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EXTENDING SOME FUNCTIONS TO STRONGLY APPROXIMATELY QUASICONTINUOUS FUNCTIONS

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ is strongly approximately quasicontinuous at a point x if for each real $r > 0$ and for each set $U \ni x$ belonging to the density topology there is an open interval I such that $I \cap U \neq \emptyset$ and $f(U \cap I) \subset (f(x) - r, f(x) + r)$. In this article we investigate the sets A such that each almost everywhere continuous bounded function may be extended from A to a bounded strongly approximately quasicontinuous function on $\mathbb{R}$.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h} \text{ respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

The family T_d of all sets A for which the implication $x \in A \implies x$ is a density point of A is true, is a topology called the density topology ([1, 6]).

The sets $A \in T_d$ are Lebesgue measurable [1, 6].

In [5] O'Malley investigates the topology

$$T_{ae} = \{A \in T_d; \mu(A \setminus \text{int}(A)) = 0\},$$

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where $\mathrm{int}(A)$ denotes the interior of the set A.

Let T_e be the Euclidean topology in $\mathbb{R}$. The continuity of functions f from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called approximate continuity ([1, 6]).

For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f and by $D(f)$ the set $\mathbb{R} \setminus C(f)$.

In [5] it is proved that a function $f : \mathbb{R} \to \mathbb{R}$ is T_{ae}-continuous (i.e., continuous as a function from $(\mathbb{R}, T_{ae})$ to $(\mathbb{R}, T_e)$) if and only if it is T_d-continuous (i.e., approximately continuous) everywhere and $\mu(D(f)) = 0$. In [2] the following property is investigated. A function $f : \mathbb{R} \to \mathbb{R}$ is strongly approximately quasicontinuous at a point x ($f \in s_0(x)$) if for each positive real r and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U$ and $|f(t) - f(x)| < r$ for all points $t \in I \cap U$.

A function f has the property (s_0), if $f \in s_0(x)$ for every point $x \in \mathbb{R}$.

For each function f having property (s_0) the set $D(f) = \mathbb{R} \setminus C(f)$ is of Lebesgue measure 0 ([2]), but it may be dense in $\mathbb{R}$.

Each approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ is of the first Baire class ([1]).

In [4] the authors investigate the family Φ_{ap} of all nonempty sets A such that for every Baire 1 function $f : \mathbb{R} \to \mathbb{R}$ there is an approximately continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f \restriction A = g \restriction A$. They prove there that $A \in \Phi_{ap}$ if and only if $\mu(A) = 0$.

In [3] I investigate the family Φ_{ae} of all nonempty sets A such that for every Baire 1 function $f : \mathbb{R} \to \mathbb{R}$ there is a T_{ae}-continuous function $g : \mathbb{R} \to \mathbb{R}$ with $f/A = g/A$. I show in this article that a nonempty set $A \in \Phi_{ae}$ if and only if $\mu(\mathrm{cl}(A)) = 0$, where $\mathrm{cl}(A)$ denotes the closure of the set A.

In this paper I investigated the family Φ_{s_0} of all nonempty sets A such that for every almost everywhere continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ there is a bounded function $g : \mathbb{R} \to \mathbb{R}$ having the property (s_0) such that $f \restriction A = g \restriction A$.

Theorem 1. *If the set $A \in \Phi_{s_0}$ then*

$$\text{for each point } x \in A \text{ we have } D_l(\mathrm{cl}(A), x) < 1. \tag{1}$$

PROOF. Assume that there is a point $x \in A$ such that the lower density $D_l(\mathrm{cl}(A), x) = 1$. Then the bounded function $f(t) = 0$ for $t \neq x$, and $f(x) = 1$ is almost everywhere continuous, but for each extension $g : \mathbb{R} \to \mathbb{R}$ of the restricted function $f \restriction A$ we obtain that g is not in $s_0(x)$. Of course, if $g : \mathbb{R} \to \mathbb{R}$ is such that $f \restriction A = g \restriction A$ and $r = \frac{1}{3}$ then the set

$$U = \{t \in \mathrm{cl}(A); D_l(\mathrm{cl}(A), t) = 1\} \in T_d \text{ and } U \ni x$$

and for every open interval I with $I \cap U \neq \emptyset$ there is a point $t \in I \cap U \cap A$ such that $t \neq x$. So,

$$|g(t) - g(x)| = |f(t) - f(x)| = |0 - 1| = 1 > \frac{1}{3} = r,$$

and consequently g is not in $s_0(x)$ and A is not in Φ_{s_0}. □

Theorem 2. *If a nonempty set $A \subset \mathbb{R}$ satisfies the condition*

$$\textit{for each point } x \in A \textit{ we have } D_l(\mathbb{R} \setminus \mathrm{cl}(A), x) > 0,$$

then $A \in \Phi_{s_0}$.

PROOF. Evidently the set A is nowhere dense. At first we suppose that A is a bounded set. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost everywhere continuous bounded function. Then the function

$$h(x) = \begin{cases} f(x) & \text{for } x \in \mathrm{cl}(A) \\ \text{linear} & \text{on the components of } [\inf A, \sup A] \setminus \mathrm{cl}(A) \\ f(\sup A) & \text{for } x \geq \sup A \\ f(\inf A) & \text{for } x \leq \inf A \end{cases}$$

is also bounded and almost everywhere continuous, $C(h) \supset \mathbb{R} \setminus \mathrm{cl}(A)$ and $f \restriction A = h \restriction A$. Since the function h is almost everywhere continuous, the set

$$B = \{y; \mu(\mathrm{cl}(h^{-1}(y))) > 0\}$$

is at most countable. Let $c = \inf h(\mathbb{R})$ and $d = \sup h(\mathbb{R})$. There are nonempty finite sets

$$B_n = \{y_{n,1}, y_{n,2}, \ldots, y_{n,j(n)}\} \subset \mathbb{R} \setminus B,\ n \geq 1,$$

such that $c = y_{n,0} < y_{n,1} < \cdots < y_{n,j(n)} < d$ for $n \geq 1$, $B_n \subset B_{n+1}$ for $n \geq 1$, $|y_{n,i+1} - y_{n,i}| < \frac{1}{2^n}$ for $n \geq 1$ and $i \leq j(n)$, where $y_{n,j(n)+1} = d + \frac{1}{8^n}$. Let

$$\phi_1(x) = y_{1,i} \text{ if } y_{1,i} \leq h(x) < y_{1,i+1} \text{ for } i = 0, 1, \ldots, j(1).$$

Since h is almost everywhere continuous and $y_{1,i} \in \mathbb{R} \setminus B$ for $i \leq j(1)$, the function ϕ_1 is almost everywhere continuous and the set $C(\phi_1)$ of all continuity points of the function ϕ_1 is open and of full measure (i.e., $\mu(\mathbb{R} \setminus C(\phi_1)) = 0$). Let $A_1 = \mathbb{R} \setminus C(\phi_1)$. From the above it follows that $\mu(A_1) = 0$ and A_1 is closed.

Now we will construct some special family of pairwise disjoint closed intervals $L_{1,i,j} \subset (\mathbb{R} \setminus \mathrm{cl}(A)) \setminus A_1$. For this let $I_{1,1,1}, I_{1,1,2}, \ldots, I_{1,1,i(1,1)}$ be the open components of the set

$$U_{1,1} = \bigcup_{x \in A_1 \cap \mathrm{cl}(A)} (x-1, x+1).$$

There are pairwise disjoint nondegenerate closed intervals

$$L_{1,1,1}, \ldots, L_{1,1,k(1,1)} \subset (U_{1,1} \setminus \mathrm{cl}(A)) \setminus A_1$$

such that for every positive integer $j \leq i(1,1)$

$$\mu(I_{1,1,j} \cap \bigcup_{i \leq k(1,1)} L_{1,1,i}) > \frac{1}{2}\mu(I_{1,1,j} \setminus \mathrm{cl}(A)).$$

In the second step put

$$r_{1,2} = \frac{\inf\{|x-y|; x \in A_1 \cap \mathrm{cl}(A),\ y \in \bigcup_{i \leq k(1,1)} L_{1,1,i}\}}{2},$$

and denote by $I_{1,2,1}, I_{1,2,2}, \ldots, I_{1,2,i(1,2)}$ the components of the set

$$U_{1,2} = \bigcup_{x \in A_1 \cap \mathrm{cl}(A)} (x - r_{1,2}, x + r_{1,2}).$$

Next we find pairwise disjoint nondegenerate closed intervals

$$L_{1,2,1}, \ldots, L_{1,2,k(1,2)} \subset (U_{1,2} \setminus \mathrm{cl}(A)) \setminus A_1$$

such that for every positive integer $j \leq i(1,2)$

$$\mu(I_{1,2,j} \cap \bigcup_{i \leq k(1,2)} L_{1,2,i}) > (1 - \frac{1}{2^2})\mu(I_{1,2,j} \setminus \mathrm{cl}(A)).$$

In general in the n^{th} step ($n > 2$) we define the positive real

$$r_{1,n} = \frac{\inf\{|x-y|; x \in A_1 \cap \mathrm{cl}(A),\ y \in \bigcup_{i \leq k(1,n-1)} L_{1,n-1,i}\}}{2},$$

and denote by $I_{1,n,1}, I_{1,n,2}, \ldots, I_{1,n,i(1,n)}$ the components of the set

$$U_{1,n} = \bigcup_{x \in A_1 \cap \mathrm{cl}(A)} (x - r_{1,n}, x + r_{1,n}).$$

Next we find pairwise disjoint nondegenerate closed intervals

$$L_{1,n,1}, \dots, L_{1,n,k(1,n)} \subset (U_{1,n} \setminus \mathrm{cl}(A)) \setminus A_1$$

such that for each positive integer $j \leq i(1,n)$

$$\mu(I_{1,n,j} \cap \bigcup_{i \leq k(1,n)} L_{1,n,i}) > (1 - \frac{1}{2^n})\mu(I_{1,n,j} \setminus \mathrm{cl}(A)) \tag{2}$$

Let $\{N(s,i)\}_{s,i=1}^{\infty}$ be a family of pairwise disjoint infinite subsets of positive integers. Observe that by (2) for each point $x \in A \cap A_1$ and for each pair (s,i) of positive integers we have

$$D_u\Big(\bigcup_{n \in N(s,i)} \bigcup_{m \leq k(1,n)} L_{1,n,m}, x) \geq D_l(\mathbb{R} \setminus \mathrm{cl}(A), x\Big) > 0. \tag{3}$$

Let $f_1(x) = y_{1,s}$ for $x \in L_{1,n,m}$, where $n \in N(s,i)$, $s \leq j(1)$, $m \leq k(1,n)$, $i = 1, 2, \dots$. For $x \in A_1 \setminus \mathrm{cl}(A)$ such that $D_u(\{(\phi_1)^{-1}(\phi_1(x)), x) = 0$ and $\phi_1(x) = y_{1,k}$ we put $f_1(x) = y_{1,k-1}$ and let $f_1(x) = \phi_1(x)$ otherwise on $\mathbb{R}$. If $x \in A_1 \cap A$ and $f_1(x) = y_{1,m_1}$, then by (3) we have

$$D_u(\mathrm{int}((f_1)^{-1}(f_1(x))), x) \geq D_u\Big(\bigcup_{i=1}^{\infty} \bigcup_{n \in N_{m_1,i}} \bigcup_{m \leq k(1,n)} L_{1,n,m}, x\Big) > 0,$$

and consequently $f_1 \in s_0(x)$. If $x \in A_1 \setminus \mathrm{cl}(A)$, then by construction $f_1 \in s_0(x)$. If $x \in (\mathbb{R} \setminus (A_1 \cap \mathrm{cl}(A))) \setminus A_1$, then f_1 is continuous or unilaterally continuous and consequently $f_1 \in s_0(x)$.

Now let $x \in (A_1 \cap \mathrm{cl}(A)) \setminus A$. Observe that $\mu(\mathbb{R} \setminus \mathrm{int}(C(f_1))) = 0$ and $f_1(\mathbb{R}) = \{y_{1,0}, y_{1,1}, \dots, y_{1,j(1)}\}$. Since $D_l(\mathrm{int}(C(f_1)), x) = 1$, there is an integer $m_2(x) \in [0, j(1)]$ such that $D_u\big(\mathrm{int}((f_1)^{-1}(y_{1,m_2(x)})), x\big) > 0$. So we fix such an $m_2(x)$ and putting

$$g_1(x) = \begin{cases} y_{1,m_2(x)} & \text{for } x \in (\mathrm{cl}(A) \cap A_1) \setminus A \\ f_1(x) & \text{otherwise on } \mathbb{R}, \end{cases}$$

we obtain a function g_1 having the property (s_0) such that $|h(x) - g_1(x)| < \frac{1}{2}$ for $x \in A$.

Now we will construct a function g_2 having property (s_0) and such that

$$|g_2 - g_1| < \frac{1}{2} \text{ and } |g_2(x) - h(x)| < \frac{1}{2^2} \text{ for } x \in A.$$

Let
$$\phi_2(x) = y_{2,i} \text{ if } y_{2,i} \le h(x) < y_{2,i+1} \text{ for } i = 0, 1, \dots, j(2).$$
Since h is almost everywhere continuous and $y_{2,i} \in \mathbb{R} \setminus B$ for $i \le j(2)$, the function ϕ_2 is almost everywhere continuous and the set $C(\phi_2)$ of all continuity points of the function ϕ_2 is open and of full measure. Let $A_2 = \mathbb{R} \setminus C(\phi_2)$. From the above it follows that $\mu(A_2) = 0$. Since $B_2 \supset B_1$, the inclusion $A_1 \subset A_2$ holds.

Now we will construct some special family of pairwise disjoint closed intervals
$$L_{2,i,j} \subset (\mathbb{R} \setminus \mathrm{cl}(A \cup A_2)) \setminus \bigcup_{n=1}^{\infty} \bigcup_{m \le k(1,n)} L_{1,n,m}.$$
For this let $I_{2,1,1}, I_{2,1,2}, \dots, I_{2,1,i(2,1)}$ be the open components of the set
$$U_{2,1} = \bigcup_{x \in A_2 \cap \mathrm{cl}(A)} (x-1, x+1).$$
There are pairwise disjoint nondegenerate closed intervals
$$L_{2,1,1}, \dots, L_{2,1,k(2,1)} \subset ((U_{2,1} \setminus \mathrm{cl}(A)) \setminus A_2) \setminus \bigcup_{n=1}^{\infty} \bigcup_{i \le k(1,n)} L_{1,n,i}$$
such that for every positive integer $j \le i(2,1)$
$$\mu(I_{2,1,j} \cap \bigcup_{i \le k(2,1)} L_{2,1,i}) > \frac{1}{2} \mu((I_{2,1,j} \setminus \mathrm{cl}(A)) \setminus \bigcup_{n=1}^{\infty} \bigcup_{m \le k(1,n)} L_{1,n,m}).$$

In the second step put
$$r_{2,2} = \frac{\inf\{|x-y|; x \in A_2 \cap \mathrm{cl}(A),\ y \in \bigcup_{i \le k(2,1)} L_{2,1,i}\}}{2},$$
and denote by $I_{2,2,1}, I_{2,2,2}, \dots, I_{2,2,i(2,2)}$ the components of the set
$$U_{2,2} = \bigcup_{x \in A_2 \cap \mathrm{cl}(A)} (x - r_{2,2}, x + r_{2,2}).$$
Next we find pairwise disjoint nondegenerate closed intervals
$$L_{2,2,1}, \dots, L_{2,2,k(2,2)} \subset ((U_2 \setminus \mathrm{cl}(A)) \setminus A_2) \setminus \bigcup_{n=1}^{\infty} \bigcup_{m \le k(1,n)} L_{1,n,m}$$

such that for every positive integer $j \leq i(2,2)$

$$\mu(I_{2,2,j} \cap \bigcup_{i \leq k(2,2)} L_{2,2,i}) > (1 - \frac{1}{2^2})\mu((I_{2,2,j} \setminus \mathrm{cl}(A)) \setminus \bigcup_{n=1}^{\infty} \bigcup_{m \leq k(1,n)} L_{1,n,m}).$$

In general in the n^{th} step ($n > 2$) we define the positive real

$$r_{2,n} = \frac{\inf\{|x-y|; x \in A_2 \cap \mathrm{cl}(A),\ y \in \bigcup_{i \leq k(2,n-1)} L_{2,n-1,i}\}}{2},$$

and denote by $I_{2,n,1}, I_{2,n,2}, \ldots, I_{2,n,i(2,n)}$ the components of the set

$$U_{2,n} = \bigcup_{x \in A_2 \cap \mathrm{cl}(A)} (x - r_{2,n}, x + r_{2,n}).$$

Next we find pairwise disjoint nondegenerate closed intervals

$$L_{2,n,1}, \ldots, L_{2,n,k(2,n)} \subset ((U_n \setminus \mathrm{cl}(A)) \setminus A_2) \setminus \bigcup_{s=1}^{\infty} \bigcup_{m \leq k(1,s)} L_{1,s,m}$$

such that for each positive integer $j \leq i(2,n)$

$$\mu(I_{2,n,j} \cap \bigcup_{i \leq k(2,n)} L_{2,n,i}) > (1 - \frac{1}{2^n})\mu((I_{2,n,j} \setminus \mathrm{cl}(A)) \setminus \bigcup_{s=1}^{\infty} \bigcup_{m \leq k(1,s)} L_{1,s,m}) \quad (4)$$

Observe that by (4) for each point $x \in A \cap (A_2 \setminus A_1)$ and for each pair (s,i) of positive integers we have

$$D_u(\bigcup_{n \in N_{s,i}} \bigcup_{m \leq k(2,n)} L_{2,n,m}, x) \geq D_l(\mathbb{R} \setminus \mathrm{cl}(A), x) > 0. \quad (5)$$

Now we will define the function g_2. For $k = 0, 1, \ldots, j(1)$ denote by

$$y_{1,k} = y_{2,s(k)} < y_{2,s(k)+1} < y_{2,s(k)+2} < \cdots < y_{2,s(k)+t(k)} < y_{2,s(k)+t(k)+1} = y_{1,k+1}$$

all numbers of the set $B_2 \cap [y_{1,k}, y_{1,k+1}]$. For $k \leq j(1)$ and $i \geq 1$ let $\{N(k,i,j)\}_{j=1}^{\infty}$ be a family of pairwise disjoint infinite subsets of integers such that $N(k,i) = \bigcup_{j=1}^{\infty} N(k,i,j)$. For $x \in L_{1,p,m}$, where $p \in N(k,i,j)$, $s(k) \leq i \leq s(k) + t(k)$ and $j \geq 1$, we put $g_2(x) = y_{2,i}$. Observe that for such points x we have

$$|g_2(x) - g_1(x)| < y_{1,k+1} - y_{1,k} < \frac{1}{2}.$$

If

$$L_{2,n,m} \subset (g_1)^{-1}(y_{1,k}) \text{ and } n \in N(k, s(k)+i),$$

where $0 \le i \le t(k)$ and $m \le k(n)$, then we put $g_2(x) = y_{2,s(k)+i}$ for $x \in L_{2,n,m}$. As above we observe that for such points x we obtain

$$|g_2(x) - g_1(x)| < y_{1,k+1} - y_{1,k} < \frac{1}{2}.$$

In the other points of the set $\mathbb{R} \setminus A_2$ we put $g_2(x) = g_1(x)$. If $x \in A_2 \cap A$, then we put $g_2(x) = \phi_2(x)$.

Now let $x \in A_2 \setminus A$ and let $g_1(x) = y_{1,k}$. Then

$$D_u((\text{int}((g_1)^{-1}(y_{1,k})), x) > 0,$$

and consequently there is an integer $i \ge 0$ $(i \le t(k))$ such that

$$D_u(\text{int}((g_2)^{-1}(y_{2,s(k)+i})), x) > 0.$$

So we fix such an i and put $g_2(x) = y_{2,s(k)+i}$. As above for such points x we obtain $|g_2(x) - g_1(x)| < \frac{1}{2}$. The function g_2 is continuous or unilaterally continuous at points $x \in \mathbb{R} \setminus A_2$; so it has the property (s_0) at these points. If $x \in A_2 \cap A$, then by (5) and the construction of g_2 we have

$$D_u(\text{int}((g_2)^{-1}(g_2(x))), x) > 0,$$

and consequently the function g_2 has the property (s_0) at x. Analogously from the construction of g_2 it follows that the function g_2 has the property (s_0) at points $x \in A_2 \setminus A$. So g_2 has the property (s_0) everywhere. Moreover

$$|g_2 - g_1| < \frac{1}{2} \text{ and } |g_2(x) - h(x)| = |\phi_2(x) - h(x)| < \frac{1}{2^2} \text{ for } x \in A.$$

Analogously, in the n^{th} step $(n > 2)$ we define a function g_n having the property (s_0) such that $|g_{n-1} - g_n| < \frac{1}{2^{n-1}}$ and $|g_n(x) - h(x)| < \frac{1}{2^n}$ for $x \in A$. The sequence $(g_n)_n$ uniformly converges to a bounded function g, which has the property (s_0) (as the uniform limit of a sequence of functions having this property [2]). For $x \in A$ we have $g(x) = \lim_{n\to\infty} g_n(x) = h(x)$.

Now we consider the general case, where the set A may be unbounded. Then there are points

$$x_0, x_{-1}, x_1, x_{-2}, x_2, \ldots \in \mathbb{R} \setminus \text{cl}(A)$$

such that

$$x_{k+1} > x_k \text{ for } k = 0, -1, 1, -2, 2, \ldots,$$

$$\lim_{k\to-\infty} x_k = -\infty, \quad \lim_{k\to\infty} x_k = \infty.$$

If $E_k = (x_{k-1}, x_k) \cap A \neq \emptyset$ then by the proved part of our theorem there is a bounded function $f_k : \mathbb{R} \to [c, d]$ having the property (s_0) such that $h(x) = f_k(x)$ for $x \in E_k$. Let

$$g(x) = \begin{cases} f_k(x) & \text{if } E_k \neq \emptyset \text{ and } x \in (x_{k-1}, x_k) \\ h(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then the bounded function g has property (s_0) and $g \restriction A = h \restriction A = f \restriction A$. $\square$

Problem. Is the following implication true?

$$A \text{ satisfies condition (1)} \Longrightarrow A \in \Phi_{s_0}$$

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