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ON THE ALEXIEWICZ TOPOLOGY OF THE DENJOY SPACE

Abstract

The paper deals with the space of all Denjoy-Perron integrable functions on a fixed interval endowed with the Alexiewicz norm and the completion of this space. The relatively weakly compact subsets of each space are characterized.

Let H be the space of all Denjoy-Perron integrable functions on $[a,b]$. If H is endowed with the Alexiewicz norm

$$\| f \|_H = \sup_x \left| \int_a^x f(t)\, dt \right|,$$

then it is called the *Denjoy space* of $[a,b]$.

The Banach dual of H is isomorphic to the space BV of all functions of bounded variation on $[a,b]$. (See [2].) and the completion $\mathcal{H}$ of H is isomorphic to the space of all distributions each of which is the distributional derivative of a continuous function. (See [3] or Theorem 6(i) below.)

A characterization of relatively weakly compact subsets of H and $\mathcal{H}$ is given in [3]. The aim of the present paper is to complete the study begun in [3] and obtain several new characterizations of these sets.

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1 Relatively Weakly Compact Subsets of $\mathcal{C}(S)$

In this section we obtain a characterization of relatively weakly compact subsets of $\mathcal{C}(S)$, the Banach space of all real valued continuous functions on a compact metric space S. To this end we first prove the following result.

Theorem 1 *Let (X,d) and (Y,d') be metric spaces and suppose that (X,d) is complete. Given a sequence $f_n : X \to Y$, $n = 1,2,\dots,$ of continuous functions, converging pointwise on X to some function f, the following assertions hold.*

(i) *(f_n) is equicontinuous on a set D dense in X.*

(ii) *If (f_n) is equicontinuous on X, then f is uniformly continuous in X.*

(iii) *If X is compact and (f_n) is equicontinuous on X, then $f_n \to f$ uniformly.*

The proof is based on the following lemma.

Lemma 2 *Let X, Y and (f_n) satisfy the conditions of Theorem 1. Then given a closed ball $\overline{B}(x_o,r) = \{x \in X : d(x,x_o) \le r\}$ in X and given $\varepsilon > 0$, there exists $w_o \in X$ and $0 < \delta < 2^{-1}r$ such that $\overline{B}(w_o,\delta) \subset B(x_o,r)$ and $d'(f_n(x), f_n(w_o)) < \varepsilon$ for each $x \in \overline{B}(w_o,\delta)$ and for each n.*

Proof. Let $0 < r' < r$. Consider the closed sets

$$X_n = \left\{ x \in \overline{B}(x_o,r') : d'(f_h(x), f_k(x)) \le \frac{\varepsilon}{3}, \ \text{ for each } h,k \ge n \right\}.$$

It is clear that $\cup_{n=1}^{\infty} X_n = \overline{B}(x_o,r')$ and hence by Baire's theorem, there exists n_o such that X_{n_o} contains a closed ball $\overline{B}(w_o,\eta)$. Let $k > n_o$. Since the functions f_n, $n = 1,2,\dots,k$, are continuous, there exists $0 < \delta < \min(\eta, 2^{-1}r)$ such that

$$d'(f_n(x), f_n(w_o)) < \frac{\varepsilon}{3}, \ \text{for each } x \in \overline{B}(w_o,\delta) \text{ and for } n = 1,2,\dots,k.$$

If $n > k$ and $x \in \overline{B}(w_o,\delta)$, then $x \in \overline{B}(w_o,\eta)$ and hence from the definition of X_{n_o} we have

$$\begin{aligned} d'(f_n(x), f_n(w_o)) &\le d'(f_n(x), f_k(x)) + d'(f_k(x), f_k(w_o)) \\ &\quad + d'(f_k(w_o), f_n(w_o)) < \varepsilon. \end{aligned}$$ □

Proof of Theorem 1. (i) Let $U = B(x_o,r)$ be an arbitrary ball in X. Then by Lemma 2 we can find a decreasing sequence $\overline{B}(w_n,r_n)$ of closed balls with $2r_n < r_{n-1}$, $n = 1,2,\dots$, where $0 < r_o < r$, such that

$$d'(f_k(x), f_k(w_n)) < \frac{1}{n}, \ \text{for all } k \in \mathbb{N} \text{ and for all } x \in \overline{B}(w_n,r_n).$$

Since X is complete, by Cantor's theorem there exists $w_o \in X$ such that

$$\bigcap_{n=1}^{\infty} \overline{B}(w_n, r_n) = \{w_o\}.$$

Given $\varepsilon > 0$, choose n_0 such that $2 < \varepsilon\, n_0$. Since

$$\{\omega_0\} = \bigcap_{n=1}^{\infty} \overline{B}(\omega_n, r_n) \supset \bigcap_{n=1}^{\infty} B(\omega_n, r_n) \supset \bigcap_{n=1}^{\infty} \overline{B}(\omega_{n+1}, r_{n+1}) = \{\omega_0\},$$

it follows that $\{\omega_0\} = \bigcap_{n=1}^{\infty} B(\omega_n, r_n)$. Then, for $x \in B(\omega_{n_0}, r_{n_0})$, we have

$$\begin{aligned} d'(f_k(x), f_k(w_0)) &\le d'(f_k(x), f_k(w_{n_0})) + d'(f_k(w_{n_0}), f_k(w_0)) \\ &< \frac{1}{n_0} + \frac{1}{n_0} < \varepsilon\,, \end{aligned}$$

for $k = 1, 2, \ldots$. Since $\omega_0 \in B(\omega_{n_0}, r_{n_0})$, there exists $\eta > 0$ such that $B(\omega_0, \eta) \subset B(\omega_{n_0}, r_{n_0})$ and hence the sequence (f_k) is equicontinuous in w_0. Thus (i) holds.

(ii) Obvious.

(iii) This follows from Lemma 29 in the proof of Ascoli's Theorem on pages 154–155 of [7]. □

Definition 3 *A sequence $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$, of continuous functions on a metric space (X, d) is said to be asymptotically continuous on X if, given $\varepsilon > 0$, there exists $\eta > 0$ such that $\overline{\lim}_n |f_n(x') - f_n(x'')| < \varepsilon$ for $x', x'' \in X$ with $d(x', x'') < \eta$.*

Theorem 4 *Let (X, d) be a separable complete metric space and let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$, be a sequence of continuous functions. Then (f_n) has a pointwise convergent subsequence with its limit f uniformly continuous in X if and only if there exists a subsequence (f_{n_k}) of (f_n) such that*

(i) *(f_{n_k}) is equicontinuous on a dense set D in X,*

(ii) *(f_{n_k}) is asymptotically continuous on X, and*

(iii) *$\{f_{n_k}(x) : k = 1, 2, \ldots\}$ is bounded for each $x \in X$.*

Proof. Suppose (f_n) has a pointwise convergent subsequence (f_{n_k}) with limit f uniformly continuous in X. Then by Theorem 1(i), (f_{n_k}) is equicontinuous on a dense set D. Moreover, as f is uniformly continuous in X, clearly (ii) holds. (iii) is obvious.

Conversely, suppose there exists a subsequence (f_{n_k}) such that conditions (i), (ii) and (iii) hold. By (i) and (iii) and by the version of Ascoli's theorem on page 155 of [7], there exists a subsequence (g_l) of (f_{n_k}) such that (g_l) converges pointwise in D to a function g continuous on D. Then the hypothesis (ii) implies that g is uniformly continuous in D and hence has a unique uniformly continuous extension to X. Let us denote this extension also by g.

Let $\varepsilon > 0$. By the uniform continuity of g in X, there exists $\eta > 0$ such that

$$|g(x') - g(x'')| < \frac{\varepsilon}{3} \tag{1}$$

for $x', x'' \in X$ with $d(x', x'') < \eta$. Moreover, choosing η sufficiently small, by (ii) there exists $l_o(\varepsilon)$ such that

$$|g_l(x') - g_l(x'')| < \frac{\varepsilon}{3} \quad \text{for } l \geq l_o(\varepsilon) \tag{2}$$

and for $x', x'' \in X$ with $d(x', x'') < \eta$. Now let $x \in X \setminus D$. Since D is dense in X, there exists $y \in D$ such that $d(x, y) < \eta$. Then by (2) and (1) we have

$$\begin{aligned} |g_l(x) - g(x)| &\leq |g_l(x) - g_l(y)| + |g_l(y) - g(y)| + |g(y) - g(x)| \\ &< \frac{\varepsilon}{3} + |g_l(y) - g(y)| + \frac{\varepsilon}{3} \end{aligned}$$

for $l \geq l_o(\varepsilon)$. Since $g_l(y) \to g(y)$, we can choose $l_1 > l_o(\varepsilon)$ such that $|g_l(y) - g(y)| < \frac{\varepsilon}{3}$ for $l \geq l_1$. Thus $|g_l(x) - g(x)| < \varepsilon$ for $l \geq l_1$ and hence $g_l(x) \to g(x)$. □

As a simple application of the above theorem and the Eberlein-Šmulian theorem we can give the following characterization of relatively weakly compact sets in $\mathcal{C}(S)$.

Theorem 5 *Let S be a compact metric space. Then a subset K of C(S) is relatively weakly compact if and only if K is bounded and each sequence (f_n) in K has a subsequence (f_{n_k}) which is equicontinuous on a dense set D and asymptotically continuous on S.*

Proof. By the Eberlein-Šmulian theorem, K is relatively weakly compact if and only if each sequence (f_n) in K has a subsequence which converges weakly to an element of $\mathcal{C}(S)$. By Corollary IV.6.4 of [5], a sequence (g_n) in $\mathcal{C}(S)$ converges weakly if and only if it is bounded and converges pointwise to a continuous function in S. Then the present theorem is an immediate consequence of Theorem 4. □

2 Distributional Derivatives of Functions in $\mathcal{C}[a, b]$

In this section we show how each $\mathbf{h} \in \mathcal{H}$ can be identified with the distributional derivative D_F of a continuous function $F \in \mathcal{C}[a, b]$. Theorem 6 given below plays a key role in the development of the subsequent sections. All the elements of the Banach space $\mathcal{H}$ will be denoted in boldface.

Let $\Omega = \{F \in \mathcal{C}[a, b] : F(a) = 0\}$. Ω is a Banach space with the sup-norm and the space AC_o of absolutely continuous functions F with $F(a) = 0$ is dense in Ω. Now for each $h \in H$, let $\Phi_o(h)$ be the Denjoy-Perron primitive of h with $\Phi_0(h)(a) = 0$. Since $\Phi_o(h)$ is an ACG_* function taking value zero in a, Φ_o is a linear isometry from H onto a dense subset of Ω. Then Φ_o has a unique isometric linear extension Φ from $\mathcal{H}$ onto Ω. (See Theorem 6 below.) Given a continuous function F we denote its distributional derivative by D_F and, when F is differentiable, its derivative by F'.

Theorem 6 *The following assertions hold.*

(i) $\mathbf{h} \in \mathcal{H}$ *if and only if* $\mathbf{h} = D_F$ *for some* $F \in \mathcal{C}[a, b]$.[1]

(ii) *For each* $\mathbf{h} \in \mathcal{H}$ *there exists a unique* $F \in \Omega$ *such that* $\mathbf{h} = D_F$.

(iii) *The mapping* $\Phi : \mathcal{H} \to \Omega$ *given by* $\Phi(\mathbf{h}) = F$ *if* $D_F = \mathbf{h}$ *and* $F \in \Omega$ *is well defined and is an onto linear isometry extending* Φ_o.

Thus the unique isometric linear extension of Φ_o *to* $\mathcal{H}$ *is precisely the map* Φ *given above.*

Proof. (i) Given $\mathbf{h} \in \mathcal{H}$, let (h_n) be a sequence of Denjoy-Perron integrable functions converging to $\mathbf{h}$ in the Alexiewicz norm. Let $F_n = \Phi_o(h_n)$. Since $||F_n - F_m||_\infty = ||h_n - h_m||_H \to 0$, the sequence (F_n) is uniformly convergent to a continuous function F. Let ϕ be an infinitely differentiable function with compact support contained in (a, b). Since $\phi \in BV$, $\phi \in H^* = \mathcal{H}^*$ (the dual of $\mathcal{H}$). (See [2].) Then, using the integration by parts formula we have

$$\begin{aligned} \langle \phi, \mathbf{h} \rangle &= \lim_n \langle \phi, h_n \rangle = \lim_n \int_a^b h_n \phi \, dt = \lim_n [\phi F_n]_a^b - \lim_n \int_a^b F_n \phi' \, dt \\ &= - \int_a^b F \phi' \, dt = D_F(\phi) \ . \end{aligned}$$

Thus $\mathbf{h} = D_F$. (See p. 35 of [8].) This shows that each $\mathbf{h} \in \mathcal{H}$ is the distributional derivative D_F of some $F \in \Omega$, as $F(a) = 0$.

[1] This assertion has already been established in [2] and we give it here for the sake of completeness.

Conversely, let $F \in \mathcal{C}[a,b]$. There exists a sequence of absolutely continuous functions F_n which converges uniformly to F. Then F_n is the Denjoy-Perron primitive of some $h_n \in H$ for each n. Thus

$$||h_n - h_m||_H = ||F_n - F_m||_\infty \to 0 \quad \text{as } n, m \to \infty$$

and hence there is some $\mathbf{h} \in \mathcal{H}$ such that $h_n \to \mathbf{h}$. Now, for each infinitely differentiable function ϕ with compact support contained in (a,b), we have

$$\begin{aligned} D_F(\phi) &= -\int_a^b F\phi'\,dt = -\lim_n \int_a^b F_n\phi'\,dt = \lim_n [F_n\phi]_a^b - \lim_n \int_a^b F_n\phi'\,dt \\ &= \lim_n \int_a^b h_n\phi\,dt = \langle \phi, \mathbf{h} \rangle\,. \end{aligned}$$

Thus $\mathbf{h} = D_F$.

(ii) The existence follows immediately by (i) and the unicity by Theorem 1 on p. 52 of [8].

(iii) Clearly Φ is well defined and linear from $\mathcal{H}$ into Ω. If $F \in \Omega$, taking the Perron-Denjoy primitives F_n in the proof of the converse part of (i) such that $F_n(a) = 0$, it follows that $F = \Phi(\mathbf{h})$ and hence Φ is onto. Now let $\mathbf{h} \in \mathcal{H}$. Then $\Phi(\mathbf{h}) = \lim_n \Phi_o(h_n)$, where $(h_n) \subset H$ and $h_n \to \mathbf{h}$. Hence

$$||\Phi(\mathbf{h})||_\infty = \lim_n ||\Phi_o(h_n)||_\infty = \lim_n ||h_n||_H = ||\mathbf{h}||_H.$$

Thus Φ is an isometry. Clearly, $\Phi\,|_H = \Phi_o$. □

The uniqueness of $\Phi(\mathbf{h})$ for $\mathbf{h} \in \mathcal{H}$ justifies the following definition.

Definition 7 *For $\mathbf{h} \in \mathcal{H}$, $\Phi(\mathbf{h})$ is called the primitive of $\mathbf{h}$ and we write*

$$\Phi(\mathbf{h}) = \int_a^x \mathbf{h}.$$

As $\Phi_o(h) = \Phi(h)$ for $h \in H$, the primitive and integral are in the sense of Denjoy-Perron if $h \in H$.

3 Relatively Compact Subsets of $\mathcal{H}$ and H

Making use of the isometric isomorphism Φ of Theorem 6, in this section we give some characterizations for a subset K of $\mathcal{H}$ to be relatively compact in $\mathcal{H}$ (resp. in H).

Theorem 8 *Let $K \subset \mathcal{H}$. The following assertions are equivalent.*

(i) *K is relatively compact in $\mathcal{H}$.*

(ii) *The primitives of K are equicontinuous.*

(iii) *Each sequence (h_n) in K contains a subsequence whose primitives are equicontinuous.*

(iv) *Each sequence (h_n) in K contains a subsequence whose primitives are uniformly convergent in $[a,b]$.*

(v) *Each sequence (h_n) in K contains a subsequence whose primitives are equicontinuous and uniformly convergent in $[a,b]$.*

Proof. Since Φ is an isometry, K is relatively compact in $\mathcal{H}$ if and only if $\Phi(K)$ is relatively compact in $\mathcal{C}[a,b]$. Moreover, by the compactness of $[a,b]$, any equicontinuous family of primitives is necessarily uniformly bounded. With this observation the theorem is immediate from Arzela-Ascoli's theorem. (See [6].) □

To characterize relatively compact sets in H we need the following definitions.

Definition 9 *A sequence (F_n) in $\mathcal{C}[a,b]$ is called asymptotically-AC_* on a set $E \subset [a,b]$ if, for each $\varepsilon > 0$, there exists a constant $\eta > 0$ such that*

$$\overline{\lim}_n \sum_{i=1}^{s} \omega(F_n, [x_i', x_i'']) < \varepsilon,$$

for each partition $\{[x_i', x_i'']; i = 1, 2, \ldots, s\}$ in $[a,b]$ with $x_i', x_i'' \in E$ and with $\sum_{i=1}^{s} |x_i' - x_i''| < \eta$.

Definition 10 *A sequence (F_n) in $\mathcal{C}[a,b]$ is called asymptotically-ACG_* on $[a,b]$ if $[a,b] = \cup_k E_k$, where (E_k) is a sequence of closed sets, and the sequence (F_n) is asymptotically-AC_* on each E_k.*

Theorem 11 *Let K be a subset of H. Then the following are equivalent:*

(i) *K is relatively compact in H (or equivalently, K is relatively compact in $\mathcal{H}$ and $\overline{K} \subset H$).*

(ii) *Given a sequence (h_n) in K, there exists a subsequence (h_{n_k}) of (h_n) such that the primitives of (h_{n_k}) converge uniformly to a function F which is ACG_* on $[a,b]$.*

(iii) *Given a sequence (h_n) in K, there exists a subsequence (h_{n_k}) of (h_n) such that the primitives of (h_{n_k}) are equicontinuous and asymptotically-ACG_*.*

Proof. Since Φ is an isometry and $\Phi(h)$ is ACG_* if and only if $h \in H$, the equivalence of (i) and (ii) holds.

(i) $\Rightarrow$ (iii) Let (h_n) be a sequence in K. By (i) and Theorem 8(v) we can choose a subsequence (h_{n_k}) of (h_n) such that their primitives (F_{n_k}) are equicontinuous and uniformly convergent to a continuous function F. Then $F(a) = 0$. If $\mathbf{h} = \Phi^{-1}(F)$, then $h_{n_k} \to \mathbf{h}$ and hence by (i), $\mathbf{h} \in H$. Consequently, F is ACG_*. Therefore there exists a sequence of closed sets (X_l) such that $[a,b] = \cup_{l=1}^{\infty} X_l$ and F is AC_* on each X_l. Thus, given $l \in \mathbf{N}$ and $\varepsilon > 0$, there exists a constant $\eta > 0$ such that $\sum_{i=1}^{s} \omega(F, [x_i', x_i'']) < \frac{\varepsilon}{3}$, for every partition $\{[x_i', x_i'']; i = 1, 2, \ldots, s\}$ in $[a,b]$ with $x_i', x_i'' \in X_l$ and with $\sum_{i=1}^{s} |x_i' - x_i''| < \eta$. Now, choose k_o such that $||F_{n_k} - F||_\infty < \frac{\varepsilon}{3s}$ for $n_k \geq n_{k_o}$. Then, for such n_k and for $x_i, y_i \in [x_i', x_i'']$ we have

$$\begin{aligned}\sum_{i=1}^{s} |F_{n_k}(x_i) - F_{n_k}(y_i)| &\leq 2\sum_{1}^{s} ||F_{n_k} - F||_\infty + \sum_{1}^{s} |F(x_i) - F(y_i)| \\ &< \frac{2}{3}\varepsilon + \sum_{1}^{s} \omega(F, [x_i', x_i'']) < \varepsilon.\end{aligned}$$

Consequently, $\sum_{i=1}^{s} \omega(F_{n_k}, [x_i', x_i'']) < \varepsilon$ for all $n_k \geq n_{k_o}$. Therefore, the sequence (F_{n_k}) is asymptotically-ACG_* and hence (iii) holds.

(iii) $\Rightarrow$ (i) By Theorem 8, (iii) implies that K is relatively compact in $\mathcal{H}$. To show that K is relatively compact in H, it suffices to show that the limit of any convergent sequence in K belongs to H. So let (h_n) be a sequence in K such that $h_n \to \mathbf{h} \in \mathcal{H}$. Then by (iii) and by Theorem 8(v) there is a subsequence (g_k) of (h_n) such that the primitives F_k of g_k satisfy the following conditions.

- (F_k) converges uniformly to a continuous function F in $[a,b]$.
- There exists a sequence of closed sets (X_ι) such that $[a,b] = \cup_{\iota=1}^{\infty} X_\iota$ and such that, given $\varepsilon > 0$ and $l \in \mathbf{N}$, there exists $\eta > 0$ such that

$$\overline{\lim}_k \sum_{i=1}^{s} \omega(F_k, [x_i', x_i'']) < \frac{1}{3}\varepsilon \tag{3}$$

for every partition $\{[x_i', x_i''], i = 1, 2, \ldots, s\}$ with $\{x_i', x_i''\} \subset X_l$ for each i and with $\sum_{i=1}^{s} |x_i' - x_i''| < \eta$. Now choose k_o such that $||F - F_k||_\infty < \frac{\varepsilon}{3s}$ for $k \geq k_o$. Then, for such k and for $x_i, y_i \in [x_i', x_i'']$, we have

$$\begin{aligned}|F(x_i) - F(y_i)| &\leq 2||F - F_k||_\infty + |F_k(x_i) - F_k(y_i)| \\ &< \frac{2}{3}\frac{\varepsilon}{s} + \omega(F_k, [x_i', x_i'']),\end{aligned}$$

so that

$$\omega(F,[x'_i,x''_i]) \le \frac{2}{3}\frac{\varepsilon}{s} + \omega(F_k,[x'_i,x''_i]), \quad i=1,2,\dots,s. \tag{4}$$

Then by (3) and (4) it follows that $\sum_{i=1}^{s}\omega(F,[x'_i,x''_i]) < \varepsilon$ and hence F is ACG_*. Therefore $\mathbf{h} \in H$ and hence (i) holds. □

4 Relatively Weakly Compact Subsets of $\mathcal{H}$ and H

As an application of the results of §1, we give some characterizations of relatively weakly compact sets in $\mathcal{H}$ and H. Some of these results have been proved in [3] by a direct argument. We need the following extension of Corollary IV.6.4 of [5].

Theorem 12 *A sequence (F_n) in Ω is weakly convergent to $F \in \Omega$ if and only if (F_n) is uniformly bounded and $F_n \to F$ pointwise in $[a,b]$. Consequently, a sequence $(\mathbf{h}_n)$ in $\mathcal{H}$ is weakly convergent to $\mathbf{h} \in \mathcal{H}$ if and only if $(\mathbf{h}_n)$ is bounded and the primitives of $(\mathbf{h}_n)$ converge pointwise to that of $\mathbf{h}$.*

Proof. By the Hahn-Banach theorem and the Riesz representation theorem, each $x^* \in \Omega^*$ is the restriction of a (regular) Borel measure μ so that

$$x^*(F) = \int_a^b F\,d\mu\,, \quad F \in \Omega.$$

Then, the Lebesgue dominated convergence theorem and the fact that the norm closed subspace Ω is also weakly closed in $\mathcal{C}[a,b]$ imply that the conditions are sufficient for (F_n) to converge to F weakly. Moreover, the mapping $T_x : \mathcal{C}[a,b] \to \mathbb{R}$ given by $T_x(F) = F(x)$ is a bounded linear functional. If $F_n \to F$ weakly in Ω, then by the uniform boundedness principle (F_n) is uniformly bounded as Ω^* is a Banach space. Moreover, for each $x \in [a,b]$, $T_x\,|_\Omega$ belongs to Ω^* and hence $F_n(x) \to F(x)$ for each $x \in [a,b]$.

The second part follows immediately from the first, as Φ is an isometric isomorphism from $\mathcal{H}$ onto Ω so that Φ is a linear homeomorphism with respect to the weak topologies. □

Corollary 13 *If K is relatively compact in H, then all sequential weak limits of K belong to H. Consequently, if K is relatively compact in H and relatively weakly compact in $\mathcal{H}$, then K is relatively weakly compact in H itself.*

Proof. Let (h_n) be a sequence in K and suppose that $h_n \to \mathbf{h} \in \mathcal{H}$ weakly. By Theorem 11 there exists a subsequence (g_k) of (h_n) such that their primitives (F_k) converge uniformly to a function $F \in \mathcal{C}[a,b]$ such that F is ACG_*.

On the other hand, as $h_n \to \mathbf{h}$ weakly, the subsequence (g_k) also converges to $\mathbf{h}$ weakly and consequently, by Theorem 12 $F_k \to G$ pointwise in $[a,b]$, where G is the primitive of $\mathbf{h}$. Thus it follows that $G = F$ and hence $\mathbf{h} = \Phi^{-1}(G) \in H$. Therefore the first part holds.

Since each element in the weak closure of a relatively weakly compact set S in a Banach space X is the weak limit of a sequence from S (See p. 45 of [4].), the second part is immediate from the first. □

Theorem 14 *Let K be a subset of $\mathcal{H}$. Then the following assertions are equivalent.*

(i) *K is relatively weakly compact in $\mathcal{H}$.*

(ii) *$\Phi(K)$ is relatively weakly compact in $\mathcal{C}[a,b]$.*

(iii) *$\Phi(K)$ is relatively weakly compact in Ω.*

(iv) *K is bounded and each sequence $(\mathbf{h}_n)$ in K contains a subsequence $(\mathbf{h}_{n_k})$ such that their primitives are equicontinuous on a dense subset of $[a,b]$ and are asymptotically continuous on $[a,b]$.*

(v) *K is bounded and each sequence $(\mathbf{h}_n)$ in K contains a subsequence $(\mathbf{h}_{n_k})$ such that their primitives converge pointwise to a continuous function.*

Proof. Since Ω is a closed linear subspace of $\mathcal{C}[a,b]$, by the Hahn-Banach theorem Ω is weakly closed and hence (ii) and (iii) are equivalent.

(i) and (iii) are equivalent as Φ is a linear homeomorphism for the weak topologies. (See the proof of Theorem 12.)

(i) and (iv) are equivalent by the equivalence of (i) and (ii), by Theorem 5 and by the Eberlein-Šmulian theorem. Finally, (iv) is equivalent to (v) by Theorem 4, since any continuous function on $[a,b]$ is uniformly continuous. □

Remark 1 *The equivalence of (i) and (iv) has already been established directly in Theorem* 11 *of* [3].

Theorem 15 *Let K be a subset of $\mathcal{H}$. Then the following assertions are equivalent.*

(i) *K is relatively weakly compact in $\mathcal{H}$ and $\overline{K}^{weak} \subset H$.*

(ii) *K is bounded and each sequence $(\mathbf{h}_n)$ in K contains a subsequence $(\mathbf{h}_{n_k})$ such that their primitives are equicontinuous on a dense subset of $[a,b]$ and are asymptotically-ACG* on $[a,b]$* [2].

[2] The property *asymptotically-ACG$_*$* implies the property *asymptotically-ACG**. See [3] for details.

(iii) *K is bounded and each sequence $(\mathbf{h}_n)$ in K contains a subsequence $(\mathbf{h}_{n_k})$ such that their primitives converge pointwise to a continuous function, which is ACG_* on $[a, b]$.*

Proof. (i) and (ii) are equivalent by Theorem 16 of [3]. Now suppose (i) holds and let $(\mathbf{h}_n)$ be a sequence in K. Then by the Eberlein-Šmulian theorem there exists a subsequence $(\mathbf{h}_{n_k})$ of $(\mathbf{h}_n)$ weakly convergent to some $\mathbf{h} \in \mathcal{H}$. Since $\overline{K}^{weak} \subset H$, it follows that $\mathbf{h} \in H$, so that $\Phi(\mathbf{h})$ is ACG_*. Then (iii) holds by Theorem 12.

Conversely, let (iii) hold. Then by Theorem 14, K is relatively weakly compact in $\mathcal{H}$. Now, let $\mathbf{h} \in \overline{K}^{weak}$. Then there exists a sequence $(\mathbf{h}_n)$ in K such that $\mathbf{h}_n \to \mathbf{h}$ weakly (See p. 45 of [4].) and consequently, by the hypothesis (iii) there exists a subsequence $(\mathbf{h}_{n_k})$ of $(\mathbf{h}_n)$ such that the primitives F_{n_k} of $\mathbf{h}_{n_k}$ converge pointwise to some function F which is continuous and ACG_* on $[a, b]$. Then by Theorem 12, it follows that (h_{n_k}) converges weakly to $\Phi^{-1}(F) \in H$ and hence $\mathbf{h} \in H$. Thus (i) holds. □

References

[1] J. B. Brown, *Totally discontinuous connectivity functions*, Coll. Math., **23** (1971), 53–60.

[2] A. Alexiewicz, *Linear functionals on Denjoy integrable functions*, Coll. Math., **1** (1948), 289–293.

[3] B. Bongiorno, *Relatively weakly compact sets in the Denjoy space*, J. Math. Study, **27** (1994), 37–43.

[4] J. Diestel, *Uniform integrability: an introduction*, Rend. Ist. Mat. Univ. Trieste, **23** (1991), 41–80.

[5] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, 1958.

[6] I. P. Natanson, *Theory of functions of a real variable*, Vol. 1, F. Ungar Publ. New York, 1964.

[7] H. L. Royden, *Real Analysis*, McMillan, New York, 1963.

[8] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.