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ORDINARY DERIVATIVES VIA SYMMETRIC DERIVATIVES AND A LIPSCHITZ CONDITION VIA A SYMMETRIC LIPSCHITZ CONDITION

Abstract

If a subset A of the real line is a countable union of closed, strongly symmetrically porous sets, then there exists a Lipschitz everywhere symmetrically differentiable function f such that A is the set of all non-differentiability points of f. Since there are closed strongly symmetrically porous sets of Hausdorff dimension 1, our construction answers a problem posed by J. Foran in 1977. We also obtain results concerning smallness of the set of points at which a continuous function fulfills the symmetric Lipschitz condition but does not fulfill the ordinary Lipschitz condition.

1 Introduction and Notation

In this article we will consider real functions defined on the real line $\mathbb{R}$. By the symmetric derivative of a function f at a point $x \in \mathbb{R}$ we mean

$$f'_s(x) := \lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h};$$

we consider here only finite symmetric derivatives.

Let us recall that f satisfies the Lipschitz condition at $x \in \mathbb{R}$ if

$$\limsup_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| < \infty.$$

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Following [9], we say that a function $f : \mathbb{R} \to \mathbb{R}$ fulfills the symmetric Lipschitz condition at a point x if

$$\limsup_{h\to 0+} \left| \frac{f(x+h) - f(x-h)}{2h} \right| < \infty.$$

We shall use the following notation.

$$\begin{aligned} C(f) &= \{x : f \text{ is continuous at } x\}, \\ D(f) &= \{x : f'(x) \in \mathbb{R} \text{ exists}\}, \\ SD(f) &= \{x : f'_s(x) \in \mathbb{R} \text{ exists}\}, \\ L(f) &= \{x : f \text{ fulfils the Lipschitz condition at } \ x\} \end{aligned}$$

and

$$SL(f) = \{x : f \text{ fulfils the symmetric Lipschitz condition at } x\}.$$

Let $E \subset \mathbb{R}, x \in \mathbb{R}$ and $r > 0$. Then we define $s(E, x, r)$ as the supremum of all numbers $h > 0$ for which there exists a $p > 0$ such that $p + h \le r$, $(x + p, x + p + h) \cap E = \emptyset$ and $(x - p - h, x - p) \cap E = \emptyset$. The symmetric porosity of E at x is defined as

$$p^s(E, x) := \limsup_{r\to 0+} \frac{s(E, x, r)}{r}.$$

We say that E is symmetrically porous at x (d-symmetrically porous at x) if $p^s(E, x) > 0$ ($p^s(E, x) \ge d$). If E is 1-symmetrically porous at x, we say that E is strongly symmetrically porous at x.

A set $E \subset \mathbb{R}$ is symmetrically porous (strongly symmetrically porous, d-symmetrically porous) if it is symmetrically porous (strongly symmetrically porous, d-symmetrically porous) at each of its points.

A set E is called σ-symmetrically porous (σ-strongly symmetrically porous, σ-d-symmetrically porous) if it is a countable union of symmetrically porous (strongly symmetrically porous, d-symmetrically porous) sets.

Khintchine [5] proved that the set $SD(f) \setminus D(f)$ is of Lebesgue measure zero for each measurable function f. Foran [4] (and independently also Ponomarev [7]) constructed a continuous function on $\mathbb{R}$ which has a finite symmetric derivative everywhere and is differentiable at no point of a nonempty perfect set. Thus the set $SD(f) \setminus D(f)$ can be uncountable also for a continuous function f. Foran in his article asked two questions.

The first question asks whether there exists a continuous function f which has a finite symmetric derivative everywhere and the set of all non-differentiability points of f has a positive Hausdorff dimension. Note that Foran observed that this set has Hausdorff dimension zero in his example. Thomson ([9], p. 266) conjectured that this question has positive answer; we will see that his intuition was right on target.

Foran's second question, which asks whether each perfect set of measure zero is the set of all non-differentiability points for a continuous function which has a finite symmetric derivative everywhere, was answered negatively by Belna, Evans and Humke [1]. They proved that, for a continuous function f, the set $SD(f) \setminus D(f)$ is σ-porous and used the fact ([10]) that there exists a perfect set of measure zero which is not σ-porous.

Evans in [2] factually proved the following result which improves the result of [1] and generalizes the previous result of (the preprint of) [12].

Theorem E. *Let $f : \mathbb{R} \to \mathbb{R}$ be given. Then the set $(SD(f) \setminus D(f)) \cap \overline{C(f)}$ is $\sigma - (1-\varepsilon)$ -symmetrically porous for each $0 < \varepsilon < 1$.*

This result was formulated in [2] in the case $SD(f) \subset \overline{C(f)}$ only, but it is obvious that the same arguments give also the above result.

In [12] this result was proved for continuous f only. The fact that Theorem E is a true improvement of the result of [1] was proved in [3].

The natural problem of a complete characterization (or at least a complete characterization of smallness) of sets $SD(f) \setminus D(f)$ for continuous f (or for symmetrically differentiable continuous f is Problem 42 of [9]) and seems to be open.

The main result (Theorem 3.2) of the present article says that if $A \subset \mathbb{R}$ is a countable union of closed strongly symmetrically porous sets, then $A = SD(f) \setminus D(f)$ for a Lipschitz everywhere symmetrically differentiable function f. The corresponding construction is similar to that of [7] but it contains also some small new ideas.

Theorem E and Theorem 3.2 suggest that, if a simple characterization discussed above exists, it must probably deal with a type of symmetric porosity. We obtain a simple characterization in the class of perfect symmetric sets only. However, this result is strong enough to easily imply a positive answer to Foran's first question mentioned above. The set $\mathbb{R} \setminus D(f)$ can have Hausdorff dimension 1 for a Lipschitz everywhere symmetrically differentiable function f.

In Section 4 we consider the size of $SL(f) \setminus L(f)$. First we show (Theorem 4.1) that the notes [12] and [2] easily give that $(SL(f) \setminus L(f)) \cap \overline{C(f)}$ is σ-strongly symmetrically porous for each $f : \mathbb{R} \to \mathbb{R}$. Thus $SL(f) \setminus L(f)$ is

σ-strongly symmetrically porous if $\overline{C(f)} = \mathbb{R}$, in particular for each Baire one function f.

The basic constructions used in the proof of Theorem 3.2 easily give that, if $F \subset \mathbb{R}$ is a countable union of closed strongly symmetrically porous sets, then there exists a continuous function f such that $F \subset SL(f) \setminus L(f)$ (even $F \subset SD(f) \setminus L(f)$). Note that we cannot demand here $SL(f) = \mathbb{R}$, see Remark 4.9.

The same constructions give a complete characterization of those symmetric, perfect sets that are of the form $SL(f) \setminus L(f)$ (or $SD(f) \setminus L(f)$) for a continuous function f. In particular, we obtain that $SD(f) \setminus L(f)$ can be of Hausdorff dimension 1 for a continuous function f.

It should be mentioned that Theorem 4.1 was originally contained in an unpublished note written (and originally also submitted for publication) in 1996. The results of Section 3 were presented on the Workshop in Real Analysis, Budapest 21.6.-24.6.1997.

We adopt the following notation.

The four Dini derivates of f at x are denoted by $D^+f(x), D_+f(x), D^-f(x)$ and $D_-f(x)$.

Lebesgue measure on $\mathbb{R}$ is denoted by λ.

If $I \subset \mathbb{R}$ is an interval, we frequently write $|I|$ instead of λI.

The symbols $\overline{A}$ and $\operatorname{int} A$ denote the closure and the interior of a set A, respectively. The distance of two sets A, B is denoted by $\operatorname{dist}(A, B)$.

We say that a function f is K-Lipschitz if f is Lipschitz with the constant K.

The support of f is $\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$.

2 Lemmas and Basic Constructions

We start with the following useful technical definitions.

Definition 2.1.

(a) By an I-system $\mathcal{I}$ we mean a finite (possibly empty) disjoint system of nonempty bounded closed intervals. We put

$$\nu(\mathcal{I}) = \sup\{|I| : I \in \mathcal{I}\}.$$

(b) Let $\mathcal{I}, \mathcal{K}$ be I-systems and let $c > 0$. We say that $\mathcal{I}$ is c-embedded in $\mathcal{K}$ if

(b1) for each $I \in \mathcal{I}$ there exists $K \in \mathcal{K}$ such that $I \subset K$ and $\operatorname{dist}(I, \mathbb{R} \setminus \bigcup \mathcal{K}) = \operatorname{dist}(I, \mathbb{R} \setminus K) > c|I|$, and (b2) $\operatorname{dist}(I, J) > c|I|$ whenever $I, J \in \mathcal{I},\ I \neq J$.

The fact that a closed set is strongly symmetrically porous can be expressed in different ways. One of them uses the notion of c-embedding of I-systems; in the following lemma we formulate and prove the only implication we need.

Lemma 2.2. *Let $F \subset \mathbb{R}$ be a nonempty bounded closed strongly symmetrically porous set and let $(c_n)_{n=1}^{\infty}$ be a sequence such that $c_n > 1$ and $c_n \to \infty$. Then there exist I-systems $(\mathcal{I}_n)_{n=0}^{\infty}$ such that, for every $n \in \mathbb{N}$,*

(i) *$\mathcal{I}_n$ is c_n-embedded in $\mathcal{I}_{n-1}$,*

(ii) *$\nu(\mathcal{I}_n) < 1/c_n$ and*

(iii) *$F = \bigcap_{k=0}^{\infty} \bigcup \mathcal{I}_k$.*

PROOF. Find $a, b \in \mathbb{R}$ such that $F \subset (a, b)$ and put $\mathcal{I}_0 = \{[a, b]\}$. Further suppose that $k \in \mathbb{N}$ and that $\mathcal{I}_0, \dots, \mathcal{I}_{k-1}$ were constructed so that, for every $0 \leq n \leq k - 1$, the following conditions hold:

(a) conditions (i) and (ii) hold whenever $n > 0$,

(b) $F \subset \operatorname{int}(\bigcup \mathcal{I}_n)$ and

(c) $F \cap I \neq \emptyset$ whenever $I \in \mathcal{I}_n$.

We want to construct $\mathcal{I}_k$ such that (i), (ii), (b) and (c) hold for $n = k$. Since $F \subset \operatorname{int}(\bigcup \mathcal{I}_{k-1})$, we have $\rho := \operatorname{dist}(F, \mathbb{R} \setminus \bigcup \mathcal{I}_{k-1}) > 0$. Since F is strongly symmetrically porous, we can assign numbers $p_x > 0, h_x > 0$ to every $x \in F$ so that

$$(x + p_x, x + p_x + h_x) \cap F = \emptyset,\ (x - p_x - h_x, x - p_x) \cap F = \emptyset, \tag{1}$$

$$h_x > 8c_k p_x \quad \text{and} \tag{2}$$

$$6p_x c_k < \min(1, \rho). \tag{3}$$

By the Borel covering lemma, we can find points $x_1, \dots, x_m \in F$ such that, putting $p_i := p_{x_i}, h_i := h_{x_i}$, the intervals $(x_i - p_i - h_i, x_i + p_i + h_i), i = 1, \dots, m$, cover the set F. By (1) we also have that the system of intervals $\Phi := \{J_i := [x_i - p_i, x_i + p_i] : i = 1, \dots, m\}$ covers F. Moreover, we may and will suppose that

$$\text{no proper subsystem of } \Phi \text{ covers } F. \tag{4}$$

Now put

$$\mathcal{I}_k = \{[x_i - 2p_i, x_i + 2p_i] : i = 1, \dots, m\}.$$

Let $1 \le i, j \le m$ and $y_i \in [x_i - 2p_i, x_i + 2p_i]$, $y_j \in [x_j - 2p_j, x_j + 2p_j]$. We may and will suppose $p_j \le p_i$. First we shall show that $y_i \ne y_j$. In fact, otherwise clearly

$$[x_j - p_j, x_j + p_j] \subset (x_i - 5p_i, x_i + 5p_i)$$

and therefore (2) implies

$$[x_j - p_j, x_j + p_j] \subset (x_i - p_i - h_i, x_i + p_i + h_i).$$

Consequently (1) gives $[x_j - p_j, x_j + p_j] \cap F \subset [x_i - p_i, x_i + p_i]$ which contradicts (4).

Thus we know that $\mathcal{I}_k$ is an I-system. Further (1) implies $|x_i - x_j| \ge p_i + h_i$. Consequently, using (2), we have

$$\begin{aligned} |y_i - y_j| &\ge p_i + h_i - 4p_i > 8c_k p_i - 3p_i > 5c_k p_i > c_k \lambda [x_i - 2p_i, x_i + 2p_i] \\ &\ge c_k \lambda [x_j - 2p_j, x_j + 2p_j]. \end{aligned}$$

If, moreover, $z \in \mathbb{R} \setminus \bigcup \mathcal{I}_{k-1}$ is given, then (3) gives $|z - x_i| \ge \rho > 6p_i c_k$. Therefore

$$|z - y_i| \ge 6p_i c_k - 2p_i > 4p_i c_k \ge c_k \lambda [x_i - 2p_i, x_i + 2p_i].$$

Thus we have shown that (i) holds for $n = k$. By (3) we obtain $\lambda[x_i - 2p_i, x_i + 2p_i] = 4p_i < 4/6c_k < 1/c_k$ which implies that (ii) holds for $n = k$ as well.

The validity of (b) and (c) for $n = k$ is obvious. Thus the sequence $(\mathcal{I}_n)_{n=0}^{\infty}$ is well defined. It clearly satisfies (i) and (ii); (iii) follows by (b),(c),(ii) and the assumption $c_n \to \infty$. $\square$

In the following construction, we build more complicated functions from basic building blocks; functions g_I which are assigned to each closed bounded interval I. We need only the following properties of g_I.

(a) g_I is 4-Lipschitz and of the class C^1 on $\mathbb{R}$.

(b) $\mathrm{supp}(g_I) \subset I$ and $g_I(x) \ge 0$ for each $x \in \mathbb{R}$.

(c) g_I attains its maximum which equals $|I|$ at the center c of I and $g_I(c+h) = g_I(c-h)$ for all $h \in \mathbb{R}$.

It is easy to see that such functions exist.

The following construction depends on a parameter $0 \le \alpha < 1$; we shall apply it in the following with $\alpha = 0$ and $\alpha = 1/2$.

Construction Let $0 \le \alpha < 1$ and $d > 1$ be given. Further let I-systems $\mathcal{I}$ and $\mathcal{K}$ such that $\mathcal{I}$ is $4d^2$-embedded in $\mathcal{K}$ be given We shall construct a function $\varphi = \varphi(\alpha, d, \mathcal{I})$ (which does not depend on $\mathcal{K}$) in the following way.

To every interval $I = [a,b] \in \mathcal{I}$, we assign the "right" interval $I^r := [b + d|I|, b + 2d|I|]$ and the "left" interval $I^l := [a - 2d|I|, a - d|I|]$. Put

$$\varphi = \varphi(\alpha, d, \mathcal{I}) := \sum_{I \in \mathcal{I}} d^\alpha (g_{I^r} + g_{I^l}).$$

We shall need the properties of φ which are proved in the following lemma.

Lemma 2.3. *The function $\varphi = \varphi(\alpha, d, \mathcal{I})$ constructed above has the following properties:*

(P1) *φ is a non-negative C^1 function on $\mathbb{R}$ with a compact support.*

(P2) $|\varphi(x)| \leq d^{\alpha+1}\nu(\mathcal{I})$ *for each $x \in \mathbb{R}$.*

(P3) $\operatorname{dist}(\operatorname{supp}(\varphi), \bigcup \mathcal{I}) > 0$ *and* $\operatorname{supp} \varphi \subset \bigcup \mathcal{K}$.

(P4) *φ is 4-Lipschitz in the case $\alpha = 0$.*

(P5) *If $x \in \bigcup \mathcal{I}$ and $h > 0$, then $|\varphi(x+h) - \varphi(x-h)|/2h \leq 4d^{\alpha-1}$.*

(P6) *For every $x \in \bigcup \mathcal{I}$ there exists $0 < h < 3\nu(\mathcal{I})d$ such that $\varphi(x+h)/h > d^\alpha/3$.*

PROOF. To each $I \in \mathcal{I}$ assign an "enlarged" interval $I^* := [a - 2d^2|I|, b + 2d^2|I|]$. Observe that

$$I^r \cup I^l \subset I^* \quad \text{and} \quad \{I^* : I \in \mathcal{I}\} \ \text{ is a disjoint system.} \tag{5}$$

The first claim of (5) is obvious. To prove the second one, suppose on the contrary that $I^* \cap J^* \neq \emptyset$ for different I, J from $\mathcal{I}$. We may and will suppose $|I| \geq |J|$. Then the distance between I and J is clearly at most $2d^2|I| + 2d^2|J| \leq 4d^2|I|$ which contradicts the assumption that $\mathcal{I}$ is $4d^2$-embedded in $\mathcal{K}$.

Using (5) and the definitions of φ and g_I we immediately obtain the properties (P1)-(P4).

To prove (P5), suppose that $x \in I = [a,b] \in \mathcal{I}$ and $h > 0$ are given. Denote $c := (a+b)/2$. If $0 < h \leq d|I|$, then clearly $\varphi(x+h) = \varphi(x-h) = 0$. If $d|I| < h \leq 2d^2|I|$, then the points $c+h, c-h, x+h, x-h$ belong to I^* and (5) implies that $\varphi = g_{I^r} + g_{I^l}$ on I^*. Thus $\varphi(c+h) - \varphi(c-h) = 0$ and

$$\begin{aligned}
|\varphi(x+h) - \varphi(x-h)| \leq& |\varphi(c+h) - \varphi(c-h)| + |\varphi(c+h) - \varphi(x+h)| \\
&+ |\varphi(c-h) - \varphi(x-h)| \\
\leq& 0 + 4d^\alpha|c-x| + 4d^\alpha|c-x| \leq 8d^\alpha|I| < 8d^{\alpha-1}h.
\end{aligned}$$

If $h > 2d^2|I|$, then (5) gives that either $\varphi(x+h) = g_{J^l}(x+h)$ or $\varphi(x+h) = g_{J^r}(x+h)$ for an interval $J \in \mathcal{I}, J \neq I$. In both cases $|\varphi(x+h)| \leq d^{1+\alpha}|J|$. If $\varphi(x+h) \neq 0$, then clearly $h + 2d|I| \geq \operatorname{dist}(I,J) \geq 4d^2|J|$. Consequently $h > 2d^2|J|$ and thus we have

$$\left|\frac{\varphi(x+h)}{2h}\right| \leq \frac{d^{\alpha+1}|J|}{4d^2|J|} = \frac{d^{\alpha-1}}{4}.$$

Similarly we obtain $|\varphi(x-h)/2h| \leq d^{\alpha-1}/4$. The inequalities proved above immediately give (P5).

To prove (P6), suppose that an $x \in I \in \mathcal{I}$ is given. Put $h := b + \frac{3}{2}d|I| - x$. Then clearly $0 < h < 3d|I| < 3\nu(\mathcal{I})d$ and

$$\frac{\varphi(x+h)}{h} = \frac{d^{\alpha+1}|I|}{h} > \frac{d^{\alpha+1}|I|}{3d|I|} = \frac{d^{\alpha}}{3}.$$

□

Lemma 2.4. *Suppose that $0 \leq \alpha < 1, (\mathcal{I})_{n=0}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ are given so that all $\mathcal{I}_n$ are I-systems, $d_n > 1$. In addition assume*

(i) *$\mathcal{I}_n$ is $4d_n^2$-embedded in $\mathcal{I}_{n-1}$ for every $n \in \mathbb{N}$,*

(ii) *$(d_n)^{\alpha+1}\nu(\mathcal{I}_n) \to 0$, $(d_n)^{\alpha+1}\nu(\mathcal{I}_n) \leq 1$ for every $n \in \mathbb{N}$ and*

(iii) *$\sum_{n=1}^{\infty}(d_n)^{\alpha-1} < \infty$.*

Denote $F := \bigcap_{n=0}^{\infty} \bigcup \mathcal{I}_n$. Then there exists a function $f = f_\alpha$ such that

(iv) *f is continuous, $|f(x)| \leq 1$ for every $x \in \mathbb{R}$ and f is 4-Lipschitz in the case $\alpha = 0$,*

(v) *f is a C^1 function on $\mathbb{R} \setminus F$,*

(vi) *$f'_s(x) = 0$ for every $x \in F$ and*

(vii) *if $x \in F$, then*

(a) $D^- f(x) \leq 0$,

(b) $D^+ f(x) \geq 1/3$ *in the case* $\alpha = 0$ *and*

(c) $D^+ f(x) = \infty$ *in the case* $\alpha > 0$.

Proof. Let $\varphi_n = \varphi(\alpha, d_n, \mathcal{I}_n)$ be the functions from the Construction. Put $f = f_\alpha = \sum_{n=1}^{\infty} \varphi_n$. By (i) and (P3) of Lemma 2.3 we have that the supports of the functions φ_n are pairwise disjoint. This fact, (P1), (P2), (P4) and (ii) easily imply (iv) and (v). To prove (vi) suppose that $x \in F$ and $\varepsilon > 0$ are given. Observe that (P3) implies that $\varphi_k'(x) = 0$ for each k. Using also (P5) and (iii) we obtain

$$\begin{aligned}\limsup_{h\to 0}\left|\frac{f(x+h)-f(x-h)}{2h}\right| &\leq \limsup_{h\to 0}\sum_{k=1}^{n}\left|\frac{\varphi_k(x+h)-\varphi_k(x-h)}{2h}\right| \\ &\quad + \limsup_{h\to 0}\sum_{k=n+1}^{\infty}\left|\frac{\varphi_k(x+h)-\varphi_k(x-h)}{2h}\right| \\ &\leq 0 + \sum_{k=n+1}^{\infty} 4d^{\alpha-1} < \varepsilon,\end{aligned}$$

if n is chosen sufficiently large. Thus $f_s'(x) = 0$.

If $x \in F$, then $f(x) = 0$, and since f is non-negative, we obtain (vii),(a).

For each index n by (P6) we can find an h_n such that $0 < h_n < 3\nu(\mathcal{I}_n)d_n$ and $\varphi(x + h_n)/h_n > (d_n)^\alpha/3$. Since $d_n > 1$, we obtain by (ii) that $h_n \to 0$. Since

$$\frac{f(x+h_n)-f(x)}{h_n} \geq \frac{\varphi_n(x+h_n)}{h_n} > \frac{(d_n)^\alpha}{3}$$

and $d_n \to \infty$ by (iii), we obtain (vii),(b) and (vii),(c). □

3 Symmetric Derivatives

Proposition 3.1. *Let $F \subset \mathbb{R}$ be a bounded closed strongly symmetrically porous set. Then there exists a non-negative 1-Lipschitz function g such that $|g(x)| \leq 1$ for every $x \in \mathbb{R}$, g is a C^1-function on $\mathbb{R} \setminus F$ and, for every $x \in F$, we have*

$$g(x) = 0,\ g_s'(x) = 0,\ D^-g(x) \leq 0 \ \textit{ and }\ D^+g(x) \geq 1/12.$$

PROOF. Put $d_n := 2n^2$ and apply Lemma 2.2 to F and $c_n := 4(d_n)^2$. The resulting I-systems $(\mathcal{I}_n)_{n=0}^\infty$ clearly satisfy assumptions (i)-(iii) of Lemma 2.4 for $\alpha = 0$. Now it is clearly sufficient to find the corresponding $f = f_0$ and put $g := f/4$. □

Theorem 3.2. *Let $A \subset \mathbb{R}$ can be written in the form $A = \bigcup_{n=1}^\infty F_n$, where each F_n is closed and strongly symmetrically porous. Then there exists a Lipschitz symmetrically differentiable function f on $\mathbb{R}$ such that A is the set of all non-differentiability points of f.*

Proof. We may suppose that all F_n are bounded. For each n, we apply Proposition 3.1 to $F = F_n$ and obtain a corresponding function $g = g_n$. Now put $f := \sum_{n=1}^{\infty}(26)^{-n} g_n$. Obviously, f is a Lipschitz function.

Let $x \in \mathbb{R}$ be given and put $D := \sum_{n=1}^{\infty}(26)^{-n}(g_n)'_s(x)$. For each $\varepsilon > 0$ find $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty}(26)^{-n} < \varepsilon/3$ and $h_0 > 0$ such that

$$\left|\sum_{n=1}^{k}(26)^{-n}\frac{g_n(x+h)-g_n(x-h)}{2h} - \sum_{n=1}^{k}(26)^{-n}(g_n)'_s(x)\right| < \frac{\varepsilon}{3}$$

for every $0 < h < h_0$. Since each g_n is 1-Lipschitz, we conclude that

$$\begin{aligned}
&\left|\frac{f(x+h)-f(x-h)}{2h} - D\right| \\
&= \left|\sum_{n=1}^{\infty}(26)^{-n}\frac{g_n(x+h)-g_n(x-h)}{2h} - \sum_{n=1}^{\infty}(26)^{-n}(g_n)'_s(x)\right| \\
&\le \left|\sum_{n=1}^{k}(26)^{-n}\frac{g_n(x+h)-g_n(x-h)}{2h} - \sum_{n=1}^{k}(26)^{-n}(g_n)'_s(x)\right| \\
&\quad + \left|\sum_{k+1}^{\infty}(26)^{-n}\frac{g_n(x+h)-g_n(x-h)}{2h}\right| + \left|\sum_{k+1}^{\infty}(26)^{-n}(g_n)'_s(x)\right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

if $0 < h < h_0$. Therefore $f'_s(x) = D$ and thus f is symmetrically differentiable.

Quite similar argument gives that $f'(x) = \sum_{n=1}^{\infty}(26)^{-n}(g_n)'(x)$ for each $x \in \mathbb{R} \setminus A$.

Let now a point $x \in A$ be fixed. Find $k \in \mathbb{N}$ such that $x \in F_k$ and $x \notin F_n$ for every $n < k$. Then the function $\sum_{n<k}(26)^{-n} g_n$ is differentiable at x and

$$D^+((26)^{-k}g_k)(x) - D^-((26)^{-k}g_k)(x) \ge 26^{-k}\frac{1}{12}.$$

Since the function $\sum_{n=k+1}^{\infty}(26)^{-n}g_n$ is Lipschitz with the Lipschitz constant $\sum_{n=k+1}^{\infty}(26)^{-n} = (26)^{-k}/25$, we conclude that

$$D^+f(x) - D^-f(x) \ge \frac{1}{12}(26)^{-k} - \frac{2}{25}(26)^{-k} > 0. \qquad \square$$

As an almost immediate consequence of this theorem and results of [3], we obtain the following result on symmetric perfect sets. We use here the notation from [6]. Namely, if a sequence $\lambda = (\lambda_n)_{n=1}^{\infty}$ with $0 < \lambda_n < \frac{1}{2}$ is given, then we consider the symmetric perfect set (the "generalized Cantor set " in [6]) $C(\lambda) \subset [0, 1]$ which is constructed like the classical Cantor ternary set is so that, after the n-th step of construction, we obtain 2^n closed "remaining" intervals with the same length $\lambda_1 \dots \lambda_n$. Symmetric perfect sets are sometimes called also "symmetric Cantor sets" and/or determined by a sequence $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$ (see [3]). Note that for $\alpha_n = 1 - 2\lambda_n$ the set $C(\alpha_n)$ from [3] coincides with the set $C(\lambda)$ from [6].

Proposition 3.3. *Let $C = C(\lambda) \subset [0, 1]$ be a symmetric perfect set. Then the following statements are equivalent.*

(i) $\liminf \lambda_n = 0$.

(ii) *There exists a Lipschitz function f on $\mathbb{R}$ which has a finite symmetric derivative at all points, is of the class C^1 outside C but $f'(x)$ exists at no point $x \in C$.*

(iii) *There exists a function f on $\mathbb{R}$ such that $C \subset (SD(f) \setminus D(f)) \cap \overline{C(f)}$.*

Proof. Theorem 3 and Theorem 5 of [3] give that (i) holds iff C is strongly symmetrically porous. Thus Proposition 3.1 immediately gives the implication $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (iii)$ is trivial. To prove the implication $(iii) \Rightarrow (i)$ suppose that (i) fails. Then we know by Theorem 3 of [3] that there exists $\varepsilon > 0$ such that

$$C \text{ is } (1 - \varepsilon) - \text{symmetrically porous at no point of } C. \tag{6}$$

By Theorem E (see Introduction) $C = \bigcup_{n=1}^{\infty} A_n$ where every A_n is $(1 - \varepsilon)$-symmetrically porous. By the Baire theorem we obtain that some A_n is dense in a portion of C, which clearly contradicts (6). $\square$

The condition (i) implies that the Lebesgue measure of C is zero but it is well-known that it implies no stronger smallness in the (Hausdorff) measure sense. In particular, there exists a symmetric perfect set C of Hausdorff dimension 1 for which (i) holds. Thus Foran's first question (see Introduction) has a negative answer.

We shall now formulate and prove a more precise statement which deals with Hausdorff measures Λ_h determined by non-decreasing functions $h : [0, \infty) \to [0, \infty), h(0) = 0$ (see [6] or [8]).

Proposition 3.4. *Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing function such that $h(0) = 0$ and $h'(0) = \infty$. Then there exists a symmetric perfect set C and a Lipschitz function f on $\mathbb{R}$ with the following properties.*

(i) *$\Lambda_h(C) = \infty$, where Λ_h is the Hausdorff measure determined by h.*

(ii) *The function f is of the class C^1 outside C, has a finite symmetric derivative at all points and $f'(x)$ exists at no point $x \in C$.*

Proof. We will need the following fact (see [6], 4.11).
Fact Let $C = C(\lambda)$ be a symmetric perfect set. Put $s_k = \lambda_1 \cdots \lambda_k$. If $g : [0,\infty) \to [0,\infty)$ is a continuous increasing function such that $g(s_k) = 2^{-k}$, then $1/4 \le \Lambda_g(C(\lambda)) \le 1$.

For each natural number k choose $\delta_k > 0$ such that

$$\frac{h(x)}{x} > (k+2)! \quad \text{whenever} \quad 0 < x \le \delta_k.$$

Further choose an increasing sequence of natural numbers $(n_k)_{k=1}^{\infty}$ such that $n_1 > 2$ and $2^{-n_k} < \delta_k$. Let $(p_n)_{n=1}^{\infty}$ be any fixed sequence such that

$$0 < p_n < 1 \quad \text{and} \quad p := \prod_1^{\infty} p_n > 0.$$

Now put $\lambda_n = 1/k$ if $n = n_k$ and $\lambda_n = p_n/2$ if no such k exists. Clearly there exists a continuous increasing function $h^* : [0,\infty) \to [0,\infty)$ such that $h^*(0) = 0$ and $h^*(\lambda_1 \cdots \lambda_n) = 2^{-n}$. By the above mentioned fact we have

$$1/4 \le \Lambda_{h^*}(C(\lambda)) \le 1.$$

To prove $\Lambda_h(C(\lambda)) = \infty$, by Theorem 40 of [8] it is suffices to establish that $\lim_{x\to 0+} \frac{h^*(x)}{h(x)} = 0$. To this end, consider $0 < x \le \lambda_1 \ldots \lambda_{n_1+1}$ and the corresponding index $n = n(x)$ for which $\lambda_1 \cdots \lambda_{n+1} < x \le \lambda_1 \cdots \lambda_n$. Since clearly $n > n_1$, there exists the unique index $k = k(x)$ such that
$n_k \le n < n_{k+1}$. Since $\lambda_1 \cdots \lambda_{n+1} \le 2^{-n_k} < \delta_k$, we obtain

$$\begin{aligned}\frac{h^*(x)}{h(x)} &\le \frac{2^{-n}}{h(\lambda_1 \cdots \lambda_{n+1})} \le \frac{2^{-n}}{(k+1)! \cdot \lambda_1 \cdots \lambda_{n+1}} \\ &\le \frac{2^{-n} \cdot (k+1)!}{(k+2)! \cdot p 2^{-(n+1)}} = \frac{2}{p(k+2)}.\end{aligned}$$

Since clearly $k(x) \to \infty$ when $x \to 0+$, we are done. □

4 A Symmetric Lipschitz Condition

In the first part of this section we show how the notes [12] and [2] give the following theorem.

Theorem 4.1. *For each function $f : \mathbb{R} \to \mathbb{R}$, the set $(SL(f) \setminus L(f)) \cap \overline{C(f)}$ is σ-strongly symmetrically porous.*

This theorem immediately implies, for example, the following result.

Proposition 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of Baire class one. Then the set of all points at which f fulfills the symmetric Lipschitz condition but does not fulfill the Lipschitz condition is σ-strongly symmetrically porous.*

Note that the above theorem is analogous to [11, Theorem 2] which asserts that, for each function $f : \mathbb{R} \to \mathbb{R}$, the set of all points at which f fulfills an one-sided Lipschitz condition but does not fulfill the Lipschitz condition is σ-strongly porous.

M. J. Evans in [2, Proposition 1] proved the following result.

Proposition 4.3. *For each function $f : \mathbb{R} \to \mathbb{R}$, the set $(SL(f) \cap \overline{C(f)}) \setminus C(f)$ is σ-strongly symmetrically porous.*

Thus to prove our Theorem 4.1 it is sufficient to prove that

$$(SL(f) \setminus L(f)) \cap C(f) \ \text{ is } \ \sigma\text{-strongly symmetrically porous.} \tag{7}$$

We will show that (7) easily follows from the following Lemma 4.4 which is essentially the main part of [12, Lemma 1].

Lemma 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $B > 0$ and $1 > \varepsilon > 0$. For a natural number m denote by S_m the set of all points $x \in \mathbb{R}$ at which $D^+ f(x) > B$ and*

$$\frac{f(x+h) - f(x-h)}{2h} < \frac{\varepsilon B}{8} \quad \textit{whenever} \quad 0 < h < \frac{1}{m}. \tag{8}$$

Then $S_m \cap C(f)$ is $(1-\varepsilon)-$symmetrically porous.

It is necessary to note that in the proof of [12, Lemma 1] it is only proved that S_m is $(1-\varepsilon)$-symmetrically porous for a continuous function f. However, as was pointed out and used in [2], the assumption of global continuity of f is not used in the proof and thus the conclusion of the above lemma holds.

To prove (7), suppose that a point $x \in M := (SL(f) \setminus L(f)) \cap C(f)$ is given. Then we can clearly find a natural number m such that

$$\left| \frac{f(x+h) - f(x-h)}{2h} \right| < m \quad \text{whenever} \quad 0 < h < \frac{1}{m}. \tag{9}$$

Thus, denoting by M_m the set of all $x \in M$ for which (9) holds, we see that $M = \bigcup_{m=1}^{\infty} M_m$ and that it is sufficient to prove that each M_m is σ-strongly symmetrically porous. Since $x \in L(f)$ clearly iff all four Dini derivates of f at x are finite, we have

$$\begin{aligned} M_m =& (M_m \cap \{x : D^+ f(x) = \infty\}) \cup (M_m \cap \{x : D_+ f(x) = -\infty\}) \\ &\cup (M_m \cap \{x : D^- f(x) = \infty\}) \cup (M_m \cap \{x : D_- f(x) = -\infty\}). \end{aligned}$$

Considering the functions $f(-x), -f(x)$ and $-f(-x)$ we easily see that it is sufficient to prove that the set

$$Z_m := M_m \cap \{x : D^+ f(x) = \infty\}$$

is strongly symmetrically porous. To this end choose an arbitrary $1 > \varepsilon > 0$ and find $B > 0$ such that $\varepsilon B/8 > m$. Then (9) and consequently also (8) is satisfied for each $x \in Z_m$. Since also $D^+ f(x) = \infty > B$ for each $x \in Z_m$, our Lemma 4.4 implies that Z_m is $(1-\varepsilon)$-symmetrically porous. Thus Z_m is 1-symmetrically porous, i.e. it is strongly symmetrically porous.

The second part of this section, which concerns the sets $SL(f) \setminus L(f)$ is analogical to Section 3 which deals with the sets $SD(f) \setminus D(f)$.

Proposition 4.5. *Let $F \subset \mathbb{R}$ be a bounded, closed, strongly symmetrically porous set. Then there exists a non-negative continuous function f such that $|f(x)| \leq 1$ for every $x \in \mathbb{R}$, f is a C^1 function on $\mathbb{R} \setminus F$ and, for every $x \in F$, we have $f(x) = 0$, $f'_s(x) = 0$ and $D^+ f(x) = \infty$. In particular, $SD(f) = SL(f) = \mathbb{R}$ and $F = \mathbb{R} \setminus L(f)$.*

Proof. Put $d_n := 2n^3$ and apply Lemma 2.2 to F and $c_n := 4(d_n)^2$. The resulting I-systems $(\mathcal{I}_n)_{n=0}^{\infty}$ obviously satisfy the assumptions (i)-(iii) of Lemma 2.4 for $\alpha = 1/2$. Then the function f from the assertion of Lemma 2.4 has clearly all required properties. □

Now we can simply prove an analogy of Proposition 3.3 on symmetric perfect sets.

Proposition 4.6. *Let $C = C(\lambda)$ be a symmetric perfect set. Then the following statements are equivalent.*

(i) $\liminf \lambda_n = 0$.

(ii) *There exists a continuous symmetrically differentiable function f which is C^1 on $\mathbb{R} \setminus C$ and $D^+ f(x) = \infty$ for each $x \in C$.*

(iii) *There exists a function f on $\mathbb{R}$ such that $C \subset (SL(f) \setminus L(f)) \cap \overline{C(f)}$.*

Proof. If (i) holds then C is strongly symmetrically porous by Theorem 5 of [3] and thus Proposition 4.5 implies (ii). The implication $(ii) \Rightarrow (iii)$ is trivial. The implication $(iii) \Rightarrow (i)$ can be easily proved, using Theorem 4.1 and Theorem 3 of [3] and imitating the proof of the implication $(iii) \Rightarrow (i)$ of Proposition 3.3. □

Quite similarly as in Proposition 3.4, we easily see that Proposition 4.6 implies that (for a continuous f) the Lebesgue null set $SD(f) \setminus L(f)$ (and thus also $SL(f) \setminus L(f)$) need not be small in any reasonable stronger (Hausdorff) measure sense.

Proposition 4.7. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing continuous function with $h(0) = 0$ and $h'_+(0) = \infty$. Then there exist a symmetric perfect set C and a continuous symmetrically differentiable function f such that $\Lambda_h(C) = \infty$, f is C^1 on $\mathbb{R} \setminus C$ and $C = \mathbb{R} \setminus L(f)$.*

Theorem 4.8. *Let $A \subset \mathbb{R}$ be written in the form $A = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed and strongly symmetrically porous. Then there exists a continuous function g on $\mathbb{R}$ such that, for every $x \in A$, $g'_s(x) \in \mathbb{R}$ exists but g is not Lipschitz at x; in particular $A \subset SL(g) \setminus L(g)$.*

Proof. We may suppose that each set F_n is bounded. For each n, let $f = f_n$ be a function which corresponds to $F = F_n$ by Proposition 4.5. It is easy to see that, for each $n \in \mathbb{N}$, there exists a closed (even discrete) set $D_n \subset \mathbb{R}$ such that $D_n \cap A = \emptyset$ and the distance function

$$d_n(x) := \operatorname{dist}(x, F_1 \cup \cdots \cup F_n \cup D_n)$$

is bounded by 1. Now put

$$g_1 := f_1, \; g_n := n^{-2} f_n (d_{n-1})^2 \text{ for } n > 1 \text{ and } g := \sum_{n=1}^{\infty} g_n.$$

The function g is clearly continuous on $\mathbb{R}$.

Now let $x \in A$ be given and let k be a natural number with $x \in F_k$ and $x \notin F_n$ for each $n < k$. Observe that each d_n has clearly finite both one-sided derivatives, and therefore a finite symmetric derivative, at any point $y \notin F_1 \cup \cdots \cup F_n \cup D_n$. The same property is satisfied also for functions $(d_n)^2$, which are clearly also bounded by 1 and Lipschitz on $\mathbb{R}$. By the above observation, the function $\sum_{n<k} g_n$ is Lipschitz on $\mathbb{R}$ and has a finite symmetric derivative at x.

Now denote $s := \sum_{n=k+1}^{\infty} g_n$. For every $n \geq k+1$, clearly $g_n(x) = d_{n-1}(x) = 0$ and $|g_n(x+h)| \leq n^{-2}(d_{n-1}(x+h))^2 \leq n^{-2}h^2$ for every $h \in \mathbb{R}$.

Consequently, for every $h \neq 0$,

$$\left|\frac{s(x+h)-s(x)}{h}\right| \leq |h|^{-1} \sum_{n=k+1}^{\infty} n^{-2}h^2 = |h| \sum_{n=k+1}^{\infty} n^{-2}.$$

Thus $s'(x) = 0$. Since both f_k and $(d_{k-1})^2$ have a finite symmetric derivative at x, we conclude that g_k and g have finite symmetric derivatives at x.

On the other hand, g_k is not Lipschitz at x. In fact, suppose that g_k is Lipschitz at x. Then, since we have observed that the function $(d_{k-1})^2$ is Lipschitz at x and $d_{k-1}(x) \neq 0$, we easily conclude that also $f_k = k^2(d_{k-1})^{-2}g_k$ is Lipschitz at x, a contradiction. Since both $\sum_{n<k} g_n$ and s are Lipschitz at x, we obtain that g is not Lipschitz at x. $\square$

Remark 4.9. If A in Theorem 4.8 is not nowhere dense, no corresponding function g is symmetrically Lipschitz at all points. In fact, suppose that f is a continuous function on $\mathbb{R}$ and $SL(f) = \mathbb{R}$. Put

$$S_n := \{x \in \mathbb{R} : \frac{|f(x+h)-f(x-h)|}{2h} \leq n \ \text{ whenever } \ 0 < h < \frac{1}{n}\}.$$

Then clearly $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$ and the continuity of f easily implies that all S_n are closed. Thus the Baire category theorem easily gives that each interval I contains a subinterval J which is contained in an S_m; it easily implies that f is Lipschitz on J. Therefore $\mathbb{R} \setminus L(f)$ is nowhere dense.

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