# THE REFLECTION PRINCIPLE FOR POLYHARMONIC FUNCTIONS

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**1. Introduction.** In this paper the reflection principle for harmonic functions is extended to the more general class of $p$-harmonic functions. The case $p = 2$ has already been treated by R. J. Duffin [1] whose method of proof will be used in part. The formula for the reflection of a biharmonic function at a straight line segment in the plane was even previously known to H. Poritsky [3]; but he did not indicate under which conditions such a continuation would have to exist.

A function $w(x_1, x_2, \cdots, x_n)$ is called $p$-harmonic in a region $D$ of the $n$-dimensional space, if it is of class $C^{2p}$ and satisfies the differential equation $\Delta^p w = 0$. We shall make use of the following well-known properties:

(I) $w$ is analytic throughout $D$.

(II) The following representation always exists:

$$w = \sum_{\nu=0}^{p-1} x_1^{\nu}\, u_{\nu}(x_1, x_2, \cdots, x_n),$$

the functions $u_1, u_2, \cdots, u_{p-1}$ being harmonic in $D$; conversely such a sum is always p-harmonic. As a consequence, the following decompositions are also possible

$$w = f(x_1, x_2, \cdots, x_n) + x_1^{p-k}\, g(x_1, x_2, \cdots, x_n), \quad k = 1, 2, \cdots, p-1,$$

$f$ denoting a $(p-k)$-harmonic, $g$ being a $k$-harmonic function.

**2. Reflection principle.**

THEOREM. *Let $G$ denote a region of the $n$-dimensional space, the boundary of which contains an open subset $S$ of $x_1 = 0$. If the function $w(x_1, x_2, \cdots, x_n)$ is p-harmonic in $G$ and if $w/x_1^{p-1}$ assumes the boundary value $0$ on $S$, then $w$ can be continued analytically across $S$ into the reflected domain $G'$ by putting*

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$$(2.1)\quad w(-x_1, x_2, \cdots, x_n) = (-1)^p \sum_{k=0}^{p-1} (-1)^k (k!)^{-2}\, x_1^{p+k}\, \Delta^k \left( \frac{w(x_1, x_2, \cdots, x_n)}{x_1^{p-k}} \right).$$

REMARKS. In the boundary condition for $w$, $x_1^{p-1}$ cannot be replaced by any smaller power of $x_1$. This is clearly indicated by the example $x_1^{p-1} \log(x_1^2 + x_2^2)$. This function is $p$-harmonic in $x_1 > 0$, but cannot be continued analytically into the origin.

For $p = 1$ (2.1) reduces to $w(-x_1, x_2, \cdots, x_n) = -w(x_1, x_2, \cdots, x_n)$, that is, the classical reflection principle of H. A. Schwarz.

In the case $p = 2$ we obtain the continuation formula of R. J. Duffin [1]:

$$w(-x_1, x_2, \cdots, x_n) = -w + 2x_1 \frac{\partial w}{\partial x_1} - x_1^2 \Delta w .$$

**3. Proof.**

We proceed in two steps:

(a) *$w$ can be continued analytically beyond $S$,*

(b) *the continuation can be extended to the whole of $G'$ and is given by* (2.1).

We can prove (a) by complete induction with respect to $p$. It is well known that (a) is true for $p = 1$. It remains to be shown that (a) holds for any positive integer $p$, provided it is known to be valid for $p - 1$. R. J. Duffin [1] has carried out this step for the special case $p = 2$. His reasoning can be extended to the general case without essential change, although there arise some technical complications (for example, the averaging operation has to be iterated $p$ times). Therefore, we refer to Duffin's work for the proof of (a).

For the proof of (b) we introduce the notation $v(x_1, x_2, \cdots, x_n)$ for the right hand side of (2.1).

From (a) we know that $w$ is analytic on $S$, and from the boundary condition we conclude further that the functions $w/x_1^{p-k}$ $(k = 0, 1, \cdots, p-1)$ are also analytic on $S$. Therefore $v$ is analytic on $G \cup S$.

Moreover, each term $x_1^{p+k}\, \Delta^k (w/x_1^{p-k})$ is a $p$-harmonic function. For we have, indicating by superscript the degree of harmonicity,

$$\Delta^p \{ x_1^{p+k}\, \Delta^k [ w^{(p)}/x_1^{p-k} ] \} = \Delta^p \{ x_1^{p+k}\, \Delta^k [ ( f^{(p-k)} + x_1^{p-k} g^{(k)} )/x_1^{p-k} ] \}$$

$$= \Delta^p \{ x_1^{p+k}\, \Delta^k [ f^{(p-k)}/x_1^{p-k} ] \} = \Delta^p \{ x_1^{p+k} ( h^{(p)}/x_1^{p+k} ) \} = 0 .$$

Consequently, $v$ is $p$-harmonic.

Now we shall prove that the Taylor developments of the functions $w(-x_1, x_2, \cdots, x_n)$ and $v(x_1, x_2, \cdots, x_n)$ are identical in an arbitrary point of $S$. This will complete the proof of the theorem. For then we can immediately draw the conclusion that $w(-x_1, x_2, \cdots, x_n)$, being identical with $v(x_1, x_2, \cdots, x_n)$, can be continued into the whole of $G$. Therefore, (2.1) yields in fact a continuation of $w(x_1, x_2, \cdots, x_n)$ into the reflected domain $G'$.

Let

$$w = \sum a_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} \tag{3.1}$$

be the Taylor development of $w$ at an arbitrary point of $S$, which we may assume to be the origin of the coordinate system without losing generality. We are left to prove that the development of $v$ at the same point is given by

$$v = \sum (-1)^{\nu_1} a_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}. \tag{3.2}$$

First we demonstrate a lemma which we shall use in the sequel:

LEMMA. (see [2, problem 137]). *Let $r$, $s$, $t$ denote arbitrary positive integers, such that $t \neq r$. Then the identity*

$$\sum_{l=0}^{r} (-1)^l \binom{l+s}{t} \binom{r}{l} = 0 \tag{3.3}$$

*holds*. (We define $\binom{l+s}{t} = 0$ for $l + s < t$).

*Proof of the lemma.* Differentiating the identity

$$(1-x)^r = \sum_{l=0}^{r} (-1)^l \binom{r}{l} x^l$$

$t$ *times with respect to $x$, putting* $x = 1$ and dividing by $t!$ we get (3.3) for the special case $s = 0$. By complete induction with respect to $s$ and making use of the relation

$$\binom{l+s}{t} = \binom{l+s-1}{t} + \binom{l+s-1}{t-1}$$

the general case is easily established.

As a first step in the proof of (3.2) we now verify that the operator (2.1) transforms the function

$$x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} \text{ into } (-1)^{\nu_1} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n};\ \nu_1, \nu_2, \cdots, \nu_n$$

being arbitrary positive integers and $p \leq \nu_1 \leq 2p-1$. Putting this function into (2.1) we obtain

$$(-1)^p \sum_{k=0}^{p-1} (-1)^k (k!)^{-2} x_1^{p+k} \Delta^k \left( x_1^{\nu_1 - p + k} x_2^{\nu_2} \cdots x_n^{\nu_n} \right) = S_1 + S_2$$

where

$$S_1 = (-1)^p \sum_{k=0}^{\nu_1 - p} (-1)^k (k!)^{-2} x_1^{p+k} \partial^{2k} \left( x_1^{\nu_1 - p + k} x_2^{\nu_2} \cdots x_n^{\nu_n} \right) / \partial^{2k} x_1$$

and

$$S_2 = (-1)^p \sum_{k=0}^{p-1} (-1)^k (k!)^{-2} x_1^{p+k} \sum_{\substack{k_1 < k;\\ k_1 + k_2 + \cdots + k_n = k}} \frac{k!}{k_1! k_2! \cdots k_n!} \frac{\partial^{2k} (x_1^{\nu_1 - p + k} x_2^{\nu_2} \cdots x_n^{\nu_n})}{\partial^{2k_1} x_1\, \partial^{2k_2} x_2 \cdots \partial^{2k_n} x_n}.$$

We have

$$(3.4) \qquad S_1 = (-1)^p \sum_{k=0}^{\nu_1 - p} (-1)^k (k!)^{-2} \frac{(\nu_1 - p + k)!}{(\nu_1 - p - k)!} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$$

$$= (-1)^{\nu_1} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$$

because of

$$\sum_{k=0}^{\nu_1 - p} (-1)^k \frac{(\nu_1 - p + k)!}{(k!)^2 (\nu_1 - p - k)!} = \sum_{k=0}^{\nu_1 - p} (-1)^k \binom{\nu_1 - p + k}{k} \binom{\nu_1 - p}{k}$$

$$= \sum_{k=0}^{\nu_1 - p} (-1)^k \binom{\nu_1 - p}{k} \sum_{l=0}^{k} \binom{k}{l} \binom{\nu_1 - p}{l}$$

$$= \sum_{l=0}^{\nu_1-p} \binom{\nu_1-p}{l} \sum_{k=l}^{\nu_1-p} (-1)^k \binom{\nu_1-p}{k}\binom{k}{l} = (-1)^{\nu_1-p},$$

since the inner sum in the last expression vanishes for $l < \nu_1 - p$ and equals $(-1)^{\nu_1-p}$ for $l = \nu_1 - p$ (see solution of problem I41 in [2]).

In order to prove that $S_2 = 0$ we consider now the sum $\sigma$ of all terms in $S_2$ belonging to an arbitrary but fixed set of values $(k_2, k_3, \cdots, k_n) \neq (0, 0, \cdots, 0)$. In the following we shall denote by $A_1$, $A_2$ and $A_3$ factors which do not depend on the summation index. Putting $k^* = k_2 + k_3 + \cdots + k_n$, we have

$$\sigma = A_1 \sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k (k!)^{-2} \frac{k!(\nu_1-p+k)!}{k_1!(\nu_1-p-k+2k^*)!} x_1^{\nu_1+2k^*} x_2^{\nu_2-2k_2} \cdots x_n^{\nu_n-2k_n}$$

$$= A_2 \sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k \binom{k}{k^*}\binom{\nu_1-p+k}{k}\binom{\nu_1-p+2k^*}{k}$$

$$= A_2 \sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k \binom{k}{k^*}\binom{\nu_1-p+2k^*}{k} \sum_{l=0}^{k}\binom{\nu_1-p}{l}\binom{k}{l} = \sigma' + \sigma'',$$

where

$$\sigma' = A_2 \sum_{l=0}^{k^*-1} \binom{\nu_1-p}{l} \sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k \binom{k}{k^*}\binom{k}{l}\binom{\nu_1-p+2k^*}{k} \tag{3.5}$$

and

$$\sigma'' = A_2 \sum_{l=k^*}^{\nu_1-p+2k^*} \binom{\nu_1-p}{l} \sum_{k=l}^{\nu_1-p+2k^*} (-1)^k \binom{k}{k^*}\binom{k}{l}\binom{\nu_1-p+2k^*}{k}. \tag{3.6}$$

But since

$$\sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k \binom{k}{k^*}\binom{k}{l}\binom{\nu_1-p+2k^*}{k} \tag{3.7}$$

$$= A_3 \sum_{k=k^*}^{\nu_1-p+2k^*} (-1)^k \binom{k}{l}\binom{\nu_1-p+k^*}{k-k^*} = 0$$

by means of (3.3), we conclude that $\sigma' = 0$. Furthermore, we observe that the summation over $k$ in (3.6) may be extended from $k = k^*$ to $k = \nu_1 - p + 2k^*$ because $\binom{k}{l} = 0$ in all terms with index $k$ lying in the interval $k^* \leq k < l$. Therefore $\sigma'' = 0$. Consequently $\sigma = 0$ and, finally, $S_2 = 0$. Combining this with (3.4), we obtain the stated property of the operator (2.1).

We now apply the transformation (2.1) to the series (3.1). This may be carried out term by term. Putting

$$v = \sum b_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n},$$

we have to prove that

$$b_{\nu_1 \nu_2 \cdots \nu_n} = (-1)^{\nu_1} a_{\nu_1 \nu_2 \cdots \nu_n}. \tag{3.8}$$

From the boundary condition we infer that $\partial^k w / \partial x_1^k = 0$ $(k = 0, 1, \cdots, p-1)$ on $S$. Since this holds in all points of $S$ it follows that $a_{\nu_1 \nu_2 \cdots \nu_n} = 0$ for $\nu_1 \leq p - 1$ and arbitrary $\nu_2, \nu_3, \cdots, \nu_n$. On the other hand, it is easy to see that the operator (2.1) never decreases the number of factors $x_1$ in a term. Therefore we have also $b_{\nu_1 \nu_2 \cdots \nu_n} = 0$ for $\nu_1 \leq p-1$ and arbitrary $\nu_2, \nu_3, \cdots, \nu_n$. This proves (3.8) for $\nu_1 \leq p - 1$.

We have verified above that the term

$$a_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$$

is transformed into

$$(-1)^{\nu_1} a_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$$

for $p \leq \nu_1 \leq 2p - 1$ and arbitrary $\nu_2, \nu_3, \cdots, \nu_n$. Since (2.1) never decreases the number of factors $x_1$, terms of order $\geq 2p$ in $x_1$ cannot contribute to the coefficients $b_{\nu_1 \nu_2 \cdots \nu_n}$, where $p \leq \nu_1 \leq 2p - 1$, and (3.8) is thus demonstrated for $p \leq \nu_1 \leq 2p - 1$.

Finally, we observe that both $w(-x_1, x_2, \cdots, x_n)$ and $v(x_1, x_2, \cdots, x_n)$ are $p$-harmonic functions. Using the differential equation (and eventually differentiating it) any derivative of order $\geq 2p$ in $x_1$ can be expressed as a linear combination of derivatives of the orders $\nu_1 - 2$, $\nu_1 - 4, \cdots, \nu_1 - 2p$ in $x_1$. If the equality of the latters has already been shown, it follows that the first ones also have to be equal. By complete induction we thus are able to assert the validity of (3.8) in the remaining range $\nu_1 \geq 2p$. This completes the proof.

## References

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