Real Analysis Exchange
Vol. (), , pp. 221–228

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REGULARITY OF LIPSCHITZ FUNCTIONS ON THE LINE

Abstract

We note a gap in Sciffer's construction of an everywhere irregular Lipschitz function on the line and provide a different simple construction of such a function, which even reaches maximal irregularity at every point.

The aim of this note is to construct an everywhere irregular real-valued Lipschitz function on the real line for the notion of regularity stemming from the study of Lipschitz functions on Banach spaces. We describe this notion in general Banach spaces, since its definition via standard notions of derivatives on the real line (which is the only Banach space we will actually use) may otherwise look somewhat artificial. Let f be a (locally) Lipschitz real-valued function on a Banach space X. The upper and lower derivatives of f at x in a direction v are:

$$D^+f(x,v) := \limsup_{h\to 0^+} \frac{f(x+hv)-f(x)}{h},$$
$$D_+f(x,v) := \liminf_{h\to 0^+} \frac{f(x+hv)-f(x)}{h}$$

and the Clarke derivative of f at x in the direction v is

$$f^0(x,v) := \limsup_{y\to x^+,\, h\to 0^+} \frac{f(y+hv)-f(y)}{h}.$$

The function f is said to be *regular at x in the direction v* if

$$f^0(x,v) = D_+f(x,v)$$

Key Words: Dini derivatives, Clarke derivative, regularity
Mathematical Reviews subject classification: 26A27
Received by the editors May 28, 2002

and f is said to be *regular at x* if it is regular in every direction.

In our case $X = \mathbb{R}$ and directional derivatives may be expressed in terms of upper and lower right and left Dini derivatives:

$$\begin{aligned} D^+f(x) &:= \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = D^+f(x,1), \\ D_+f(x) &:= \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = D_+f(x,1), \\ D^-f(x) &:= \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = -D_+f(x,-1), \\ D_-f(x) &:= \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = -D^+f(x,-1). \end{aligned}$$

Similarly, Clarke derivatives may be expressed in terms of upper and lower right and left sharp derivatives:

$$\begin{aligned} S^+f(x) &:= \limsup_{y\to x^+} D^+f(y) = f^0(x,1), \\ S_+f(x) &:= \liminf_{y\to x^+} D^+f(y) = -(-f)^0(x,1), \\ S^-f(x) &:= \limsup_{y\to x^-} D^+f(y) = (-f)^0(x,-1), \\ S_-f(x) &:= \liminf_{y\to x^-} D^+f(y) = -f^0(x,-1). \end{aligned}$$

This follows immediately from the fact that f, being Lipschitz, is differentiable almost everywhere and satisfies

$$f(x+h)-f(x) = \int_x^{x+h} f'(y)\,dy = \int_x^{x+h} D^+f(y)\,dy.$$

The use of the upper right Dini derivative in the above definitions may seem artificial until one notes that it may equivalently be replaced by any other Dini derivative or that one may take the upper or lower limits of $f'(y)$ over those y at which f is differentiable, etc. Note also that we always have

$$S^+f(x) \geq D^+f(x) \geq D_+f(x) \geq S_+f(x)$$

and

$$S^-f(x) \geq D^-f(x) \geq D_-f(x) \geq S_-f(x).$$

For Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ we have therefore four natural notions of one-sided regularity: f is upper regular at x from the right if $S^+ f(x) = D_+ f(x)$, lower regular at x from the right if $S_+ f(x) = D^+ f(x)$, upper regular at x from the left if $S^- f(x) = D_- f(x)$, and lower regular at x from the left if $S_- f(x) = D^- f(x)$. Then upper right regularity means that $f^0(x, 1) = D_+ f(x, 1)$, i.e. f is regular in the direction 1, and, similarly, lower left regularity means that f is regular in the direction -1. In other words, f is regular at x if and only if $S^+ f(x) - D_+ f(x) = D^- f(x) - S_- f(x) = 0$ or, equivalently, if $\max(S^+ f(x) - D_+ f(x), D^- f(x) - S_- f(x)) = 0$. Similarly, the remaining two concepts correspond to the regularity of $-f$.

S. Sciffer [3] proposed a construction of an everywhere irregular Lipschitz function on the real line. It was based on a Cantor-type set K (nowhere dense compact subset of $\mathbb{R}$ without isolated points) of positive measure such that every point $x \in K$ with the exception of the right isolated points satisfies $\lim_{h \to 0^+} \dfrac{\lambda([x, x+h] \cap K)}{h} = 1$. (We will denote by λ the Lebesgue measure.) However, such a set does not exist. This can be shown directly or one can use the Baire Category Theorem. Indeed, assume that K is such a set. Let (a_n, b_n) be the sequence of bounded intervals contiguous to K. Then the set $S = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} K \cap (a_n - (b_n - a_n), a_n)$ is non-empty, since it is residual in K. Moreover, S contains neither $\max(K)$ nor any of the a_n, hence it contains no right isolated points of K. But

$$\liminf_{h \to 0^+} \frac{\lambda([x, x+h] \cap K)}{h} \leq 1/2$$

for every $x \in S$, which contradicts our initial assumption. (See [2] for a more detailed analysis of Sciffer's argument.)

Our aim is to construct a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is not only everywhere irregular but is irregular in the strongest possible sense. Natural quantities measuring irregularities of f are the non-negative differences $S^+ f(x) - D_+ f(x)$, $D^+ f(x) - S_+ f(x)$, $S^- f(x) - D_- f(x)$ and $D^- f(x) - S_- f(x)$. Each of these values is bounded by $\omega(f) := \sup_{x \in \mathbb{R}} D^+ f(x) - \inf_{x \in \mathbb{R}} D^+ f(x)$ (which must be positive, since otherwise f would be affine and hence regular); an ideal example would therefore make them all equal to $\omega(f)$ at every point. But this is impossible: given any $\varepsilon > 0$, there are points $x \in \mathbb{R}$ at which f is differentiable and satisfies $f'(x) > \sup_{y \in \mathbb{R}} D_+ f(y) - \varepsilon$. At every such point we have $S^+ f(x) \geq D_+ f(x) > S^+ f(x) - \varepsilon$ and $S^- f(x) \geq D_- f(x) > S^- f(x) - \varepsilon$. Similarly, there are points at which $S_+ f(x) \leq D^+ f(x) < S_+ f(x) + \varepsilon$ and $S_- f(x) \leq D^- f(x) < S^- f(x) + \varepsilon$. We can therefore try to find a function f which is, at every point, irregular from the right as well as from the left, from above as well as from below. However, there must exist points at which both upper irregularities are as small as we wish, and similarly for lower irregularities.

Although we cannot have all four irregularities maximal, we could at least aim for two of the irregularities to reach $\omega(f)$. In particular, recalling that a natural measure of irregularity is $\max(S^+f(x) - D_+f(x), D^-f(x) - S_-f(x))$, we may wish to have this quantity equal to $\omega(f)$ at every point. However, at certain points we may reach only $\omega(f)/2$. To see this, define $g(x) := f(x) - ax$, where a is chosen so that $\sup_{x\in\mathbb{R}} D^+g(x) = \omega(f)/2$ and $\inf_{x\in\mathbb{R}} D^+g(x) = -\omega(f)/2$. If g is monotonic in some interval, all four measures of irregularity of f are bounded by $\omega(f)/2$. In the opposite case there are points at which g attains a local minimum as well as points at which it attains a local maximum. At a minimum we have $D_+g(x) \geq 0$ and $D^-g(x) \leq 0$, so $S^+f(x) - D_+f(x) = S^+g(x) - D_+g(x) \leq \omega(f)/2$ and $D^-f(x) - S_-f(x) \leq \omega(f)/2$. Similarly, at a maximum we have $D^+f(x) - S_+f(x) \leq \omega(f)/2$ and $S^-f(x) - D_-f(x) \leq \omega(f)/2$.

We come now to a construction of a Lipschitz everywhere irregular function. Our function will have Lipschitz constant 1 and hence $\omega(f) \leq 2$; then the measure of irregularity obtained in the second statement of the Theorem is in fact the largest achievable one.

Theorem . *There is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is at every point irregular from the right as well as from the left, from below as well as from above. In addition, we can require that the Lipschitz constant of f is one and that at every $x \in \mathbb{R}$ either*

$$\min(S^+f(x) - D_+f(x), S^-f(x) - D_-f(x)) \geq 1 \tag{1}$$

or

$$\min(D^+f(x) - S_+f(x), D^-f(x) - S_-f(x)) \geq 1. \tag{2}$$

From (1) and (2) we get that $\max(S^+f(x)-D_+f(x), D^-f(x)-S_-f(x)) \geq 1$ for every $x \in \mathbb{R}$, which confirms that the irregularity of f is the largest possible. Similarly, $-f$ has irregularity at least one at every point.

Before embarking on the proof of the Theorem, we recall the following version of the Lusin-Menchoff Lemma (see, for example, [1, page 156]). It will be used only in the special case when M is a Lebesgue null set and $c = 1$.

Lemma 1. *Let $M \subset \mathbb{R}$ be a measurable set, $\emptyset \neq F \subset \mathbb{R} \setminus M$ a closed set and let $c > 0$. Then there is a closed set $\tilde{F}$ such that $F \subset \tilde{F} \subset \mathbb{R} \setminus M$ and $\lambda(I \setminus (\tilde{F} \cup M)) < c\lambda(I)^2$ for every interval I such that $I \cap F \neq \emptyset$.*

PROOF OF THEOREM. Let $\mathbb{Q} = \{q_1, q_2, \dots\}$ denote the set of rational numbers and let M be a Lebesgue null set dense in $\mathbb{R}$ and disjoint from $\mathbb{Q}$. For example, we may take $M = \{q + \pi : q \in \mathbb{Q}\}$.

Define $F_1 = \{q_1\}$. Then $F_1 \subset \mathbb{Q} \subset \mathbb{R} \setminus M$ is non-empty and closed. Applying Lemma 1 with $F = F_1$, we obtain a closed set $\tilde{F}$ such that $F_1 \subset \tilde{F} \subset \mathbb{R} \setminus M$ and $\lambda(I \setminus \tilde{F}) < \lambda(I)^2$ for every interval I such that $I \cap F_1 \neq \emptyset$. Now set $F_2 = \tilde{F} \cup \{q_2\}$. Then $F_2 \subset \mathbb{R} \setminus M$ is closed, $F_1 \subset F_2 \subset \mathbb{R} \setminus M$ and since $I \setminus F_2 = I \setminus (\tilde{F} \cup \{q_2\}) \subset I \setminus \tilde{F}$, we have $\lambda(I \setminus F_2) \leq \lambda(I \setminus \tilde{F}) < \lambda(I)^2$ for every interval I such that $I \cap F_1 \neq \emptyset$.

Repeating this process with $F = F_2$, and so on, we obtain a sequence of closed sets $F_1, F_2, \dots$ with $F_1 \subset F_2 \subset \cdots \subset \mathbb{R} \setminus M$ such that $q_n \in F_n$ and

$$\lambda(I \setminus F_{n+1}) < \lambda(I)^2 \quad \forall I,\ I \cap F_n \neq \emptyset,\ (n = 1, 2, \dots). \tag{3}$$

Set $F_0 = \emptyset$ and define

$$g(x) = \begin{cases} (-1)^n\left(1 - \frac{1}{n}\right) & \text{if } x \in F_{2n} \setminus F_{2n-1} \text{ for some } n = 1, 2, \dots \\ 0 & \text{if } x \notin \bigcup_{i=0}^{\infty} F_i \text{ or } x \in \bigcup_{i=0}^{\infty} (F_{2i+1} \setminus F_{2i}) \end{cases}$$

and $f(x) = \int g(x)\,dx$. Since $|g| \leq 1$, we have

$$|f(y) - f(z)| = \left| \int_z^y g(x)\,dx \right| \leq |z - y|.$$

Therefore f is Lipschitz with Lipschitz constant not exceeding one and all its Dini and sharp derivatives are between -1 and 1. We will now establish more precise estimates of Dini derivatives.

If $x \in F_{2n} \setminus F_{2n-2}$ for some even n, we use that F_{2n-2} is closed to choose $\delta > 0$ such that $[x - \delta, x + \delta] \cap F_{2n-2} = \emptyset$. Then for every $0 < h < \delta$,

$$\begin{aligned} f(x+h) - f(x) &= \int_x^{x+h} g(t)\,dt \\ &= \int_{[x,x+h) \cap F_{2n+1}} g(t)\,dt + \int_{[x,x+h) \setminus F_{2n+1}} g(t)\,dt. \end{aligned}$$

Since every $t \in [x, x+h) \cap F_{2n+1}$ at which $g(t) \neq 0$ belongs to F_{2n}, we have $0 \leq g(t) \leq 1 - \frac{1}{n}$ if $t \in [x, x+h) \cap F_{2n+1}$. Also, $|g(t)| \leq 1$ always; so

$$f(x+h) - f(x) \geq -\lambda([x, x+h) \setminus F_{2n+1})$$

and

$$f(x+h) - f(x) \leq \left(1 - \frac{1}{n}\right)\lambda([x, x+h) \cap F_{2n+1}) + \lambda([x, x+h) \setminus F_{2n+1}).$$

Certainly $\lambda([x, x+h) \cap F_{2n+1}) \le \lambda([x, x+h)) = h$, and by (3),

$$\lambda([x, x+h) \setminus F_{2n+1}) < h^2$$

since $[x, x+h) \cap F_{2n} \supset \{x\} \ne \emptyset$. Then

$$-h^2 < f(x+h) - f(x) < \Big(1 - \frac{1}{n}\Big)h + h^2,$$

so

$$-h < \frac{f(x+h) - f(x)}{h} < \Big(1 - \frac{1}{n}\Big) + h.$$

Taking $\limsup$ and $\liminf$ as $h \to 0^+$, and using analogous estimates for $-\delta < h < 0$ we get

$$0 \le D_+ f(x) \le D^+ f(x) \le 1 - \frac{1}{n} \tag{4}$$

and

$$0 \le D_- f(x) \le D^- f(x) \le 1 - \frac{1}{n} \tag{5}$$

whenever $x \in F_{2n} \setminus F_{2n-2}$ for some even n. Symmetric arguments give

$$-\Big(1 - \frac{1}{n}\Big) \le D_+ f(x) \le D^+ f(x) \le 0 \tag{6}$$

and

$$-\Big(1 - \frac{1}{n}\Big) \le D_- f(x) \le D^- f(x) \le 0 \tag{7}$$

whenever $x \in F_{2n} \setminus F_{2n-2}$ for some odd n.

If $x \notin \bigcup_{i=1}^{\infty} F_i$ and $\delta > 0$, choose $N \in \mathbb{N}$ so that $q_N \in (x, x+\delta) \cap F_N$. Then for any $n \ge N$ there is $h \in (0, \delta)$ such that $x + h = \min([x, x+\delta] \cap F_n)$. So

$$\begin{aligned} f(x+h) - f(x) &= \int_x^{x+h} g(t)\,dt \\ &= \int_{[x,x+h)\cap F_{n+1}} g(t)\,dt + \int_{[x,x+h)\setminus F_{n+1}} g(t)\,dt. \end{aligned}$$

Now $[x, x+h) \cap F_n = \emptyset$ so $[x, x+h) \cap F_{n+1} \subset F_{n+1} \setminus F_n$. If we pick $n \ge N$ so that $n+1$ is divisible by four, our definition gives $g(t) = 1 - \frac{2}{n+1}$ for $t \in F_{n+1} \setminus F_n$. Also, $g(t) \ge -1$ always, so

$$\begin{aligned} &f(x+h) - f(x) \\ &\qquad \ge \Big(1 - \frac{2}{n+1}\Big)\lambda([x, x+h) \cap F_{n+1}) - \lambda([x, x+h) \setminus F_{n+1}) \\ &\qquad = \Big(1 - \frac{2}{n+1}\Big)\lambda([x, x+h)) - \Big(2 - \frac{2}{n+1}\Big)\lambda([x, x+h) \setminus F_{n+1}). \end{aligned}$$

Certainly $\lambda([x, x+h]) = h$, and by (3),

$$\lambda([x, x+h) \setminus F_{n+1}) = \lambda([x, x+h] \setminus F_{n+1}) < h^2$$

since $[x, x+h] \cap F_n = \{x+h\} \neq \emptyset$. Then

$$f(x+h) - f(x) > \Big(1 - \frac{2}{n+1}\Big)h - \Big(2 - \frac{2}{n+1}\Big)h^2;$$

so

$$\frac{f(x+h) - f(x)}{h} > \Big(1 - \frac{2}{n+1}\Big) - \Big(2 - \frac{2}{n+1}\Big)h > \Big(1 - \frac{2}{n+1}\Big) - 2h.$$

In short, for n large enough and such that $n+1$ is divisible by four, we have found $0 < h < \delta$ for which

$$\frac{f(x+h) - f(x)}{h} > \Big(1 - \frac{2}{n+1}\Big) - 2h.$$

Hence

$$\sup_{0<h<\delta} \frac{f(x+h) - f(x)}{h} \geq 1 - 2\delta$$

and so

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \geq 1.$$

Recalling that the Dini derivatives are between 1 and -1 and using arguments symmetric to the above, we obtain that

$$D^+ f(x) = D^- f(x) = 1 \text{ and } D_+ f(x) = D_- f(x) = -1 \tag{8}$$

whenever $x \notin \bigcup_{i=0}^{\infty} F_i$. Since M is dense in $\mathbb{R}$, we now see from (8) that for every interval I, $\sup_{y \in I} D^+ f(y) \geq 1$. Together with the already established bound $S^+ f(x) \leq 1$ this gives that $S^+ f(x) = 1$ for every $x \in \mathbb{R}$. Using symmetric arguments for the remaining sharp derivatives, we conclude that

$$S^+ f(x) = S^- f(x) = 1 \text{ and } S_+ f(x) = S_- f(x) = -1 \text{ for all } x \in \mathbb{R}. \tag{9}$$

The Theorem now follows by recapitulation of the above estimates of the Dini and sharp derivatives. Let $x \in \mathbb{R}$. Then either $x \notin \bigcup_{i=0}^{\infty} F_i$ and $D_+ f(x) = -1 < 1 = S^+ f(x)$ by (8) and (9), or $x \in \bigcup_{i=0}^{\infty} F_i$, in which case we use that $x \in F_{2n} \setminus F_{2n-2}$ for some n to infer from (4) or (6) that $D_+ f(x) < 1 = S^+ f(x)$. In addition, if $x \in F_{2n} \setminus F_{2n-2}$ for some odd n, (9), (6) and (7) give that $S^+ f(x) - D_+ f(x) \geq 1$ and $S^- f(x) - D_- f(x) \geq 1$, hence (1) holds; if $x \in F_{2n} \setminus F_{2n-2}$ for some even n, (9), (4) and (5) give that (2) holds. Finally, if $x \notin \bigcup_{i=0}^{\infty} F_i$, (9) and (8) give that both (1) and (2) hold. $\square$

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