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FINITELY CONTINUOUS, DARBOUX FUNCTIONS

Abstract

Some properties of finitely continuous functions are investigated. In particular we show that, in the class of Darboux functions, the family of 2-continuous functions is the same as the family B_1^{**} and the set of all discontinuity points of finitely continuous, Darboux functions is nowhere dense.

Our terminology is standard. By $\mathbb{R}$ we denote set of all real numbers. If X is a metric space and $A \subset X$, then $card(A), (der(A))$ stand for the cardinality (derivative respectively) of A. The cardinality of the set of natural numbers is denoted by ω. We consider only real-valued functions. If X is a metric space and $f \in \mathbb{R}^X$ then by $C_f(D_f)$ we denote the set of all continuity (discontinuity) points of the function f. The function f is said to be Darboux if, for every connected set $A \subset X$ the image $f(A)$ is a connected subset of $\mathbb{R}$ (i.e.,an interval). The class $\mathcal{F} \subset \mathbb{R}^X$ is an ordinary system (in the sense of Aumann) if $\mathcal{F}$ contains all constants and for $f, g \in \mathcal{F}$, $max f, g \in \mathcal{F}, min f, g \in \mathcal{F}, f+g \in \mathcal{F}, f \cdot g \in \mathcal{F}$ and (if $\{x : g(x) = 0\} = \emptyset$) $\frac{f}{g} \in \mathcal{F}$. The class $\mathcal{F}$ is a complete system (in the sense of Aumann) if $\mathcal{F}$ is an ordinary system and uniform limit of a sequence of functions from $\mathcal{F}$ belongs to $\mathcal{F}$.

Let $\mathcal{A}$ be a covering of a metric space X (i.e. $\bigcup \mathcal{A} = X$). The function $f \in \mathbb{R}^X$ is said to be $\mathcal{A} -$ *continuous* if, for all $A \in \mathcal{A}$, the restriction $f \restriction A$ is continuous. The function $f \in \mathbb{R}^X$ is said to be *n-continuous* (*finitely continuous*, *countable continuous*) if there exists a covering $\mathcal{A}$ of X such that $card(\mathcal{A}) = n$ $(card(\mathcal{A}) < \omega$, $card(\mathcal{A} \leq \omega)$ and f is $\mathcal{A}$-continuous.

R. Pawlak in [5] introduced the notions of the class functions B_1^{**}- intermediate between the family of continuous functions and the class Baire*1 functions. We say that the function f belongs to the class B_1^{**} if either $D_f = \emptyset$ or $f \restriction D_f$ is continuous function.

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Theorem 1. *Let X be a locally connected metric space. If $f \in \mathbb{R}^X$ is a Darboux function, then f is 2-continuous if and only if f belongs to the class B_1^{**}.*

PROOF. If f is Darboux and belongs to the class B_1^{**} then f is of course 2-continuous, since $f \restriction C_f$ is a continuous function.

Now we suppose that f is Darboux, $X = A \cup B$ and $f \restriction A$, $f \restriction B$ are continuous. We can assume that $D_f \neq \emptyset$. Fix $x \in D_f$. Let for instance $x \in A$. Let $\epsilon > 0$. There is a connected, open neighborhood U of x such that $f(U \cap A) \subset (f(x) - \frac{\epsilon}{2}, f(x) + \frac{\epsilon}{2})$. If $U \cap B \subset C_f$ then $U \cap D_f \subset U \cap A$ and consequently $f(U \cap D_f) \subset (f(x) - \epsilon, f(x) + \epsilon)$. Thus, we can assume that $U \cap D_f \cap B \neq \emptyset$. Let $b \in U \cap D_f \cap B$. There is a connected, open neighborhood $U_1 \subset U$ of b, such that $f(U_1 \cap B) \subset (f(b) - \frac{\epsilon}{2}, f(b) + \frac{\epsilon}{2})$. Since $f \restriction B$ is continuous and $b \in D_f$, then b is an accumulation point of A. So, consequently $U_1 \cap A \neq \emptyset \neq U_1 \cap B$. According to the Darboux property of f, we may infer that $f(U_1)$ is a connected set. So, $(f(x) - \frac{\epsilon}{2}, f(x) + \frac{\epsilon}{2}) \cap (f(b) - \frac{\epsilon}{2}, f(b) + \frac{\epsilon}{2}) \neq \emptyset$. Thus $f(b) \in (f(x) - \epsilon, f(x) + \epsilon)$. This means that $f(U \cap D_f) \subset (f(x) - \epsilon, f(x) + \epsilon)$. We conclude that $f \restriction D_f$ is continuous at x. □

Remark 1. *There is a 3-continuous, Darboux function that is not in the first class of Baire.*

PROOF. Let C be the ternary Cantor set in $[0, 1]$. Let $\{\ (a_n, b_n) : n = 1, 2, \dots\}$ be the family of all components of $[0, 1] \setminus C$. Define the function $f \in \mathbb{R}^{[0,1]}$ by $f(a_n) = 0$, $f(b_n) = 1$ and linear on $[a_n, b_n]$, for $n = 1, 2, \dots$. Otherwise let $f(x) = 0$. Then f is Darboux function. By Lemma 1 from [3] the function f is not almost continuous in the sense of Stallings, so it is not Baire 1. If $A_1 = [0, 1] \setminus C$, $A_2 = \{b_n : n = 1, 2, \dots\}$, $A_3 = C \setminus A_2$, then $f \restriction A_i$ is a continuous function, for $i = 1, 2, 3$. □

R. Pawlak in [5] proved that if f belongs to the class B_1^{**} then the set of all discontinuity points of f is nowhere dense. Moreover this is true also in case when f is a finitely continuous Darboux function:

Theorem 2. *Let X be a locally connected space. If $f \in \mathbb{R}^X$ is a finitely continuous, Darboux function, then the set of all discontinuity points of f is nowhere dense.*

PROOF. Let $f \in \mathbb{R}^X$ be a Darboux function and $\mathcal{A} = \{A_1, \dots A_k\}$ be a finite covering of X, such that $f \restriction A$ is continuous function, for $A \in \mathcal{A}$. Let $G \subset X$ be a nonempty open set. We shall show that there is a nonempty open set $U \subset G$, such that f is continuous on U.

This fact is obvious if G contains some isolated point. So, we assume that G contains no isolated points of X.

Observe, that there is an open, nonempty set U, such that (for $i = 1, \dots, k$) if $U \cap A_i \neq \emptyset$ then $U \subset der(A_i)$. Indeed, let $V_0 = G$. If $V_0 \subset der(A_1)$ let $V_1 = V_0$, otherwise exists an open nonempty set $V_1 \subset V_0$ such that $V_1 \cap A_1 = \emptyset$. Similarly, if $V_1 \subset der(A_2)$ let $V_2 = V_1$, otherwise let $V_2 \subset V_1$ be an open nonempty set, such that $V_2 \cap A_2 = \emptyset$. In this way we show, that there are nonempty open sets $V_1, \dots V_k$ such that, for $i = 1, \dots k$, $V_i \subset V_{i-1}$ and if $V_i \cap A_i \neq \emptyset$ then $V_i \subset der(A_i)$. Now we can take $U = V_k$.

Let $\mathcal{B} = \{A \in \mathcal{A} : U \cap A \neq \emptyset\}$ and $n = card(\mathcal{B})$. If $n \leq 2$, then by Theorem 1 the function $f \restriction U$ belongs to the class $B_1^{**}(U, \mathbb{R})$. Thus by Lemma 2 in [5] the set of discontinuity points of $f \restriction U$ is nowhere dense. Since U is an open nonempty set, then there exists an open nonempty set $W \subset U \subset G$ such that f is continuous on W.

We also can assume that $n > 2$. We shall now prove that f is continuous on U. Let $x \in U$ and $\epsilon > 0$. There is $B_1 \in \mathcal{B}$, such that $x \in B_1$. Let $W_1 \subset U$ be an open connected neighborhood of x such that $f(W_1 \cap B_1) \subset (f(x) - \frac{\epsilon}{2n}, f(x) + \frac{\epsilon}{2n})$. We shall show that $f(W_1) \subset (f(x) - \epsilon, f(x) + \epsilon)$.

Suppose, to the contrary, that there is $t \in W_1$ such that $f(t) \notin (f(x) - \epsilon, f(x) + \epsilon)$. Without loss of generality we may assume that $f(t) \leq f(x) - \epsilon$. There is $B_2 \in \mathcal{B}$, such that $t \in B_2$. Let $W_2 \subset W_1$ be a open connected neighborhood of t such that $f(W_2 \cap B_2) \subset (f(t) - \frac{\epsilon}{2n}, f(t) + \frac{\epsilon}{2n})$.

Observe, that if $V \subset U$ is an nonempty open set and $B \in \mathcal{B}$, then $V \cap B \neq \emptyset$. In particular $B_1 \cap W_2 \neq \emptyset \neq B_2 \cap W_2$. By Darboux property of f we can found $s_3 \in W_2$ such that $f(s_3) = f(x) - \frac{2\epsilon}{n}$. Let $B_3 \in \mathcal{B}$ be such that $s_3 \in B_3$ and $W_3 \subset W_2$ be a nonempty connected neighborhood of s_3 such that $f(B_3 \cap W_3) \subset (f(s_3) - \frac{\epsilon}{2n}, f(s_3) + \frac{\epsilon}{2n})$. Similarly (continuing this process) if $3 < i \leq n$ then $W_{i-1} \cap B_1 \neq \emptyset \neq W_{i-1} \cap B_2$. We can also found $s_i \in W_{i-1}$ such that $f(s_i) = f(x) - \frac{(i-1)\epsilon}{n}$. Let B_i be such that $s_i \in B_i \in \mathcal{B}$, $W_i \subset W_{i-1}$ be a nonempty connected neighborhood of s_i such that $f(B_i \cap W_i) \subset (f(s_i) - \frac{\epsilon}{2n}, f(s_i) + \frac{\epsilon}{2n})$.

We obtain the points $s_3, \dots s_n$ and sets $W_3, \dots W_n$, $B_3, \dots B_n$, such that $f(s_i) = f(x) - \frac{(i-1)\epsilon}{n}$, W_i is an open connected neighborhood of s_i, $W_i \subset W_{i-1}$, $s_i \in B_i \in \mathcal{B}$, $f(W_i \cap B_i) \subset (f(s_i) - \frac{\epsilon}{2n}, f(s_i) + \frac{\epsilon}{2n})$.

Observe, that $f(W_i \cap B_i)$ are nonempty and pairwise disjoint. This means that $B_i \neq B_j$ for $i \neq j$, $i, j \in \{1, \dots n\}$ so $\{B_1, \dots B_n\} = \mathcal{B}$.

Let $H = W_n$. Then H is an open, nonempty, connected set, $B_1 \cap H \neq \emptyset \neq B_2 \cap H$, $f(H) = f(H \cap \bigcup \mathcal{B}) \subset \bigcup_{i=1}^{n} f(W_i \cap B_i) \subset (-\infty, f(x) - \frac{3\epsilon}{2n}) \cup (f(x) - \frac{\epsilon}{2n}, +\infty)$. Observe that $B_1 \cap H \neq \emptyset \neq B_2 \cap H$, also $f(H) \cap (-\infty, f(x) - \frac{3\epsilon}{2n}) \neq \emptyset \neq f(H) \cap (f(x) - \frac{\epsilon}{2n}, +\infty)$. Thus $f(H)$ is not connected. This contradicts Darboux property of f. □

Proposition 1. *The family of finitely continuous, real valued functions is an ordinary system in the sense of Aumann.*

PROOF. If f is $\mathcal{A}$ continuous and g is $\mathcal{B}$ continuous, where $\mathcal{A}$, $\mathcal{B}$ are finite covering of X then $\mathcal{C} = \{A \cap B : A \in \mathcal{A} \wedge B \in \mathcal{B}\}$ is a finite covering of X and $min(f, g), max(f, g)$, $f + g$, $f \cdot g$, $\frac{f}{g}$ (when $\{x : g(x) = 0\} = \emptyset$), are $\mathcal{C}$-continuous. □

Remark 2. *The family of finitely continuous, real valued functions is not an complete system in the sense of Aumann.*

PROOF. The uniform limit of finitely continuous functions may not be finitely continuous. In fact, if $f \in \mathbb{R}^{\mathbb{R}}$ is bounded Lebesgue measurable function then it is uniform limit of simple functions [2, Theorem 4.19] On the other hand family of all bounded functions of first Baire class contains a dense G_δ subset consisting of functions of Sierpiński-Adjan-Nowikow type which are even not countably continuous [4, Theorem 4.3. and Lemma 4.1]. □

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