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UNIFORMLY ANTISYMMETRIC FUNCTIONS WITH BOUNDED RANGE

Abstract

The goal of this note is to construct a uniformly antisymmetric function $f: \mathbb{R} \rightarrow \mathbb{R}$ with a bounded countable range. This answers Problem 1(b) of Ciesielski and Larson [6]. (See also the list of problems in Thomson [9] and Problem 2(b) from Ciesielski's survey [5].) A problem of existence of uniformly antisymmetric function $f: \mathbb{R} \rightarrow \mathbb{R}$ with finite range remains open.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be *uniformly antisymmetric* [6] (or *nowhere weakly symmetrically continuous* [9]) provided for every $x \in \mathbb{R}$ the limit $\lim_{n \rightarrow \infty} (f(x + s_n) - f(x - s_n))$ equals 0 for no sequence $\{s_n\}_{n < \omega}$ converging to 0. Uniformly antisymmetric functions have been studied by Kostyrko [7], Ciesielski and Larson [6], Komjáth and Shelah [8], and Ciesielski [1, 2]. (A connection of some of these results to the paradoxical decompositions of the Euclidean space \mathbb{R}^n is described in Ciesielski [3].) In particular in [6] the authors constructed a uniformly antisymmetric function $f: \mathbb{R} \rightarrow \mathbb{N}$ and noticed that the existence of a uniformly antisymmetric function cannot be proved without an essential use of the axiom of choice.

The terminology and notation used in this note is standard and follows [4]. In particular for a set X we will write $|X|$ for its cardinality and $\mathcal{P}(X)$ for its

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power set. Also 2^ω will stand for the set of all functions from $\omega = \{0, 1, 2, \dots\}$ into $2 = \{0, 1\}$. We consider 2^ω as ordered lexicographically.

Theorem 1. *There exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ with countable bounded range such that for every $x \in \mathbb{R}$ there exists an $\varepsilon_x > 0$ with the property that the set*

$$S_x = \{s \in \mathbb{R}: |f(x-s) - f(x+s)| < \varepsilon_x\}$$

is finite. In particular f is uniformly antisymmetric.

PROOF. First notice that it is enough to find a compact zero-dimensional metric space $\langle T, d \rangle$ and a function g from \mathbb{R} into a countable subset T_0 of T such that for every $x \in \mathbb{R}$ there is a $\delta_x > 0$ for which the set

$$\hat{S}_x = \{s \in \mathbb{R}: d(g(x-s), g(x+s)) < \delta_x\}$$

is finite.

To see this assume that such a function $g: \mathbb{R} \rightarrow T$ exists and take a homeomorphic embedding h of T into \mathbb{R} . We claim that $f = h \circ g: \mathbb{R} \rightarrow \mathbb{R}$ is as desired. Indeed, $f[\mathbb{R}] = h[g[\mathbb{R}]]$ is countable, as it is a subset of a countable set $h[T_0]$, and it is bounded, since it is a subset of a compact set $h[T]$. So take $x \in \mathbb{R}$ and $\delta_x > 0$ for which \hat{S}_x is finite. Since $h^{-1}: h[T] \rightarrow T$ is uniformly continuous, we can find an $\varepsilon_x > 0$ such that

$$|y_1 - y_2| < \varepsilon_x \text{ implies } d(h^{-1}(y_1), h^{-1}(y_2)) < \delta_x$$

for every $y_1, y_2 \in h[T]$. But for such a choice of ε_x we have

$$S_x = \{s \in \mathbb{R}: |h(g(x-s)) - h(g(x+s))| < \varepsilon_x\} \subset \hat{S}_x$$

proving that S_x is finite.

Thus, we proceed to construct a function g described above. The value of $g(x)$ will be defined with help of a representation of x in a Hamel basis; i.e., a linear basis of \mathbb{R} over \mathbb{Q} . For this we will use the following notation. Let $\{y_\eta: \eta \in 2^\omega\}$ be a one-to-one enumeration of a Hamel basis \mathcal{H} . For every $x \in \mathbb{R}$ let $\sum_{\eta \in 2^\omega} q_{x,\eta} y_\eta$, with $q_{x,\eta} \in \mathbb{Q}$ for $\eta \in 2^\omega$, be the unique representation of x in basis \mathcal{H} and let $w_x = \{\eta \in 2^\omega: q_{x,\eta} \neq 0\}$. Thus w_x is finite and

$$x = \sum_{\eta \in w_x} q_{x,\eta} y_\eta.$$

The definition of the space T is considerably more technical since it reflects several different cases of the proof that the sets \hat{S}_x are indeed finite. To this

end let $\{q_j: j < \omega\}$ be a one-to-one enumeration of \mathbb{Q} with $q_0 = 0$. For $i < \omega$ let $\mathcal{P}_i = \mathcal{P}(\{q_j: j < i\})$ and put $P_i = \mathcal{P}(2^i \times \{0, 1\} \times \mathcal{P}_i \times \mathcal{P}_i)$. Note that each P_i is finite; so $T = \prod_{i < \omega} P_i$, considered as the standard product of discrete spaces, is compact zerodimensional. We equip T with a distance function d defined between different $s, t \in T$ by $d(s, t) = 2^{-\min\{i < \omega: s(i) \neq t(i)\}}$ and let

$$T_0 = \{t \in T: (\exists n < \omega)(\forall i \geq n) t(i) = \emptyset\}.$$

Clearly T_0 is countable.

Now we are ready to define $g: \mathbb{R} \rightarrow T_0 \subset T$. For this, however, we will need few more definitions. For $x \in \mathbb{R}$, $q \in \mathbb{Q}$, $i < \omega$, and $\zeta \in 2^i$ such that $\zeta \in \{(\eta \upharpoonright i): \eta \in w_x\}$ we define:

- $p(i) \in \{0, 1\}$ as the parity of i ; i.e., $p(i) = i \bmod 2$;
- $k_i(q) = \{q_j \in \mathbb{Q}: q_j < q \ \& \ j < i\} \in \mathcal{P}_i$;
- $\eta(x, \zeta)$ to be the minimum of $\{\eta \in w_x: \zeta \subset \eta\}$ (in the lexicographical order);
- $\xi(x, \zeta)$ to be the minimum of $\{\eta \in w_x: \zeta \subset \eta\} \setminus \{\eta(x, \zeta)\}$ provided $|\{\eta \in w_x: \zeta \subset \eta\}| \neq 1$; otherwise we put $\xi(x, \zeta) = \eta(x, \zeta)$;
- $n_x < \omega$ to be the smallest number $n > 0$ such that
 - (i) $\eta \upharpoonright n \neq \xi \upharpoonright n$ for any different $\eta, \xi \in w_x$, and
 - (ii) $q_{x,\eta} \in \{q_j: j < n\}$ for every $\eta \in w_x$.

Consider the function $g: \mathbb{R} \rightarrow T_0$ defined as follows. For every $x \in \mathbb{R}$ and $i < \omega$ we define $g(x)(i) \in P_i$ as

$$\{\langle \zeta, p(|\{\eta \in w_x: \zeta \subset \eta\}|), k_i(q_{x,\eta(x,\zeta)}), k_i(q_{x,\xi(x,\zeta)}) \rangle: \zeta \in \{(\eta \upharpoonright i): \eta \in w_x\}\}$$

provided $i \leq n_x$ and we put $g(x)(i) = \emptyset$ for $n_x < i < \omega$. In the argument below the key role will be played by the function k_i in general, and the coordinate $k_i(q_{x,\eta(x,\zeta)})$ in particular.

The key step in the proof that g has the desired property is that for every $x \in \mathbb{R}$ and $s \neq 0$

$$\text{if } n_x \leq \max\{n_{x-s}, n_{x+s}\} \text{ then } g(x-s)(n_x) \neq g(x+s)(n_x). \quad (1)$$

To see (1) assume that $n_x \leq n_{x+s}$. If $n_{x-s} < n_x$, then $g(x-s)(n_x) = \emptyset \neq g(x+s)(n_x)$, where $g(x+s)(n_x) \neq \emptyset$ since $w_{x+s} \neq \emptyset$ as $n_{x-s} < n_x \leq n_{x+s}$ implies $x+s \neq 0$. Thus, we can assume that $n_x \leq \min\{n_{x-s}, n_{x+s}\}$. Take an

$\hat{\eta} \in w_{x-s} \cup w_{x+s}$ such that $q_{x-s,\hat{\eta}} \neq q_{x+s,\hat{\eta}}$ and let $\zeta = \hat{\eta} \upharpoonright n_x$. Note that, by the definition of n_x , the set $S = \{\eta \in w_x : \zeta \subset \eta\}$ has at most one element.

If $S = \emptyset$, then $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \neq \emptyset$ and so $\eta(x-s, \zeta) = \eta(x+s, \zeta) \notin w_x$ while $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$. Thus $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$ implying that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$. Therefore $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, assume that $S \neq \emptyset$ and let η' be the only element of S . Then $\eta' \in w_{x-s} \cup w_{x+s}$. If η' belongs to precisely one of the sets w_{x+s} and w_{x-s} , say w_{x+s} , then $\{\eta \in w_{x+s} : \zeta \subset \eta\} = \{\eta \in w_{x-s} : \zeta \subset \eta\} \cup \{\eta'\}$. In particular, $p(|\{\eta \in w_{x+s} : \zeta \subset \eta\}|) \neq p(|\{\eta \in w_{x-s} : \zeta \subset \eta\}|)$ implying that $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, we can assume that $\eta' \in w_{x-s} \cap w_{x+s}$. Then $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\}$ and $\eta(x-s, \zeta) = \eta(x+s, \zeta)$. We will consider three cases.

CASE 1: $\eta' \neq \eta(x-s, \zeta) = \eta(x+s, \zeta)$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$; so $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 2: $\eta' = \eta(x-s, \zeta) = \eta(x+s, \zeta)$ and $q_{x-s,\eta(x-s,\zeta)} \neq q_{x+s,\eta(x+s,\zeta)}$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 2q_{x,\eta'}$ and, by the definition of n_x , $q_{x,\eta'} \in \{q_j : j < n_x\}$. Since $q_{x,\eta'}$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$, we conclude that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 3: $\eta' = \eta(x-s, \zeta) = \eta(x+s, \zeta)$ and $q_{x-s,\eta(x-s,\zeta)} = q_{x+s,\eta(x+s,\zeta)}$. Then $Z = \{\eta \in w_{x-s} : \zeta \subset \eta\} \setminus \{\eta(x-s, \zeta)\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \setminus \{\eta(x+s, \zeta)\}$ is non-empty, since it contains $\hat{\eta}$, and so $\xi(x-s, \zeta) = \xi(x+s, \zeta) \notin w_x$. Therefore, as in Case 1, $q_{x-s,\xi(x-s,\zeta)} + q_{x+s,\xi(x+s,\zeta)} = 0$; so $q_0 = 0$ separates $q_{x-s,\xi(x-s,\zeta)}$ and $q_{x+s,\xi(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\xi(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\xi(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

This finishes the proof of (1).

Next, for every $x \in \mathbb{R}$ put $\delta_x = 2^{-n_x}$. To finish the proof of the theorem it is enough to show that every \hat{S}_x defined for such a choice of δ_x is a subset of a finite set

$$Z_x = \left\{ s \in \mathbb{R} : w_{x+s} \subset w_x \ \& \ n_{x+s} < n_x \right\} = \left\{ \sum_{\eta \in w_x} p_\eta y_\eta : p_\eta \in \{q_j : j < n_x\} \right\}.$$

Indeed, take an $s \in \hat{S}_x$. Then, by (1) and the definition of the distance function d , we have $\max\{n_{x-s}, n_{x+s}\} < n_x$. Notice also that if $n_{x-s} \neq n_{x+s}$, say $n_{x-s} < n_{x+s}$, then $g(x-s)(n_{x+s}) = \emptyset \neq g(x+s)(n_{x+s})$ implying that $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$, which contradicts $s \in \hat{S}_x$. So, we have $n_{x-s} = n_{x+s}$. To prove that $s \in Z_x$ it is enough to show that

$w_{x+s} \subset w_x$. But if it is not the case, then there exists an $\eta \in w_{x+s} \setminus w_x$. Moreover, $q_{x+s,\eta} = -q_{x-s,\eta} \neq 0$ and $\eta = \eta(x+s, \zeta) = \eta(x-s, \zeta)$, where $\zeta = \eta \upharpoonright n_{x+s}$. In particular, $q_0 = 0$ separates $q_{x+s,\eta(x+s,\zeta)}$ and $q_{x-s,\eta(x-s,\zeta)}$. Therefore $k_{n_{x+s}}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_{x+s}}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_{x+s}) \neq g(x+s)(n_{x+s})$. So $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$ again contradicting $s \in \hat{S}_x$. Thus, $w_{x+s} \subset w_x$ and $s \in Z_x$. \square

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