RETRACTIONS IN SEMIGROUPS

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Let S be a semigroup (that is, a Hausdorff space together with a continuous associative multiplication) and let E denote the set of idempotents of S. If $x \in S$ let

$$L_x = \{y | y \cup Sy = x \cup Sx\}$$

and

$$R_x = \{y | y \cup yS = x \cup xS\} .$$

Put $H_x = L_x \cap R_x$ and for $e \in E$ let

$$H = \cup \{H_e | e \in E\} ,$$

$$M_e = \{x | ex \in H \text{ and } xe \in H\} ,$$

$$Z_e = H_e \times (R_e \cap E) \times (L_e \cap E)$$

and

$$K_e = (L_e \cap E) \cdot H_e \cdot (R_e \cap E) .$$

Under the assumption that S is compact we shall prove that K_e is a retract of M_e and that K_e and Z_e are equivalent, both algebraically and topologically. This latter fact sharpens a result announced in [6] and the former settles several questions raised in [7].

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LEMMA 1. *Let $Z = S \times S \times S$ and define a multiplication in Z by*

$$(t, x, y) \cdot (t', x', y') = (txy't', x', y) ;$$

then Z is a semigroup and, with this multiplication, the function f: $Z \to S$ defined by $f(t, x, y) = ytx$ is a continuous homomorphism.

The proof of this is immediate. *We use only the above defined multiplication in Z and not coordinatewise multiplication.* It is clear that $f(Z_e) = K_e$.

Since the sets H_e, $e \in E$, are pairwise disjoint groups [1] it is legitimate to define functions

$$\eta : H \to E , \qquad \theta : H \to H$$

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by "$\eta(x)$ is the unit of the group H_e which contains x" and "$\theta(x)$ is the inverse of x in the group H_e which contains x". If $x \in M_e$ then ex, $xe \in H$ so that $\eta(ex)$, $\eta(xe)$ are defined. Define $g : M_e \to Z$ by

$$g(x)=(exe,\ \eta(ex),\ \eta(xe))$$

and note that the continuity of η implies the continuity of g. For $x \in M_e$ let

$$\rho(x)=\eta(xe)x\eta(ex)$$

so that ρ is continuous if η is continuous.

LEMMA 2. *For any $x \in K_e$ we have $fg(x)=x=\rho(x)$ and $g(K_e)=Z_e$. The function $f|Z_e$ takes Z_e onto K_e in a one-to-one way and is a homeomorphism if η is continuous. If η is continuous then ρ retracts M_e onto K_e.*

Proof. Let $t \in H_e, e_1 \in R_e \cap E$ and $e_2 \in L_e \cap E$. Since $L_{e_2}=L_e$ it is immediate that $ee_2=e$ and since t is an element of the group H_e whose unit is e (Green [3]) we also have $et=t=te$. Similarly we see that $e_1e=e$. It is important to observe that the sets $\{L_x|x \in S\}$, $\{R_x|x \in S\}$ and $\{H_x|x \in S\}$ are disjointed covers of S so that, for example $L_x \cap L_y \neq \square$ implies $L_x=L_y$. We see that $ee_2te_1=te_1$ and $e_2te_1e=e_2t$ so that $ee_2te_1e=t$. We note next that $te_1 \in H_{e_1}$ and thus $\eta(te_1)=e_1$. For $e \in R_e \cap L_e=R_{e_1} \cap L_t$ and $e^2=e$, give $te_1 \in R_t \cap L_{e_1}$ in view of Theorem 3 of [2]. But

$$R_t \cap L_{e_1}=R_e \cap L_{e_1}=R_{e_1} \cap L_{e_1}=H_{e_1}$$

and H_{e_1} being a group with unit e_1 we have, from the definition of η, $\eta(te_1)=e_1$. In a similar fashion we show that $\eta(e_2t)=e_2$. If $x \in K_e$ then we have $x=e_2te_1$ with the above notation and

$$\begin{aligned} fg(x)&=f(exe,\ \eta(ex),\ \eta(xe))=\eta(xe)exe\,\eta(ex) \\ &=\eta(e_2t)t\,\eta(te_1)=e_2te_1=x\ . \end{aligned}$$

It will suffice to show in addition that $gf(z)=z$ for $z \in Z$ since $fg(x)=x$ gives $x=\rho(x)$. Now let $z=(t,\ e_1,\ e_2) \in Z_e$ so that $f(z)=e_2te_1 \in K_e$ and

$$g(f(z))=(ef(z)e,\ \eta(ef(z)),\ \eta(f(z)e))=(t,\ e_1,\ e_2)$$

in virtue of the computation given earlier.

It remains to prove the continuity of η when S is compact. This was announced in [7] but no proof of this fact has been published. Let

$$\mathscr{L}=\{(x,\ y)\,|\,L_x=L_y\}\ , \qquad \mathscr{R}=\{(x,\ y)\,|\,R_x=R_y\}$$

and let $\mathscr{H}=\mathscr{L} \cap \mathscr{R}$.

LEMMA 3. *If S is compact then $\mathscr{H}$, $\mathscr{L}$ and $\mathscr{R}$ are closed.*

Proof. Let

$$\mathscr{L}' = \{(x, y) \mid Sx \subset Sy\}$$

and assume that $(a, b) \in S \times S \setminus \mathscr{L}'$. Then $Sb \subset S \setminus a$ and hence $Sb \subset S \setminus U^*$ for some open set U about a since Sb is closed and S is regular. Again from the compactness of S we can find an open set V about b such that $SV \subset S \setminus U^*$. Hence $(U \times V) \cap \mathscr{L}' = \square$ and we may infer that $\mathscr{L}'$ is closed. There is no loss of generality in assuming that S has a unit [3]. Hence if $h: S \times S \to S \times S$ is defined by $h(x, y) = (y, x)$ then $h(\mathscr{L}')$ is closed and thus $\mathscr{L} = \mathscr{L}' \cap h(\mathscr{L}')$ is closed. In a similar way it may be shown that $\mathscr{R}$ is closed. Moreover, $\mathscr{H}$ is closed because $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

THEOREM 1 [7]. *If S is compact then H is closed, $\eta: H \to E$ is a retraction and $\theta: H \to H$ is a homeomorphism.*

Proof. Define $p: S \times S \to S$ by $p(x, y) = x$. Then

$$H = \bigcup \{H_e \mid e \in E\} = p(\mathscr{H} \cap (S \times E))$$

is closed since $\mathscr{H}$ and E are closed. We show next that θ is continuous and to this end it is enough to prove that $G = \{(x, \theta(x)) | x \in H\}$ in virtue of the fact that H is compact Hausdorff. If $m: S \times S \to S$ is defined by $m(x, y) = xy$ then $\mathscr{H} \cap (H \times H) \cap m^{-1}(E)$ is closed and we will show that this set is the same as G. For $(x, \theta(x))$ in G implies $m(x, \theta(x)) = x\theta(x) \in E$ in virtue of the definition of θ. Since x and $\theta(x)$ are in the same set H_e, $e \in E$, it is clear that $(x, \theta(x)) \in H \times H$ and it is easily seen from the definition of $H_x = L_x \cap R_x$, and $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ that also $(x, \theta(x)) \in \mathscr{H}$. Now take x, y such that $xy = e \in E$, $x, y \in H$ and $(x, y) \in \mathscr{H}$. The last fact shows that $H_x = H_y$ and the penultimate condition, together with this shows that $x, y \in H_{e_1}$ for some $e_1 \in E$. But $e = xy \in H_{e_1}$ and the fact that H_{e_1} is a group implies that $e = e_1$. Now the uniqueness of inversion in the group H_e shows that $y = \theta(x)$. Hence θ is continuous and η is continuous because $\eta(x) = x\theta(x)$ from the definition of η and θ.

G. B. Preston raised the question as to the continuity of a certain generalized "inversion"—Suppose that there is a unique function $\alpha: S \to S$ such that $x\alpha(x)x = x$ and $\alpha(x)x\alpha(x) = \alpha(x)$ for each $x \in S$. If S is compact then α is continuous. To see this let $\mathscr{N}$ be the set of all $(x, y) \in S \times S$ such that $xyx = x$ and $yxy = y$ and define $\varphi: S \times S \to S \times S$ by $\varphi(x, y) = (xyx, x)$. If D is the diagonal of $S \times S$ then $\varphi^{-1}(D)$ is closed. Similarly $\psi^{-1}(D)$ is closed where $\psi(x, y) = (y, yxy)$ and $\mathscr{N} = \varphi^{-1}(D) \cap \psi^{-1}(D)$ is therefore closed. The uniqueness of α implies that $\{(x, \alpha(x)) | x \in S\} = \mathscr{N}$

so that α is continuous if S is compact. For a discussion of the existence and uniqueness of such functions as α, see [2, pp. 273–274] as well as references therein to Liber, Munn and Penrose, Thierrin, Vagner and the papers of Preston in London Math. Soc., 1954.

From Theorem 1 and Lemma 2 we obtain at once

THEOREM 3. *Let S be compact and let $e \in E$; then K_e is topologically isomorphic with*

$$Z_e = H_e \times (L_e \cap E) \times (R_e \cap E)$$

and K_e is a retract of M_e.

It is not asserted that K_e is a subsemigroup of S. The first corollary is a topologized form of the Rees-Suschkewitsch theorem, see [6], [7] and [2] for a bibliography of relevant algebraic results.

COROLLARY 1. *If S is compact, if K is the minimal ideal of S and if $e \in E \cap K$ then K is topologically isomorphic with $eSe \times (Se \cap E) \times (es \cap E)$ and K and each "factor" of K is a retract of S.*

Proof. We rely, without explicit citation, on the results of [1]. It is immediate that $M_e = S$. Now $L_e = Se$, $Re = eS$ and $H_e = eSe$ so that (by definition and [1]) $K_e = Se \cdot eSe \cdot eS \subset K$ and, being an ideal, $K_e = K$. Clearly $x \to exe$ retracts S onto eSe. Now $Se \subset K \subset H$ and $\eta | Se$ retracts Se onto $Se \cap E$.

It is clear, when S is compact, that K enjoys all the retraction invariants of S, for example, if S is locally connected so is K. We do not list these nor do we give here the applications of Corollary 1 that were mentioned in [6].

COROLLARY 2. *If S is a clan* [7], *if $K \subset E$ and if $H^n(S) \neq 0$ for some $n > 0$ and some coefficient group, then* $\dim K \geqq 2$.

Proof. If $K \subset E$ then $H_e = \{e\}$ and K is thus topologically the product $Se \times eS$ since $Se, eS \subset K$. Now $H^n(Se) \approx H^n(S) \approx H^n(eS)$ [9] and hence Se, eS are non-degenerate continua. It follows that $\dim K \geqq 2$.

It is possible to put some of the above in a more general framework. Let T be a closed subsemigroup of S and let

$$L_x = \{y | x \cup Tx = y \cup Ty\} ,$$

with similar definitions for R_x and H_x. If $e \in E$ then H_e is a semigroup and H_e is a group if $eT \cup Te \subset T$. If $\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$ are defined analogously then $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Moreover we have $\mathscr{L} \circ \mathscr{R} = \mathscr{J}$, where

$$\mathscr{J}=\{(x, y)|x \cup Tx \cup xT \cup TxT=y \cup Ty \cup yT \cup TyT\} ,$$

when S is compact [5]. In this case $\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{L}\circ\mathscr{R}$ and $\mathscr{J}$ are closed. It is easy to see that many of the results of [3] and [2] are valid in this setting. If we define a left T-ideal as a non-void set A such that $TA \subset A$, then the basic propositions about ideals are also available. Many of these results follow from general theorems on structs [8].

References

1. A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math. **70** (1948), 521–526.

2. ——— and D. D. Miller, *Regular D-classes in semigroups*, Trans. Amer. Math. Soc. **82** (1956), 270–280.

3. J. A. Green, *On the structure of semigroups*, Ann. Math. **54** (1951), 163–172.

4. R. J. Koch and A. D. Wallace, *Maximal ideals in compact semigroups*, Duke Math. J., **21** (1954), 681–685.

5. ——— and ———, *Stability in semigroups*, Duke Math. J., **24**, (1957), 193–196.

6. A. D. Wallace, *The Rees-Suschkewitsch theorem for compact semigroups*, Proc. Nat. Acad. Sci. **42** (1956), 430–432.

7. ———, *The structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112.

8. ———, *Struct ideals*, Proc. Amer. Math. Soc. **6** (1955), 634–642.

9. ———, *Inverses in Euclidean mobs*, Math. J. Okayama Univ. **3** (1953), 23–28.

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