d_j = \frac{P'_j + \lceil (1 - \frac{b}{B}) m_j \rceil - (Q'_j + \lfloor (1 - \frac{a}{A}) n_j \rfloor)}{P'_j - Q'_j}$$

and

$$e_j = \frac{Q_{j+1} + \lfloor (1 - \frac{a}{A}) n_{j+1} \rfloor - (P_j + \lceil (1 - \frac{b}{B}) m_j \rceil)}{Q_{j+1} - P_j}.$$

Define $\pi : \mathbb{N} \to \mathbb{N}$ by

$$\pi(i) = \begin{cases} i + \lfloor (1 - \frac{a}{A}) n_j \rfloor & \text{if } Q_j \leq i < Q'_j \\ i + \lceil (1 - \frac{b}{B}) m_j \rceil & \text{if } P'_j < i \leq P_j \\ Q'_j + \lfloor (1 - \frac{a}{A}) n_j \rfloor + \lfloor k d_j \rfloor & \text{if } i = Q'_j + k, k = 0, \ldots, P'_j - Q'_j \\ P_j + \lceil (1 - \frac{b}{B}) m_j \rceil + \lceil k e_j \rceil & \text{if } i = P_j + k, k = 1, \ldots, Q_{j+1} - P_j - 1. \end{cases}$$

We define $t_i = s_{\pi(i)}$. The choice of n_j, m_j and γ ensure that $Q_j < Q_j' < P_j' < P_j < Q_{j+1}$. We also have $\pi(Q_j) = n_j$ and $\pi(P_j) = m_j$. It is straightforward, but quite tedious, to check that π is injective and also that

$$\frac{B}{b} \le \frac{\pi(i)}{i} \le \frac{A}{a} \tag{5}$$

for all large i. We remark that the strict inequality $a/A < b/B$ is necessary in order to show (5) for the last two cases in the definition of π.

Thus, for $\epsilon > 0$ small and all large enough i, we have

$$N^i t_i^a = N^i s_{\pi(i)}^a \ge N^{\pi(i)(a/A)} s_{\pi(i)}^{(a/A)A} = \left(N^{\pi(i)} s_{\pi(i)}^A\right)^{a/A} \ge (I-\epsilon)^{a/A} > 0,$$

and thus $\mathcal{H}^a(C_{t,\mathcal{D}}) > 0$. For $i = Q_j$, we have $\pi(i) = n_j$ and $Q_j \le (a/A)n_j + 1$ and so

$$N^i t_i^a = N^{Q_j} s_{n_j}^a \le N^{(a/A)n_j} s_{n_j}^a N = N\left(N^{n_j} s_{n_j}^A\right)^{a/A} \le N(I+\epsilon)^{a/A} < \infty.$$

Hence, $\mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$. The argument that $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$ is similar. □

Example 11. *The simple Example 1 will show that in general we cannot find a subsequence which will give a subset of arbitrary measure. Recall that it was* $s_n = 2/3^n$ *and* $\mathcal{D}^n = \{0,1\}$ *for all* n. *It is known that* $\dim_H C_{s,\mathcal{D}} = d = \ln(2)/\ln(3)$ *and* $\mathcal{H}^d(C_{s,\mathcal{D}}) = 1$. *The key observation is that if* t_n *is a subsequence of* s_n *constructed by removing only* K *terms from* s_n, *then* $\mathcal{H}^d(C_{t,\mathcal{D}}) = 2^{-K}$, *since* $C_{s,\mathcal{D}}$ *is the union of* 2^K *disjoint copies of* $C_{t,\mathcal{D}}$ *(these copies correspond to the possible subsums of the removed terms). But this means that it is impossible to find a subsequence* t_n *with* $\mathcal{H}^d(C_{t,\mathcal{D}}) = 1/3$.

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