THE MONGE-AMPERE PARTIAL DIFFERENTIAL EQUATION
$rt - s^2 + \lambda^2 = 0$

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Introduction. Recently the study of the propagation of a plane shock wave moving into a quiet atmosphere, and leaving a nonisentropic disturbance behind it, has been reduced [6] to the solution of a Problem of Cauchy for a Monge-Ampère equation of the type

$$(1) \qquad rt - s^2 + \lambda^2 = 0, \ \lambda = X(x)Y(y).$$

The present paper is devoted to a study of the Problem of Cauchy for this partial differential equation with a view to later applications to shock propagation.

In the first section we determine those functions $X(x)$, $Y(y)$ for which (1) has intermediate integrals. A summary of the results will be found in the seven cases in Theorem 1.

The linearization (without approximation) of the seven equations found in § 1 is carried out in § 2 with results summarized in Theorem 2. The individual results (particularly on cases 3, 5) are of interest for the applications in mind.

The solution of the Problem of Cauchy is taken up in § 3 and reduced to the solution of the Problem of Cauchy for linear partial differential equations. A summary of the results will be found in Theorem 3.

1. Intermediate integrals. In this section we investigate the intermediate integrals of (1).

If either X or Y is zero, or both are constant, then $\lambda = \text{const.}$, and (1) has intermediate integrals [3, pp. 154-155]

$$q - \lambda x = \phi(p + \lambda y), \ q + \lambda x = \psi(p - \lambda y),$$

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involving arbitrary functions. These simple cases will be excluded in our search for functions X, Y for which (1) has intermediate integrals.

According to classical theory [3, p. 58], based on the differential systems

$$(2)\quad \begin{aligned} dz - pdx - qdy &= 0, \\ dp + \lambda dy &= 0, \\ dq - \lambda dx &= 0, \end{aligned} \qquad (2')\quad \begin{aligned} dz - pdx - qdy &= 0, \\ dp - \lambda dy &= 0, \\ dq + \lambda dx &= 0, \end{aligned}$$

for the two families of characteristic strips, (1) will have an intermediate integral

$$V(x, y, z, p, q) = \text{const.}$$

if, and only if, V is a simultaneous solution either of

$$(3)\quad \begin{aligned} V_x + pV_z + \lambda V_q &= 0, \\ V_y + qV_z - \lambda V_p &= 0, \end{aligned} \qquad \text{or of } (3')\quad \begin{aligned} V_x + pV_z - \lambda V_q &= 0, \\ V_y + qV_z + \lambda V_p &= 0. \end{aligned}$$

Any solution of the first equation in (3) must have the form

$$V = F(u, v, y, p), \quad u = z - px, \quad v = q - X_1 Y, \quad X_1 = \int X dx,$$

and will be a solution of the second equation if and only if

$$(4)\qquad xX \cdot YF_u - X \cdot YF_p + X_1 \cdot (YF_u - Y'F_v) + F_y + vF_u \equiv 0$$

is an identity in the independent variables x, u, v, y, p.

The manifolds

$$(5)\quad \begin{aligned} &\mathrm{M}: \quad F_1 = xX, \quad F_2 = X, \ F_3 = X_1, \ F_4 = 1, \\ &\mathrm{N}: \quad G_1 = YF_u, \ G_2 = -YF_p, \ G_3 = YF_u - Y'F_v, \ G_4 = F_y + vF_u, \end{aligned}$$

are at most one and four dimensional, respectively, with the bilinear condition

$$F_1 G_1 + F_2 G_2 + F_3 G_3 + F_4 G_4 = 0$$

imposed on them, in view of (4). For this condition to hold, it is necessary and sufficient[1] that MCS_m, NCT_n, where S_m, T_n are linear orthogonal subspaces defined by

[1] This follows as a special case of a general theorem which will be proved elsewhere [7].

$$S_m: \sum_{\alpha=1}^{4} b_{k\alpha} F_\alpha = 0 \qquad (k = 1, \cdots, n);$$

$$T_n: \sum_{\alpha=1}^{4} a_{i\alpha} G_\alpha = 0 \qquad (i = 1, \cdots, m;\ m + n = 4),$$

the matrices

$$\mathbf{A} = \|a_{i\alpha}\|, \quad \mathbf{B} = \|b_{k\alpha}\|,$$

having ranks m, n, respectively. The linear subspaces S_m, T_n are orthogonal if, and only if, the composite matrix **C** formed by taking the m rows of **A** followed by the n rows of **B** has the following property; the m-rowed minors in **A** are all proportional to their complimentary minors in **B**, the indices of the columns of **A** followed by the indices of the columns of **B** forming an even permutation of 1, 2, 3, 4.

There are fives cases to consider as $m = 0, 1, 2, 3, 4$.

I. $m = 0$, $n = 4$. This case does not arise, as it requires

$$F_1 = F_2 = F_3 = F_4 = 0,$$

whereas actually $F_4 = 1$ in (5).

II. $m = 1$, $n = 3$. Here

$$S_1: \frac{F_1}{a_{11}} = \frac{F_2}{a_{12}} = \frac{F_3}{a_{13}} = \frac{F_4}{a_{14}}; \qquad T_3: \sum_{\alpha=1}^{4} a_{1\alpha} G_\alpha = 0,$$

and T_3 is a hyperplane with S_1 the normal to S_3 at the origin. From (5) the equations of S_1 imply that X, X_1 are both constant and therefore $X \equiv 0$, $\lambda \equiv 0$, an excluded case.

III. $m = 2$, $n = 2$. Here

$$S_2: \sum_{\alpha=1}^{4} b_{k\alpha} F_\alpha = 0 \quad (k = 1, 2); \qquad T_2: \sum_{\alpha=1}^{4} a_{i\alpha} G_\alpha = 0 \quad (i = 1, 2)$$

are linear orthogonal two-spaces, and

$$\frac{A_{12}}{B_{34}} = \frac{A_{13}}{B_{42}} = \frac{A_{14}}{B_{23}} = \frac{A_{23}}{B_{14}} = \frac{A_{42}}{B_{13}} = \frac{A_{34}}{B_{12}},$$

where A_{ij}, B_{ij} denote the determinants formed from the ith and jth columns of **A**, **B**. From (5) the equations of S_2 yield simultaneous equations

$$(b_{11}x + b_{12})X + b_{13}X_1 + b_{14} = 0, \quad (b_{21}x + b_{22})X + b_{23}X_1 + b_{24} = 0,$$

for X, X_1, so that

$$X = \frac{B_{34}}{B_{13}x + B_{23}}, \qquad X_1 = \frac{B_{41}x + B_{42}}{B_{13}x + B_{23}},$$

the case in which all B_{ij}, except B_{12}, are zero being rejected, since it implies $X \equiv 0$. When the second equation is differentiated, two simultaneous equations result for X which can subsist only if X = const.

We accordingly place

$$X = a \neq 0, \quad X_1 = a_0 + ax \qquad (a_0, a = \text{const.}),$$

in (4), which implies two simultaneous partial differential equations

$$2YF_u - Y'F_v = 0, \quad (v + a_0Y)F_u - a_0Y'F_v + F_y - aYF_p = 0,$$

for F. If $Y \not\equiv 0$, the general solution of the first is

$$F = G(r, y, p), \quad r = 2v + Lu, \quad L = \frac{Y'}{Y} \not\equiv 0.$$

and the result

$$\left(\frac{1}{2}rL - a_0Y'\right)G_r + G_y - aYG_p + u\left(L' - \frac{1}{2}L^2\right)G_r \equiv 0,$$

of substituting this in the second leads to

$$\left(\frac{1}{2}rL - a_0Y'\right)G_r + G_y - aYG_p = 0; \quad G_r = 0 \text{ or } 2L' - L^2 = 0.$$

Under the alternative $G_r = 0$, we arrive at an obvious intermediate integral

$$V = p + aY_1 = \text{const.}, \quad Y_1 = \int Y\,dy, \tag{6}$$

valid for any function $Y(y)$; with the second alternative, necessarily

$$Y = b(y - y_0)^{-2} \qquad (b,\ y_0 = \text{const.}),$$

and we have two intermediate integrals

$$p - ab(y - y_0)^{-1} = \text{const.}, \quad (y - y_0)r + 2a_0 b(y - y_0)^{-1} = \text{const.},$$

the first of which repeats (6), with the second becoming

$$px + q(y - y_0) - z - \frac{abx}{y - y_0} = \text{const.}$$

in the original variables. If the first is multiplied by an arbitrary constant x_0, and subtracted from the second, a more symmetric form

$$p(x - x_0) + q(y - y_0) - z - ab\,\frac{x - x_0}{y - y_0} = \text{const.}$$

results for the second.

With $ab = k$, for $\lambda = k(y - y_0)^{-2}$ equation (1) consequently has the intermediate integrals

$$p(x - x_0) + q(y - y_0) - z - k\,\frac{x - x_0}{y - y_0} = \phi\left(p - \frac{k}{y - y_0}\right),$$

$$p(x - x_0) + q(y - y_0) - z + k\,\frac{x - x_0}{y - y_0} = \psi\left(p + \frac{k}{y - y_0}\right),$$

involving arbitrary functions ϕ, ψ, the second intermediate integral arising from (3′).

IV. $m = 3$, $n = 1$. Here

$$S_3 : \sum_{\alpha=1}^{4} b_{1\alpha} F_\alpha = 0; \qquad T_1 : \frac{G_1}{b_{11}} = \frac{G_2}{b_{12}} = \frac{G_3}{b_{13}} = \frac{G_4}{b_{14}},$$

and T_1 is the normal to the hyperplane S_3 at the origin. From (5) the functions X, Y, F must meet the conditions

$$(b_{11}x + b_{12})X_1' + b_{13}X_1 + b_{14} = 0, \quad \frac{YF_u}{b_{11}} = \frac{-YF_p}{b_{12}} = \frac{YF_u - Y'F_v}{b_{13}} = \frac{F_y + vF_u}{b_{14}},$$

and we begin with the case

(i) $b_{11} \neq 0$.

There is no loss in generality if we write

$$b_{11} = 1, \; b_{12} = -x_0, \; b_{13} = -m, \; b_{14} = -a_0,$$

to put the above equations in the form

$$(x - x_0)X_1' - mX_1 - a_0 = 0,$$

$$x_0F_u - F_p = 0, \; (m+1)F_u - LF_v = 0, \; (v + a_0Y)F_u + F_y = 0,$$

where we recall $L = Y'/Y$. We find

$$X = a_0(x - x_0)^{-1} \quad (m = 0); \qquad X = a(x - x_0)^{m-1} \quad (a = \text{const.}, \; m \neq 0),$$

from the first of these equations. The last three are simultaneous equations for Y, F. Any solution of the first must have the form

$$F = G(r, v, y), \quad r = u + x_0p,$$

which, substituted in the remaining two, yields simultaneous equations for G:

$$(m+1)G_r - LG_v = 0, \quad (v + a_0Y)G_r + G_y = 0.$$

If $m = -1$, either $G_v \equiv 0$ to imply $G = \text{const.}$, or $L = 0$ to imply

$$X = a(x - x_0)^{-2}, \; Y = b, \quad \lambda = k(x - x_0)^{-2}, \; k = ab,$$

and the existence of intermediate integrals

$$p(x - x_0) + q(y - y_0) - z + k\frac{y - y_0}{x - x_0} = \phi\left(q + \frac{k}{x - x_0}\right),$$

$$p(x - x_0) + q(y - y_0) - z - k\frac{y - y_0}{x - x_0} = \psi\left(q - \frac{k}{x - x_0}\right),$$

analogous to those found in III.

If $m \neq -1$, for the general solution

$$G = H(s, y), \qquad s = (m+1)v + Lr,$$

of the first equation for G to yield a solution of the second, requires that the identity

$$[s + a_0(m+1)Y]LH_s + (m+1)H_y + r[(m+1)L' - L^2]H_s \equiv 0$$

hold. For $H \neq$ const. this implies

$$(m+1)L' - L^2 = 0, \qquad [s + a_0(m+1)Y]LH_s + (m+1)H_y = 0,$$

the first of which yields

$$Y = \frac{b}{(y - y_0)^{m+1}} \qquad (b, y_0 = \text{const.}),$$

so that the second becomes

$$H_y - \left[\frac{s}{y - y_0} + \frac{a_0 b(m+1)}{(y - y_0)^{m+2}}\right] H_s = 0.$$

This yields the intermediate integrals

$$(y - y_0)s - \frac{a_0 b(m+1)}{m(y - y_0)^m} = \text{const.} \qquad (m \neq 0),$$

$$(y - y_0)s + a_0 b \log (y - y_0) = \text{const.} \qquad (m = 0),$$

or, in the original variables, the intermediate integrals[2]

[2]If we write the intermediate integral for $m \neq 0$ in the form $p(x - x_0) + q(y - y_0) - z - kn(r^{1/n} - 1) = \text{const.}$, $r = (x - x_0)/(y - y_0)$, $n = 1/m$, and let $n \longrightarrow \infty$, this intermediate integral approaches the one for $m = 0$, in view of

$$\lim_{n \to \infty} n(r^{1/n} - 1) = \log r,$$

for which see [4, pp. 139-140].

$$p(x-x_0)+q(y-y_0)-z-\frac{k}{m}\left(\frac{x-x_0}{y-y_0}\right)^m=\text{const.}\qquad (m\neq 0),$$

$$p(x-x_0)+q(y-y_0)-z+\frac{k}{m}\left(\frac{x-x_0}{y-y_0}\right)^m=\text{const.}\qquad (m\neq 0),$$

$$p(x-x_0)+q(y-y_0)-z-k\log\frac{x-x_0}{y-y_0}=\text{const.}\qquad (m=0),$$

$$p(x-x_0)+q(y-y_0)-z+k\log\frac{x-x_0}{y-y_0}=\text{const.}\qquad (m=0),$$

of (1) for

$$\lambda=k\,\frac{(x-x_0)^{m-1}}{(y-y_0)^{m+1}}\qquad (k=ab,\ m\neq 0),$$

$$\lambda=k\,\frac{(x-x_0)^{-1}}{y-y_0}\qquad (k=a_0b,\ m=0),$$

the second arising from (3′).

The next case to consider is

(ii) $b_{11}=0,\ b_{12}\neq 0.$

Here $F=F(v,y,p)$, and we can write

$$b_{12}=1,\ b_{13}=-m,\ b_{14}=c,$$

so that X, Y, F satisfy the equations

$$X_1'-mX_1+c=0,\ F_y+cYF_p=0,\ LF_v+mF_p=0.$$

Since $m=0$ implies $X=\text{const.}$, treated in III, we assume $m\neq 0$, and obtain

$$X=ae^{mx}\qquad (a=\text{const.})$$

from the first equation. From the second equation we get

$$F = G(r, v) \text{ with } r = p - cY_1,$$

and for this to satisfy the third, necessarily $L = n = \text{const.}$, so that we have

$$Y = be^{ny} \qquad (b = \text{const.}),$$

and the intermediate integral $nr - mv = \text{const.}$; or, in the original variables,

$$np - mq + ke^{mx}e^{ny} = \text{const.}, \quad np - mq - ke^{mx}e^{ny} = \text{const.},$$

are intermediate integrals of (1) for

$$\lambda = ke^{mx}e^{ny} \qquad (k = ab).$$

The last possibility is

(iii) $b_{11} = b_{12} = 0$, $b_{13} \neq 0$.

Here $X_1 = \text{const.}$, so $X \equiv 0$, which has been excluded.

V. $m = 4$, $n = 0$. This requires $G_1 = G_2 = G_3 = G_4 = 0$; if we exclude $Y = \text{const.}$, as previously treated, this can only arise if $F \equiv \text{const.}$

If we observe that the form of (1) is left invariant under the transformations in the group

$$x' = x - x_0, \; y' = y - y_0; \quad x' = y, \; y' = x; \quad x' = kx, \; y' = ly,$$

the results of this section can be summed up in the theorem:

THEOREM 1. *The Monge-Ampère equation*

$$rt - s^2 + \lambda^2 = 0, \quad \lambda = X(x)Y(y),$$

has intermediate integrals only in the following cases:

λ	*Intermediate Integrals*
1. 0	$q = \phi(p)$
2. 1	$q - x = \phi(p + y)$, $q + x = \psi(p - y)$
3. $Y(y) \neq 0$	$p + Y_1 = \text{const.}$, $p - Y_1 = \text{const.}$, $Y_1 = \int Y\,dy$,
4. y^{-2}	$px + qy - z - \frac{x}{y} = \phi\left(p - \frac{1}{y}\right)$, $px + qy - z + \frac{x}{y} = \psi\left(p + \frac{1}{y}\right)$,

5. $\dfrac{x^{m-1}}{y^{m+1}}$ $(m \neq 0)$ $px + qy - z - \dfrac{1}{m}\left(\dfrac{x}{y}\right)^m = \text{const.}$, $px + qy - z + \dfrac{1}{m}\left(\dfrac{x}{y}\right)^m = \text{const.}$,

6. $\dfrac{x^{-1}}{y}$ $px + qy - z - \log\dfrac{x}{y} = \text{const.}$, $px + qy - z + \log\dfrac{x}{y} = \text{const.}$,

7. $e^x e^y$ $p - q + e^x e^y = \text{const.}$, $p - q - e^x e^y = \text{const.}$,

and in those which arise from these under translations, reflections in the line $y = x$, and dilations in the (x, y)-plane.

That each solution of the partial differential equation of first order represented by an intermediate integral is actually a solution of the appropriate Monge-Ampère equation may be verified directly by differentiating the intermediate integral partially with respect to x and y, and calculating $rt - s^2$.

2. Linearization of the Monge-Ampère equation. The integration of (1) for all cases except the first in Theorem 1 will be reduced to the integration of linear partial differential equations of at most the second order and quadratures.

The differential system (2), (2′) is replaced by an apparently over-determined system

$$(7) \qquad \begin{aligned} z_\beta - px_\beta - qy_\beta &= 0, & z_\alpha - px_\alpha - qy_\alpha &= 0, \\ p_\beta + \lambda y_\beta &= 0, & p_\alpha - \lambda y_\alpha &= 0, \qquad \lambda = \lambda(x, y, z, p, q), \\ q_\beta - \lambda x_\beta &= 0, & q_\alpha + \lambda x_\alpha &= 0, \end{aligned}$$

of six equations for five unknown functions

$$(8) \qquad x = x(\alpha, \beta),\ y = y(\alpha, \beta),\ z = z(\alpha, \beta),\ p = p(\alpha, \beta),\ q = q(\alpha, \beta),$$

of the characteristic variables[3] α, β. Actually, if one supposes that the quantity

$$A = z_\alpha - px_\alpha - qy_\alpha$$

vanishes along a curve C drawn on a solution surface S defined parametrically by the first three equations in (8), provided the curves $\alpha =$ const. on S cut C,

[3]The idea of introducing the characteristic variables α, β as independent variables to replace the characteristic differential system (2) by the system of partial differential equations (7) has been ascribed to Hans Lewy in a footnote on p. 327 of [1]. According to Goursat [3, pp. 106-116], the idea goes back to Ampère. Lewy was the first to use them, however, in the proof of the existence and uniqueness of the solution to Cauchy's problem when intermediate integrals do not exist.

it is easy to show that $A \equiv 0$ on S; for the remaining equations in (7) imply [1, pp. 329-330] $A_\beta \equiv 0$, and therefore, since $A = 0$ on C, it vanishes everywhere on S.

If $\lambda = \lambda(x, y, p, q)$, the four equations in the last two rows of (7) form a determined system for the unknown functions x, y, p, q of α, β. Corresponding to any solution

$$(9) \qquad x = x(\alpha, \beta), \quad y = y(\alpha, \beta), \quad p = p(\alpha, \beta), \quad q = q(\alpha, \beta)$$

of this system, the first row in (7) yields

$$(10) \qquad z = \int \frac{p^2}{\lambda}\left\{-\left(\frac{q}{p}\right)_\alpha d\alpha + \left(\frac{q}{p}\right)_\beta d\beta\right\} = z(\alpha, \beta),$$

when x_α, y_α, x_β, y_β are eliminated with the aid of the remaining equations. Consequently

$$\left(\frac{q}{p}\right)_\alpha = -\Lambda z_\alpha, \quad \left(\frac{q}{p}\right)_\beta = \Lambda z_\beta \qquad (\Lambda = \lambda/p^2),$$

so that $z(\alpha, \beta)$ is a solution of the linear equation

$$(11) \qquad z_{\alpha\beta} + \frac{\Lambda_\beta}{2\Lambda} z_\alpha + \frac{\Lambda_\alpha}{2\Lambda} z_\beta = 0,$$

the function $\Lambda(\alpha, \beta)$ depending, in general, on the selection of the solution (9).

It is worth noting that $z(\alpha, \beta)$ is a solution of the linear equation (11) for $\lambda = \lambda(x, y, p, q)$, i.e., this result is not restricted to the special form $\lambda = X(x) Y(y)$.

The treatment of the various cases will be based on the following lemma.

LEMMA. *The integration of the system*

$$v_\beta - \kappa u_\beta = 0, \quad v_\alpha + \kappa u_\alpha = 0, \quad \kappa = \kappa(\alpha - \beta),$$

is equivalent to the integration of either one of the pair of conjugate [5] *linear equations*

$$L(u) = u_{\alpha\beta} - \frac{\kappa'}{2\kappa}(u_\alpha - u_\beta) = 0, \quad M(v) = v_{\alpha\beta} + \frac{\kappa'}{2\kappa}(v_\alpha - v_\beta) = 0.$$

If $u[v]$ is a solution of $L(u) = 0$ $[M(v) = 0]$, its conjugate function $v[u]$ may be obtained by quadratures.

We begin with:

Case 1. Here (1) is the equation for the developable surfaces, and its integration is well known.

Case 2. Instead of (7) we have

$$(12)\qquad \begin{aligned} z_\beta - px_\beta - qy_\beta &= 0, \\ p + y &= \alpha, \\ q_\beta - x_\beta &= 0, \end{aligned} \qquad \begin{aligned} z_\alpha - px_\alpha - qy_\alpha &= 0, \\ p - y &= \beta, \\ q_\alpha + x_\alpha &= 0. \end{aligned}$$

From the intermediate integrals

$$p + y = \alpha_0, \quad p - y = \beta_0, \quad \alpha_0, \beta_0 = \text{const.},$$

we obtain solutions

$$(13)\qquad z = \alpha_0 x - xy + G(y), \qquad z = \beta_0 x + xy + G(y),$$

involving an arbitrary function $G(y)$.

The integration of the system in the third row of (12) is equivalent to the integration of either of the linear equations

$$L(x) = x_{\alpha\beta} = 0, \quad M(q) = q_{\alpha\beta} = 0,$$

and for (9) we find the formulas of Goursat [3, pp. 154-155]

$$x = \frac{1}{2}[\psi'(\beta) - \phi'(\alpha)], \ y = \frac{1}{2}(\alpha - \beta), \ p = \frac{1}{2}(\alpha + \beta), \ q = \frac{1}{2}[\phi'(\alpha) + \psi'(\beta)],$$

where $\phi(\alpha)$, $\psi(\beta)$ are arbitrary functions. Carrying out the quadratures in (10) yields

$$(15)\qquad z = \frac{1}{4}(\alpha + \beta)[\psi'(\beta) - \phi'(\alpha)] + \frac{1}{2}[\phi(\alpha) - \psi(\beta)],$$

a solution[4], containing two arbitrary functions, of

$$z_{\alpha\beta} - \frac{z_\alpha + z_\beta}{\alpha + \beta} = 0,$$

to which (11) reduces.

The solutions (13) are not contained among the solutions (14), (15). For example, for the first solution in (13) the sum $p + y$ is constant over the entire solution surface; but for a solution (14), (15), while the sum $p + y = \alpha$ is constant along each characteristic α = const., it varies from one characteristic to another.

Case 3. System (7) is replaced by

$$\begin{array}{lll} z_\beta - px_\beta - qy_\beta = 0, & z_\alpha - px_\alpha - qy_\alpha = 0, & \\ p + Y_1 = \alpha, & p - Y_1 = \beta, & (Y_1 = \int Y\,dy) \\ q_\beta - Yx_\beta = 0, & q_\alpha + Yx_\alpha = 0, & \end{array} \tag{16}$$

The intermediate integrals yield solutions

$$z = \alpha_0 x - Y_1 x + G(y), \quad z = \beta_0 x + Y_1 x + G(y) \tag{17}$$

containing an arbitrary function $G(y)$.

To obtain other solutions we observe that

$$Y_1 = \frac{1}{2}(\alpha - \beta),$$

and consequently Y is a function κ of $\alpha - \beta$. In view of the lemma, (9) becomes

$$x = x(\alpha, \beta), \quad y = y(\alpha - \beta), \quad p = \frac{1}{2}(\alpha + \beta), \quad q = q(\alpha, \beta), \tag{18}$$

where x $[q]$ is any solution of the linear equation $L(x) = 0$ $[M(q) = 0]$, and q $[x]$ is determined by quadratures. From (10) the function z is obtained by carrying out the quadratures.

$$z = \frac{1}{2}\int \frac{(\alpha + \beta)^2}{\kappa} \left\{ -\left(\frac{q}{\alpha + \beta}\right)_\alpha d\alpha + \left(\frac{q}{\alpha + \beta}\right)_\beta d\beta \right\} = z(\alpha, \beta), \tag{19}$$

[4]This reduces to the partial differential equation $E(-1, -1)$ of Euler-Darboux when β is replaced by $-\beta$. See [2, pp. 54-70].

and is a solution of the linear equation

$$(20)\qquad z_{\alpha\beta}-\left(\frac{\kappa'}{2\kappa}+\frac{1}{\alpha+\beta}\right)z_\alpha+\left(\frac{\kappa'}{2\kappa}-\frac{1}{\alpha+\beta}\right)z_\beta=0.$$

Case 4. This is a special case of the above in which

$$L(x)=x_{\alpha\beta}-\frac{x_\alpha-x_\beta}{\alpha-\beta}=0,\quad M(q)=q_{\alpha\beta}+\frac{q_\alpha-q_\beta}{\alpha-\beta}=0,$$

are the Euler-Darboux equations $E(1, 1)$, $E(-1,-1)$ respectively, and it is possible to give [2, p. 64] explicit formulas for (18), (19):

$$(18')\quad x=\frac{\phi'-\psi'}{\alpha-\beta},\ y=\frac{2}{\beta-\alpha},\ p=\frac{1}{2}(\alpha+\beta),\ q=\frac{1}{4}(\beta-\alpha)(\phi'+\psi')+\frac{1}{2}(\phi-\psi),$$

$$(19')\qquad z=\frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta}(\phi'-\psi')-\frac{\phi-\psi}{\alpha-\beta},$$

containing arbitrary functions $\phi(\alpha)$, $\psi(\beta)$, with (20) becoming

$$(20')\qquad z_{\alpha\beta}-\frac{2\alpha}{\alpha^2-\beta^2}z_\alpha+\frac{2\beta}{\alpha^2-\beta^2}z_\beta=0.$$

Case 5. System (7) becomes

$$(21)\qquad \begin{aligned} &z_\beta-px_\beta-qy_\beta=0, && z_\alpha-px_\alpha-qy_\alpha=0,\\ &p_\beta+\frac{x^{m-1}}{y^{m+1}}y_\beta=0, && p_\alpha-\frac{x^{m-1}}{y^{m+1}}y_\alpha=0,\\ &q_\beta-\frac{x^{m-1}}{y^{m+1}}x_\beta=0, && q_\alpha+\frac{x^{m-1}}{y^{m+1}}x_\alpha=0.\end{aligned}$$

From the intermediate integrals

$$(22)\qquad px+qy-z-\frac{1}{m}\left(\frac{x}{y}\right)^m=\alpha_0,\quad px+qy-z+\frac{1}{m}\left(\frac{x}{y}\right)^m=\beta_0,$$

we obtain solutions

$$(23) \qquad z = xG(r) - \frac{r^m}{m} - \alpha_0, \quad z = xG(r) + \frac{r^m}{m} - \beta_0 \quad (r = x/y),$$

containing an arbitrary function $G(r)$.

To obtain other solutions, we observe from Theorem 1 that

$$(24) \qquad \left(\frac{x}{y}\right)^m = \frac{m}{2}(\beta - \alpha), \quad px + qy - z = \frac{1}{2}(\alpha + \beta),$$

and use the first of these equations to eliminate x from the second row, y from the third row of (21), to replace these rows by

$$(25) \qquad p_\beta - \kappa_1 \eta_\beta = 0, \; p_\alpha + \kappa_1 \eta_\alpha = 0, \; \eta = \frac{1}{y}, \; \kappa_1 = \left[\frac{m}{2}(\beta - \alpha)\right]^{1-1/m},$$

$$(26) \qquad q_\beta + \kappa_2 \xi_\beta = 0, \; q_\alpha - \kappa_2 \xi_\alpha = 0, \; \xi = \frac{1}{x}, \; \kappa_2 = \left[\frac{m}{2}(\beta - \alpha)\right]^{1+1/m}.$$

In view of the lemma, the integration of (25) is equivalent to the integration of either one of

$$(27) \qquad L(\eta) = \eta_{\alpha\beta} - \frac{A}{\alpha - \beta}(\eta_\alpha - \eta_\beta) = 0, \; M(p) = p_{\alpha\beta} + \frac{A}{\alpha - \beta}(p_\alpha - p_\beta) = 0,$$

$$\left(A = \frac{1}{2}(1 - 1/m)\right),$$

and the intègration of (26) to the integration of either one of

$$(28) \qquad L(\xi) = \xi_{\alpha\beta} - \frac{B}{\alpha - \beta}(\xi_\alpha - \xi_\beta) = 0, \; M(q) = q_{\alpha\beta} + \frac{B}{\alpha - \beta}(q_\alpha - q_\beta) = 0,$$

$$\left(B = \frac{1}{2}(1 + 1/m)\right),$$

all of which are Euler-Darboux equations of the type $E(k, k)$. A solution (9) of the last two rows of (21) may be obtained by starting with either (25) or (26). To fix the ideas, let $p = p(\alpha, \beta)$, $y = y(\alpha, \beta)$ be any solution of (25). The first equation in (24) yields $\xi = \xi(\alpha, \beta)$, with $q = q(\alpha, \beta)$ determined by quadratures from (26). Finally we obtain $z = z(\alpha, \beta)$ from the second equation in

(24). Thus the integration of (1) is reduced to the integration of any one of the linear equations (27), (28) followed by quadratures.

For $m = 1$ this case reduces to the preceding one; and (27), (28) simplify to

$$L(\eta) = \eta_{\alpha\beta} = 0, \quad M(p) = p_{\alpha\beta} = 0;$$

$$L(\xi) = \xi_{\alpha\beta} - \frac{\xi_\alpha - \xi_\beta}{\alpha - \beta} = 0, \quad M(q) = q_{\alpha\beta} + \frac{q_\alpha - q_\beta}{\alpha - \beta} = 0.$$

By carrying out the process outlined above we find a solution containing two arbitrary functions

$$x = \frac{\alpha - \beta}{\phi' - \psi'}, \; y = \frac{2}{\psi' - \phi'}, \quad p = \frac{1}{2}(\phi' + \psi'), \quad q = \frac{1}{4}(\alpha - \beta)(\phi' + \psi') - \frac{1}{2}(\phi - \psi),$$

with

$$z = \frac{\phi - \psi}{\phi' - \psi'} - \frac{1}{2}(\alpha + \beta),$$

to which (18′), (19′) reduce under the change of parameter

$$\bar{\alpha} = \phi'(\alpha), \quad \bar{\beta} = \psi'(\beta); \qquad \alpha = \bar{\phi}'(\bar{\alpha}), \quad \beta = \bar{\psi}'(\bar{\beta}),$$

provided one observes that

$$\phi = \bar{\alpha}\bar{\phi}'(\bar{\alpha}) - \bar{\phi}(\bar{\alpha}), \qquad \psi = \bar{\beta}\bar{\psi}'(\bar{\beta}) - \bar{\psi}(\bar{\beta}).$$

Case 6. System (7) is the same as (21) for $m = 0$, but the intermediate integrals (22) are replaced by

$$px + qy - z - \log\frac{x}{y} = \alpha_0, \quad px + qy - z + \log\frac{x}{y} = \beta_0,$$

with the solutions

$$(29) \qquad z = xG(r) - \log r - \alpha_0, \quad z = xG(r) + \log r - \beta_0 \qquad (r = x/y),$$

in place of (23), and the relations

$$\log \frac{x}{y} = \frac{1}{2}(\beta - \alpha), \qquad px + qy - z = \frac{1}{2}(\alpha + \beta),$$

replacing (24). The quantities κ_1, κ_2 in (25), (26) are now

$$\kappa_1 = e^{(\alpha-\beta)/2}, \qquad \kappa_2 = e^{(\beta-\alpha)/2},$$

and instead of the Euler-Darboux equations (27), (28) we find linear equations with constant coefficients

$$(30) \qquad \begin{aligned} L(\eta) &= \eta_{\alpha\beta} - \frac{1}{4}(\eta_\alpha - \eta_\beta) = 0, \quad M(p) = p_{\alpha\beta} + \frac{1}{4}(p_\alpha - p_\beta) = 0, \\ L(\xi) &= \xi_{\alpha\beta} + \frac{1}{4}(\xi_\alpha - \xi_\beta) = 0, \quad M(q) = q_{\alpha\beta} - \frac{1}{4}(q_\alpha - q_\beta) = 0. \end{aligned}$$

Once a solution of any one of these has been obtained, the integration of (1) proceeds as in the previous case.

Case 7. The intermediate integrals are

$$p - q + e^x e^y = \alpha_0, \quad p - q - e^x e^y = \beta_0,$$

and have the solutions

$$(31) \qquad z = G(r) - xe^r + \alpha_0 x, \quad z = G(r) + xe^r + \beta_0 x, \quad r = x + y.$$

In place of (24) we find

$$(32) \qquad x + y = \log \frac{1}{2}(\alpha - \beta), \quad p - q = \frac{1}{2}(\alpha + \beta),$$

and (25), (26) are replaced by

$$(33) \qquad p_\beta - \kappa_1 y_\beta = 0, \quad p_\alpha + \kappa_1 y_\alpha = 0, \quad \kappa_1 = \frac{1}{2}(\beta - \alpha),$$

$$(34) \qquad q_\beta - \kappa_2 x_\beta = 0, \quad q_\alpha + \kappa_2 x_\alpha = 0, \quad \kappa_2 = \frac{1}{2}(\alpha - \beta),$$

so that instead of (27), (28) we obtain Euler-Darboux equations of the special, forms [2, pp. 69-70] $E(\pm 1/2, \pm 1/2)$, specifically

$$(35)\quad L(y) = y_{\alpha\beta} - \frac{1/2}{\alpha-\beta}(y_\alpha - y_\beta) = 0, \qquad M(p) = p_{\alpha\beta} + \frac{1/2}{\alpha-\beta}(p_\alpha - p_\beta) = 0,$$

$$L(x) = x_{\alpha\beta} - \frac{1/2}{\alpha-\beta}(x_\alpha - x_\beta) = 0, \qquad M(q) = q_{\alpha\beta} + \frac{1/2}{\alpha-\beta}(q_\alpha - q_\beta) = 0.$$

Starting with a solution of any one of these, say $p = p(\alpha, \beta)$, one determines $q = q(\alpha, \beta)$ from (32) and calculates the functions $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ by quadratures based on (33), (34). A final quadrature (10) then yields $z = z(\alpha, \beta)$.

The results obtained may be summarized in the theorem:

THEOREM 2. *Whenever the partial differential equation*

$$rt - s^2 + \lambda^2 = 0, \quad \lambda = X(x)Y(y),$$

has intermediate integrals, its integration can be reduced to the integration of linear partial differential equations of the first and second order, and quadratures.

3. The Problem of Cauchy. In the Problem of Cauchy one requires a solution $z = z(x, y)$ of (1) such that along a prescribed curve (the *carrier*)

$$C:\ x = x(t), \quad y = y(t),$$

in the (x, y)-plane, the partial derivatives $p = z_x$, $q = z_y$ take preassigned values (the *Cauchy data*)

$$C_1:\ p = p(t), \quad q = q(t).$$

We begin with the simplest case.

Case 2. Here the Problem of Cauchy imposes the following conditions

$$\alpha + \beta = 2p(t), \quad \alpha - \beta = 2y(t), \quad \phi'(\alpha) + \psi'(\beta) = 2q(t), \quad \phi'(\alpha) - \psi'(\beta) = -2x(t),$$

on the arbitrary functions $\phi(\alpha)$, $\psi(\beta)$ entering into the formulas (14) of Goursat. These imply

$$\alpha = p(t) + y(t), \quad \beta = p(t) - y(t); \quad \phi'(\alpha) = q(t) - x(t), \quad \psi'(\beta) = q(t) + x(t),$$

of which the first pair determine a curve (the *carrier*)

$$\Gamma:\ \alpha = \alpha(t) = p(t) + y(t),\quad \beta = \beta(t) = p(t) - y(t),$$

in the *characteristic* (α, β)-plane, the horizontal and vertical lines of which are termed *characteristic lines*.

Provided[5] $\dot{\alpha} \neq 0$, $\dot{\beta} \neq 0$ we can invert the equations defining Γ to obtain

$$t = f(\alpha),\quad t = g(\beta),$$

and obtain the required functions $\phi(\alpha)$, $\psi(\beta)$ in (14), up to arbitrarily additive constants, by quadratures from

$$\phi'(\alpha) = q(f(\alpha)) - x(f(\alpha)),\quad \psi'(\beta) = q(g(\beta)) + x(g(\beta)).$$

If Γ is a segment of a characteristic line, say of $\alpha = \alpha_0 =$ const., the Cauchy data C_1 cannot be taken arbitrarily, but must fulfill the conditions

$$p(t) + y(t) \equiv \alpha_0,\quad q(t) - x(t) \equiv k = \text{const.},$$

so that up to the additive constant k, the carrier C prescribes the Cauchy data C_1. The first equation for Γ fails to determine $f(\alpha)$, and $\phi(\alpha)$ may accordingly be taken arbitrarily in Goursat's formulas, subject to the single condition $\phi'(\alpha_0) = k$. Due to the arbitrariness in $\phi(\alpha)$, the solution of the Problem of Cauchy is not unique. In addition, the general solution (13) of the intermediate integral $p + y = \alpha_0$ offers an additional solution, for we have $q + x = G'(y)$, and if Γ is not a point, $y = y(t) \neq$ const., so that $t = t(y)$, and $G(y)$ can be determined from

$$G'(y) = q(t(y)) + x(t(y))$$

up to an additive constant.

The carrier Γ will be a point if, and only if, C, C_1 have the form

$$C:\ x = x(t),\ y = y_0 = \text{const.},\quad C_1:\ p = p_0 = \text{const.},\quad q = q(t),$$

and therefore necessarily, as follows from (1) with $\lambda = 1$,

$$q(t) \pm x(t) \equiv k = \text{const.}$$

The solutions (13) of the intermediate integrals

$$p + y = \alpha_0 = p_0 + y_0,\quad p - y = \beta_0 = p_0 - y_0,$$

[5]We forego consideration of the difficult case in which $\dot{\alpha}$ or $\dot{\beta}$ changes sign.

offer solutions for these Problems of Cauchy, containing an arbitrary function $G(y)$, provided $G'(y_0) = k$.

Case 3. The carrier Γ in the characteristic plane is defined by

$$\Gamma: \quad \alpha = \alpha(t) = p(t) + Y_1(y(t)), \quad \beta = \beta(t) = p(t) - Y_1(y(t)).$$

By differentiating the first and fourth equations (18) with respect to t we obtain

$$x_\alpha \dot{\alpha} + x_\beta \dot{\beta} = \dot{x}, \quad -x_\alpha \dot{\alpha} + x_\beta \dot{\beta} = \kappa^{-1} \dot{q}.$$

Along an arc of Γ for which $\dot{\alpha} \neq 0$, $\dot{\beta} \neq 0$ these equations specify Cauchy data

$$x_\alpha = \frac{\dot{x} - \kappa^{-1}\dot{q}}{2\dot{\alpha}}, \quad x_\beta = \frac{\dot{x} + \kappa^{-1}\dot{q}}{2\dot{\beta}},$$

on the carrier Γ for a Problem of Cauchy for the linear equation $L(x) = 0$. The solution of this Problem of Cauchy can be effected by quadratures, once the resolvent [5], a properly chosen two-parameter family of solutions of the conjugate equation $M(q) = 0$, is known, and, up to an arbitrarily additive constant, is unique.

If Γ is a segment of a characteristic line $\alpha = \alpha_0$, the Cauchy data C, C_1 must meet the conditions

$$p(t) + Y_1(y(t)) \equiv \alpha_0, \quad \dot{q}(t) - Y(y(t))\dot{x}(t) \equiv 0,$$

so C implies C_1 up to an additive constant. The Cauchy data on Γ reduce to

$$x_\beta = \frac{\dot{x}}{\dot{\beta}} = g(\beta)$$

and fail to determine a unique solution to $L(x) = 0$, so that the solution to the original problem of Cauchy C, C_1 framed for (1) is likewise no longer unique. In addition to this multiplicity of solutions, the general solution (17) of the intermediate integral $p + Y_1 = \alpha_0$ provides another, for if Γ is not a point, $t = t(y)$, the arbitrary function $G(y)$ is determined by

$$G'(y) = q(t(y)) + x(t(y))\, Y(y)$$

up to an additive constant.

The carrier Γ will be a point if, and only if, C, C_1 have the form

$$C: \ x = x(t), \ y = y_0 = \text{const.}, \quad C_1: \ p = p_0 = \text{const.}, \ q = q(t),$$

and therefore necessarily, as follows from (1) with $\lambda = Y(y)$,

$$q(t) \pm Y(y_0)x(t) \equiv k = \text{const.}$$

The solutions (17) of the intermediate integrals

$$p + Y_1(y) = \alpha_0 = p_0 + Y_1(y_0), \quad p - Y_1(y) = \beta_0 = p_0 - Y_1(y_0),$$

offer solutions to these Problems of Cauchy containing an arbitrary function $G(y)$, provided $G'(y_0) = k$.

Case 4. This case comes under Case 3, and also Case 5 for $m = 1$.

Case 5. For the carrier Γ we have

$$\Gamma: \ \alpha = px + qy - z - \frac{1}{m}\left(\frac{x}{y}\right)^m = \alpha(t), \ \beta = px + qy - z + \frac{1}{m}\left(\frac{x}{y}\right)^m = \beta(t),$$

where

$$z = \int (p\dot{x} + q\dot{y})\,dt = z(t).$$

Recalling that $\eta = y^{-1}$ we write

$$\eta_\alpha \dot{\alpha} + \eta_\beta \dot{\beta} = \dot{\eta}, \quad -\eta_\alpha \dot{\alpha} + \eta_\beta \dot{\beta} = \kappa_1^{-1}\dot{p},$$

where κ_1 is defined in (25).

As long as $\dot{\alpha} \neq 0$, $\dot{\beta} \neq 0$ these equations specify Cauchy data

$$\eta_\alpha = \frac{\dot{\eta} - \kappa_1^{-1}\dot{p}}{2\dot{\alpha}}, \quad \eta_\beta = \frac{\dot{\eta} + \kappa_1^{-1}\dot{p}}{2\dot{\beta}},$$

on the carrier Γ for a Problem of Cauchy for the linear equation $L(\eta) = 0$ in (27). The resolvent of this equation is known [5, pp. 401-406], and the solution of the Problem of Cauchy can be carried out by quadratures. The solution is unique up to an arbitrarily additive constant.

If Γ is a segment of a characteristic line $\alpha = \alpha_0$, the Cauchy data C, C_1

must meet the conditions

$$\dot{p} + \frac{x^{m-1}}{y^{m+1}} \dot{y} = 0, \qquad \dot{q} - \frac{x^{m-1}}{y^{m+1}} \dot{x} = 0,$$

with C_1 determined by C up to translations. The Cauchy data

$$\eta_\beta = \frac{\dot{\eta}}{\dot{\beta}} = g(\beta)$$

are insufficient to determine a unique solution to $L(\eta) = 0$, and the solution of the original problem of Cauchy again is no longer unique. The general solution (23) of the first intermediate integral in (22) offers an additional solution, provided the arbitrary function $G(r)$ can be properly determined. This will be possible, as long as $r = r(t) \neq$ const. (C not a radial straight line), or, what is equivalent, Γ is not a point. Indeed, from (23) we determine G by integrating

$$G'(r) = \frac{r^{m-2}}{y} - \frac{q}{r^2},$$

the given functions y, q of t becoming functions of r, since $t = t(r)$.

For Γ to be a point (α_0, β_0) it is necessary and sufficient that

$$px + qy - z = k = \frac{1}{2}(\alpha_0 + \beta_0), \quad \frac{x}{y} = r_0 = \left[\frac{m}{2}(\beta_0 - \alpha_0)\right]^{1/m},$$

conditions which are equivalent to the requirements

$$\dot{p}x + \dot{q}y = 0, \quad \frac{x}{y} = r_0,$$

on the Cauchy data C, C_1. It is obvious from the very form of the intermediate integrals (22) that any one of general solutions (23) meets the required conditions.

The treatment of the remaining cases does not differ substantially from this case and we may sum up our results as follows:

THEOREM 3. *Whenever the partial differential equation*

$$rt - s^2 + \lambda^2 = C \qquad (\lambda = X(x)Y(y))$$

has intermediate integrals

$$\alpha(x, y, z, p, q) = \alpha, \qquad \beta(x, y, z, p, q) = \beta,$$

the Problem of Cauchy

$$C: \ x = x(t), \ y = y(t); \qquad C_1: \ p = p(t), \ q = q(t),$$

for this partial differential equation defines a carrier

$$\Gamma: \ \alpha = \alpha(x(t), y(t), z(t), p(t), q(t)) = \alpha(t),$$

$$\beta = \beta(x(t), y(t), z(t), p(t), q(t)) = \beta(t),$$

in the characteristic plane, and reduces to a Problem of Cauchy with Cauchy data on this carrier for a linear partial differential equation of the second order.

If Γ is a segment of one of the characteristic lines α = const., or β = const., or a point, the solution is not unique.

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