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ITERATIVE ROOTS WITH BIG GRAPH

Abstract

Let $g : X \to X$ be a bijection and $n \geq 2$ be a fixed integer. We consider the equation of iterative roots $\varphi^n(x) = g(x)$ and we look for its solution with big graph: big from the point of view both of topology and measure theory.

1 Introduction

Let X and Y be two sets and $\mathcal{R}$ be a family of subsets of $X \times Y$. We say that $\varphi : X \to Y$ has a *big graph* with respect to $\mathcal{R}$ if the graph $\mathrm{Gr}\varphi$ of φ meets every set of $\mathcal{R}$. We are interested in finding conditions under which the functional equation of iterative roots

$$\varphi^n(x) = g(x) \tag{1}$$

has a solution with big graph with respect to a sufficiently large family. Well known results on additive functions with big graph are due to F. B. Jones [6] (see also [12]). Solutions with big graph for some iterative functional equation were obtained in [8], [16], [1]–[4].

2 Main Result

Let X be a nonempty set, $g : X \to X$ be a given bijection (one-to-one and onto) and $n \geq 2$ be an integer. We start with recalling the well known theorem of S. Łojasiewicz ([15], [11], [19]) concerning the iterative roots. To formulate it, for every positive integer k let L_k denote the (cardinal) number of k-cycles of g and L_0 denote the number of infinite orbits of g. Note that any infinite

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orbit of g is simply the sequence of the form $(g^k(x): \; k \in \mathbb{Z})$ and every finite orbit of g is a cycle. Put $d_0 = n$ and $d_k = \frac{n}{n_k}$ for $k \in \mathbb{N} \setminus \{0\}$, where n_k is the largest divisor of n relative prime to k.

Theorem (S. Łojasiewicz). *Equation* (1) *has a solution* $\varphi : X \to X$ *iff for every nonnegative integer* k *either* L_k *is infinite or* L_k *is divisible by* d_k*.*

We are interested in finding special solutions of equation (1) and it is obvious that our assumptions must be stronger than Łojasiewicz ones: they read as follows.

(H_1) The set X is uncountable and $g : X \to X$ is a bijection.

(H_2) There exists a $k_0 \in \mathbb{N}$ such that

$$\mathrm{card} X = L_{k_0}, \tag{2}$$

$$\sum_{k \neq k_0} L_k < \mathrm{card} X \tag{3}$$

and for $k \neq k_0$ either L_k is infinite or L_k is divisible by d_k.

For any set $R \subset X \times X$ and $y \in X$, R^y denotes horizontal section of R, i.e., the set $\{x \in X : (x, y) \in R\}$. The following is the main result of this paper.

Theorem 1. *Assume* (H_1), (H_2) *and let* $\mathcal{R}$ *be a family of subsets of* $X \times X$ *such that*

$$\mathrm{card} \mathcal{R} \leq \mathrm{card} X \tag{4}$$

and

$$\mathrm{card}\{y \in X : \mathrm{card} R^y = \mathrm{card} X\} = \mathrm{card} X \textit{ for } R \in \mathcal{R}. \tag{5}$$

Then there exists a solution $\varphi : X \to X$ *of* (1) *which has a big graph with respect to* $\mathcal{R}$.

PROOF. We start with some notations. The set of all *periodic* points of g with *period* p will be denoted by $\mathrm{Per}(g, p)$, i.e.,

$$\mathrm{Per}(g, p) = \{x \in X : g^p(x) = x, \; g^k(x) \neq x \text{ for } k = 1, \dots, p-1\};$$

moreover we put $\mathrm{Per} g = \bigcup_{p=1}^{\infty} \mathrm{Per}(g, p)$. Let

$$A = \begin{cases} \mathrm{Per}(g, k_0), & \text{if } k_0 \neq 0, \\ X \setminus \mathrm{Per} g, & \text{if } k_0 = 0, \end{cases}$$

where $k_0 \in \mathbb{N}$ satisfies the requirements stated in (H_2). According to (2) and (H_1) we have

$$\mathrm{card}A = \begin{cases} k_0 \cdot L_{k_0}, & \text{if } k_0 \neq 0, \\ \aleph_0 \cdot L_0, & \text{if } k_0 = 0, \end{cases} = L_{k_0} = \mathrm{card}X. \tag{6}$$

Hence and from (3) for $A_{-1} = X \setminus A$ we get

$$\mathrm{card}A_{-1} < \mathrm{card}X. \tag{7}$$

Applying Łojasiewicz's theorem to the function $g|_{A_{-1}}$ we obtain a function $\varphi_{-1} : A_{-1} \to A_{-1}$ such that $\varphi_{-1}^n = g|A_{-1}$.

Let γ be the smallest ordinal such that its cardinal $\bar{\gamma}$ equals that of $\mathcal{R}$ and let $(R_\alpha : \alpha < \gamma)$ be a one-to-one transfinite sequence of all the elements of $\mathcal{R}$. Similarly, let δ be the smallest ordinal with $\bar{\delta} = \mathrm{card}A$ and $(x_\alpha : \alpha < \delta)$ be a one-to-one sequence of all the elements of A. According to (4) and (6), $\gamma \leq \delta$. Define now a sequence $(A_\alpha : \alpha < \delta)$ of countable subsets of A and a sequence $(\varphi_\alpha : \alpha < \delta)$ of functions $\varphi_\alpha : A_\alpha \to A_\alpha$ such that for every $\alpha < \delta$ the following conditions (8)-(11) hold:

$$g(A_\alpha) = A_\alpha, \tag{8}$$

$$\varphi_\alpha^n = g|_{A_\alpha}, \tag{9}$$

$$A_\beta \cap A_\alpha = \emptyset \text{ for } \beta < \alpha, \tag{10}$$

$$\{x_\beta : \beta \leq \alpha\} \subset \bigcup_{\beta \leq \alpha} A_\beta, \tag{11}$$

and, for every $\alpha < \gamma$,

$$\mathrm{Gr}\varphi_\alpha \cap R_\alpha \neq \emptyset. \tag{12}$$

Suppose $\alpha < \delta$ and that we have already defined suitable A_β's and φ_β's for every $\beta < \alpha$. According to (H_1) we have

$$\mathrm{card}(\bigcup_{\beta<\alpha} A_\beta) \leq \bar{\alpha} \cdot \aleph_0 = \max\{\aleph_0, \bar{\alpha}\} < \bar{\delta} = \mathrm{card}A. \tag{13}$$

Let $\eta < \delta$ be the first ordinal such that $x_\eta \notin \bigcup_{\beta<\alpha} A_\beta$. Then $\eta \geq \alpha$. Let C_1 be the orbit generated by x_η, i.e., $C_1 = \{g^k(x_\eta) : k \in \mathbb{Z}\}$. Since $g(\bigcup_{\beta<\alpha} A_\beta) = \bigcup_{\beta<\alpha} A_\beta$, we have $\bigcup_{\beta<\alpha} A_\beta$ and C_1 disjoint, and according to (13) we can choose $(n-1)$ different orbits $C_2, \dots, C_n$ disjoint with $\bigcup_{\beta<\alpha} A_\beta \cup C_1$. Note that all the orbits $C_1, \dots, C_n$, as the subset of A, are simultaneously either k_0–cycles (if $k_0 \neq 0$) or infinite (if $k_0 = 0$). Put $C = \bigcup_{i=1}^n C_i$.

Assume first that $\alpha \geq \gamma$ and define $A_\alpha = C$. Then (8), (10) and (11) hold, and we construct suitable φ_α in this case $\alpha \geq \gamma$ as follows. Fix u_i in C_i for $i = 1, \dots, n$ and define $\varphi_\alpha : A_\alpha \to A_\alpha$ by putting (in both cases: $k_0 \neq 0$ and $k_0 = 0$)

$$\varphi_\alpha(g^k(u_i)) = g^k(u_{i+1}),\ \ \varphi_\alpha(g^k(u_n)) = g^{k+1}(u_1)$$

for $i = 1, \dots, n-1$ and for $k \in \mathbb{Z}$. Clearly, (9) holds.

Consider now the case where $\alpha < \gamma$. According to (5) and (7) we have

$$\mathrm{card}\{y \in A :\ \mathrm{card} R_\alpha^y = \mathrm{card} X\} = \mathrm{card} X$$

whereas (13) gives $\mathrm{card}(\bigcup_{\beta<\alpha} A_\beta \cup C) < \mathrm{card} A$. This allows us to fix a $y \in A \setminus (\bigcup_{\beta<\alpha} A_\beta \cup C)$ such that $\mathrm{card} R_\alpha^y = \mathrm{card}\, X$. Consequently, denoting $D_2 = \{g^k(y) : k \in \mathbb{Z}\}$, we can find an

$$x \in R_\alpha^y \setminus (\bigcup_{\beta<\alpha} A_\beta \cup C \cup D_2 \cup A_{-1}). \tag{14}$$

Put $D_1 = \{g^k(x) : k \in \mathbb{Z}\}$ and choose now $(n-2)$ different orbits $D_3, \dots, D_n$ disjoint from $\bigcup_{\beta<\alpha} A_\beta \cup C \cup D_1 \cup D_2 \cup A_{-1}$. Let $A_\alpha = C \cup \bigcup_{i=1}^n D_i$ and construct now (similarly to the case $\alpha \geq \gamma$ fixing additionally $v_i \in D_i$ with $v_1 = x,\ v_2 = y$) a function $\varphi_\alpha : A_\alpha \to A_\alpha$ such that

$$\varphi_\alpha(x) = y \tag{15}$$

and (9) hold. Then (12) follows from (14) and (15).

According to (11) we have $X = A_{-1} \cup \bigcup_{\alpha<\delta} A_\alpha$, and since the summands are disjoint, the formula $\varphi = \varphi_{-1} \cup \bigcup_{\alpha<\delta} \varphi_\alpha$ defines a function $\varphi : X \to X$ which is clearly a solution of (1) and has a big graph with respect to $\mathcal{R}$. □

Following [10] consider now a more general equation

$$\varphi^{n+m}(x) = g(\varphi^m(x)). \tag{16}$$

Clearly, every n-th iterative root of g is a solution of (16). This simple observation allows us to derive from Theorem 1 the following corollary.

Corollary 1. *Assume $m \in \mathbb{N}$. If* (H_1), (H_2) *hold and a family $\mathcal{R}$ of subsets of $X \times X$ satisfies* (4) *and* (5), *then there exists a solution $\varphi : X \to X$ of* (16) *which has a big graph with respect to $\mathcal{R}$.*

Applying Theorem 1 in the case where g is simply the identity we obtain the following corollary concerning the Babbage equation

$$\varphi^n(x) = x \tag{17}$$

which belongs to the oldest functional equations (see [11, Ch. XV, §1], [13, 11.7]).

Corollary 2. *Assume X is an uncountable set and $\mathcal{R}$ is a family of subsets of $X \times X$ which satisfies* (4) *and* (5)*. Then for every integer $n \geq 2$ there exists a solution $\varphi : X \to X$ of the Babbage equation* (17) *which has a big graph with respect to $\mathcal{R}$.*

In the case there g is an involution, i.e., $g^2(x) = x$ we have the following corollary which in the very special case of $X = (0, +\infty)$, $g(x) = 1/x$ and $n = 2$ generalizes Proposition 5.1 of [5].

Corollary 3. *Assume X is uncountable and $g : X \to X$ is an involution with $L_1 \neq L_2 \geq \aleph_0$. Let $\mathcal{R}$ be a family of subsets of $X \times X$ which satisfies* (4) *and* (5)*. Then for every integer $n \geq 2$ there exists a solution $\varphi : X \to X$ of* (1) *which has a big graph with respect to $\mathcal{R}$.*

The following two remarks give some other conditions which ensure that (H_2) holds.

Remark 1. Assume that a nonempty set X is equipped with an order $\leq$. If $g : X \to X$ is a bijection and $g(x) < x$ for every $x \in X$, then $L_k = 0$ for $k \geq 1$; consequently (H_2) holds.

Remark 2. Assume that an uncountable set X is equipped with a linear order and $g : X \to X$ is a bijection.

(i) If g is strictly increasing, then $L_k = 0$ for $k \geq 2$; if moreover g has less than card$X$ of fixed points, then (H_2) holds.

(ii) If g is strictly decreasing, then $L_k = 0$ for $k \geq 3$; if moreover n is odd and g^2 has less than card$X$ of fixed points, then (H_2) holds.

Remark 3. Clearly (cf. also [11, Lemma 15.5]) any iterative root φ of a bijection $g : X \to X$ maps k–cycles onto k–cycles and infinite orbits onto infinite orbits and so its graph is a subset of

$$(X \setminus \mathrm{Per}g)^2 \cup \bigcup_{k=1}^{\infty} \mathrm{Per}(g, k)^2. \tag{18}$$

Therefore, looking for theorems on the existence of n–th iterative roots with big graph in $X \times X$ we have to assume that the set (18) is big in $X \times X$. Our assumption (H_2) is in this direction.

On the other hand, having a bijection for which (H_2) does not hold we can ask whether there are solutions having big graph in a subset of $X \times X$ only, e.g., in (18). It turns out that our Theorem 1 jointly with Łojasiewicz's theorem may help in this, as the following corollary shows.

Corollary 4. *Assume* (H_1)*, and for every $k \in \mathbb{N}$, either L_k is infinite or divisible by d_k. Let $\mathcal{R}$ be a family of subsets of* (18) *such that* (4) *holds and for every $R \in \mathcal{R}$ we have either*

$$\mathrm{card}\{y \in X \setminus \mathrm{Per}g : \ \mathrm{card}R^y = \mathrm{card}X\} = \mathrm{card}X$$

or there exists a k such that

$$\mathrm{card}\{y \in \mathrm{Per}(g,k) : \ \mathrm{card}R^y = \mathrm{card}X\} = \mathrm{card}X.$$

Then there exists a solution $\varphi : X \to X$ of (1) *which has a big graph with respect to $\mathcal{R}$.*

PROOF. Put

$$\mathcal{R}_0 = \{R \cap (X \setminus \mathrm{Per}g)^2 : R \in \mathcal{R}, \mathrm{card}\{y \in X \setminus \mathrm{Per}g : \mathrm{card}R^y = \mathrm{card}X\} = \mathrm{card}X\}$$

and

$$\mathcal{R}_k = \{R \cap \mathrm{Per}(g,k)^2 : R \in \mathcal{R}, \mathrm{card}\{y \in \mathrm{Per}(g,k) : \mathrm{card}R^y = \mathrm{card}X\} = \mathrm{card}X\}$$

for $k \geq 1$. Fix $k \in \mathbb{N}$. If $\mathcal{R}_k \neq \emptyset$ then an application of Theorem 1 to the function $g|_{\mathrm{Per}(g,k)}$ (or to $g|_{X \setminus \mathrm{Per}g}$ if $k = 0$) gives a function φ_k which has a big graph with respect to $\mathcal{R}_k$ and such that $\varphi_k^n = g|_{\mathrm{Per}(g,k)}$ (or $\varphi_k^n = g|_{X \setminus \mathrm{Per}g}$ if $k = 0$). If $\mathcal{R}_k = \emptyset$, then using Łojasiewicz's theorem we get a function φ_k such that $\varphi_k^n = g|_{\mathrm{Per}(g,k)}$ (or $\varphi_k^n = g|_{X \setminus \mathrm{Per}g}$ if $k = 0$). Putting now $\varphi = \bigcup_{k=0}^{+\infty} \varphi_k$ we obtain a solution $\varphi : X \to X$ of (1) which has a big graph with respect to $\mathcal{R}$. $\square$

Example 1. Consider the bijection g on $(-1, 1)$ given by $g(x) = x$ for $x \in (-1, 0]$ and $g(x) = x^2$ for $x \in (0, 1)$. According to Corollary 4 there exists a solution $\varphi : (-1, 1) \to (-1, 1)$ of (1) which has a big graph with respect to any fixed family $\mathcal{R}$ of subsets of $(-1, 0]^2 \cup (0, 1)^2$ for which (4) and (5) holds with $X = (-1, 1)$.

3 Properties of Functions with Big Graph

Given a topological space X, consider the family

$$\{R \in \mathcal{B}(X \times X) : \{y \in X : R^y \text{ is uncountable}\} \text{ is uncountable}\} \tag{19}$$

where $\mathcal{B}(X \times X)$ denotes the σ-algebra of all Borel subsets of $X \times X$. The following remark is a consequence of the theorem of Alexandrov–Hausdorff ([14, p. 427], [9, 13.6]), the theorem of Mazurkiewicz–Sierpiński ([9, 29.19]) and the fact that there are not more than $2^{\aleph_0}$ many Borel sets in a Polish space.

Remark 4. If X is an uncountable Polish space, then the family (19) satisfies all the requirements of Theorem 1.

The following observation shows that if a function $\varphi : X \to X$ has a big graph with respect to the family (19), then its graph is big from the topological point of view.

Proposition 1. *Assume* T_1*–space* X *has a countable base and has no isolated point. If* $\varphi : X \to X$ *has a big graph with respect to the family* (19)*, then the set* $(X \times X) \setminus \mathrm{Gr}\varphi$ *contains no subset of* $X \times X$ *of second category having the property of Baire.*

PROOF. Assume $(X \times X) \setminus \mathrm{Gr}\varphi$ contains a set F of second category having the property of Baire. Let G be a second category $\mathcal{G}_\delta$ subset of $X \times X$ contained in F. Consider the sets

$$\bigcup\{G^y \times \{y\} : G^y \text{ is countable}\}, \quad \bigcup\{G^y \times \{y\} : G^y \text{ is uncountable}\}$$

summing up to G. Since G is not in (19), the second one is a countable sum of Borel sets. Consequently both these sets are Borel. Since all the sections of the first one are of first category, the set itself is of first category according to the Kuratowski-Ulam Theorem (see [17, Theorem 15.4], [9, 8.41]). Hence $\{x \in X : \ G_x \text{ is uncountable}\}$ is uncountable, i.e., G belongs to the family (19), a contradiction. $\square$

Making use of the Fubini Theorem, instead of that of Kuratowski-Ulam, (and the fact that $\mathcal{B}(X \times X) = \mathcal{B}(X) \times_\sigma \mathcal{B}(X)$ if X has a countable base) we obtain the following measure-theoretic analogue of Proposition 1.

Proposition 2. *Assume* X *is the* T_1*–space with a countable base. Let* μ *and* ν *be* σ*-finite Borel measures on* X *vanishing on all the singletons. If* $\varphi : X \to X$ *has a big graph with respect to the family* (19)*, then the set* $(X \times X) \setminus \mathrm{Gr}\varphi$ *contains no Borel subset of* $X \times X$ *of positive product measure* $\mu \times \nu$*.*

In other words $(\mu \times \nu)_*(X \times X \setminus \mathrm{Gr}\varphi) = 0$ and, consequently, $(\mu \times \nu)^*(B \cap \mathrm{Gr}\varphi) = (\mu \times \nu)(B)$ for every $B \in \mathcal{B}(X \times X)$. Here λ_* and λ^* denote inner and outer measures, respectively, generated by a Borel measure λ; cf. [7, Sec. 14].

It is worth while to mention that if a Polish space has no isolated point then there are lot of Borel measures on it vanishing on all the singletons [18, p. 55].

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