ON A CRITERION FOR THE WEAKNESS OF AN IDEAL BOUNDARY COMPONENT

KÔTARO OIKAWA

1. Exhaustion. Let F be an open Riemann surface. An *exhaustion* $\{F_n\}$ of F is an increasing (i.e., $\overline{F}_n \subset F_{n+1}$) sequence of subregions with compact closures such that $\bigcup_{n=1}^{\infty} F_n = F$. We assume that ∂F_n consists of a finite number of closed analytic curves and that each component of $F - F_n$ is noncompact. This is the most common definition used in the theory of open Riemann surfaces. Sometimes, however, we shall add the restriction that each component of ∂F_n is a dividing cycle; if this is the case we shall call the exhaustion *canonical*.

2. Weak boundary component. Let γ be an ideal boundary component of F, and let $\{F_n\}$ be a canonical exhaustion of F. Then there exists a component γ_n of ∂F_n which separates γ from F_n. Let n_0 be a fixed number and consider the component G_n of $\overline{F}_n - F_{n_0}$ $(n > n_0)$ such that $\gamma_n \subset \partial G_n$. There exists a harmonic function $s_n(p)$ on $\overline{G}_n$ which satisfies the following conditions:

(i) $s_n = 0$ on γ_{n_0} and $\int_{\gamma_{n_0}} *ds_n = 2\pi$, $(\gamma_{n_0} = \partial F_{n_0} \cap \partial G_n)$

(ii) $s_n = \log r_n = \text{const.}$ on γ_n,

(iii) $s_n = \text{const.}$ on each component $\beta_{n\nu}$ of $\partial G_n - \gamma_n - \gamma_{n_0}$ and $\int_{\beta_{n\nu}} *ds_n = 0$.

The condition $\lim_{n\to\infty} r_n = \infty$ depends neither on n_0 nor on the exhaustion. If it is satisfied, γ is said to be *weak*.

Weak boundary components were introduced for plane regions by Grötzch [1] in connection with the so-called Kreisnormierungsproblem. He called them vollkommen punktförmig. They were generalized for open Riemann surfaces by Sario [6] and discussed also by Savage [7] and Jurchescu [2]. The above definition was given by Jurchescu [2].

A noncompact subregion N whose relative boundary ∂N consists of a finite number of closed analytic curves is called a *neighborhood of* γ if γ is an ideal boundary component of N as well. Let $\{c\}$ be the family of all cycles c (i.e., unions of finite numbers of closed curves) which are in N and separate γ from ∂N. Jurchescu [2] showed that $\lambda\{c\} = 0$ if and only if γ is weak, where $\lambda\{c\}$ is the extremal length of the family $\{c\}$.

Received October 22, 1958. This paper was prepared under Contract No. DA-04-495-ORD-722, OOR Project No. 1517 between the University of California, Los Angeles and the Office of Ordnance Research, U. S. Army.

3. Savage's criterion. Let $\{F_n\}$ be an arbitrary exhaustion. Let E_n be the smallest union of components of $F_n - \overline{F}_{n-1}$ such that $\gamma_{n-1} = \partial E_n \cap \partial F_{n-1}$ is a cycle which separates γ from F_{n-1} $(n = 2, 3, \cdots)$. Evidently $\gamma_n \subset \partial E_n$. If $\{F_n\}$ is canonical, E_n is connected and γ_n is a closed analytic curve.

There exists a harmonic function $u_n(p)$ on $\overline{E}_n$ such that

(i) $u_n = 0$ on γ_{n-1} and $\int_{\gamma_{n-1}} *du_n = 2\pi$,

(ii) $u_n = \log \mu_n = \text{const.}$ on $\partial E_n - \gamma_{n-1} = \partial E_n \cap \partial F_n$.

The quantity $\log \mu_n$ is called the *modulus of* E_n (cf. Sario [4,5], who called μ_n the modulus). It is expressed in terms of extremal length as follows:

$$\log \mu_n = \frac{2\pi}{\lambda\{c\}_n},$$

where $\{c\}_n$ is the family of cycles in E_n homologous to γ_{n-1}.

Since $\sum^\infty 1/\lambda\{c\}_n \leqq 1/\lambda\{c\}$, we get the following criterion:

THEOREM 1 (Savage [7]). *If there exists an exhaustion such that* $\prod_{n=2}^\infty \mu_n = \infty$, *then* γ *is weak.*

The purpose of the present note is to discuss the converse of this theorem.

4. Jurchescu's criterion. Suppose the exhaustion $\{F_n\}$ is canonical. There exists a harmonic function $U_n(p)$ on $\overline{E}_n$ such that

(i) $U_n = 0$ on γ_{n-1} and $\int_{\gamma_{n-1}} *dU_n = 2\pi$,

(ii) $U_n = \log M_n = \text{const.}$ on γ_n,

(iii) $U_n = \text{const.}$ on each component $\beta_{n\nu}$ of $\partial E_n - \gamma_n - \gamma_{n-1}$ and $\int_{\beta_{n\nu}} *dU_n = 0$.

Jurchesch's paper [2] contains implicitly the following result:

THEOREM 2 (Jurchescu). *A boundary component* γ *is weak if and only if there exists a canonical exhaustion such that* $\prod_{n=2}^\infty M_n = \infty$.

Proof. Sufficiency: Let $\{c'\}_n$ be the family of cycles in E_n separating γ_n from γ_{n-1}. It is not difficult to see that $\log M_n = 2\pi/\lambda\{c'\}_n$. Since $\sum^\infty 1/\lambda\{c'\}_n \leqq 1/\lambda\{c\}$, we conclude that $\sum_{n=2}^\infty \log M_n = \infty$ implies $\lambda\{c\} = 0$.

Necessity: Consider a canonical exhaustion $\{F_n^0\}$. The desired exhaustion $\{F_n\}$ is obtained by taking its subsequence as follows:

$F_1 = F_1^0$. To define F_2, consider the quantity r_n introduced in No. 2 with respect to $F_n^0 - \overline{F}_1^0$ $(n = 2, 3, \cdots)$. Take n_2 so large that $r_{n_2} \geqq 2$, and put $F_2 = F^0_{n_2}$. Evidently $M_2 = r_{n_2}$. Similarly, $F_3 = F^0_{n_3}$ is defined by considering $F^0_n - \overline{F}^0_{n_2}$ $(n = n_2 + 1, n_2 + 2, \cdots)$ and by taking $n_3 > n_2$ so large that $r_{n_3} \geqq 2$ where r_{n_2} is the quantity r_n introduced in No. 2 with respect to $F^0_n - \overline{F^0_{n_2}}$. We have $M_3 = r_{n_3}$. On continuing this process, we obtain a canonical exhaustion such that $\sum_{n=2}^{\infty} \log M_n \geqq \sum_{n=2}^{\infty} \log 2 = \infty$. The idea of this proof was first used by Noshiro [3].

5. The converse of Savage's criterion. We shall now show that Savage's criterion in Theorem 1 is also necessary.

THEOREM 3. *If γ is weak, then there exists an exhaustion such that $\prod_{n=2}^{\infty} \mu_n = \infty$. It is not necessarily canonical.*

Proof. By Theorem 2 there exists a canonical exhaustion $\{F^0_n\}$ such that $\prod_{n=2}^{\infty} M^0_n = \infty$. From this we construct a canonical exhaustion $\{F^*_n\}$ as follows:

$F^*_1 = F^0_1$. To construct F^*_2, let $\partial E^0_2 - \gamma^0_1 - \gamma^0_2 = \beta_{21} \cup \beta_{22} \cup \cdots \cup \beta_{2k_2}$ be the decomposition into components, and let H^ν_3 be the component of $F^0_3 - F^0_2$ such that $\partial H^\nu_3 \cap \overline{F}^0_2 = \beta_{2\nu}$ $(\nu = 1, 2, \cdots, k_2)$. F^*_2 is the union of $F^*_1, E^0_2 \cup \gamma^0_1$, all the other components of $F^0_2 - F^0_1$, and $\bigcup_{\nu=1}^{k_2} H^\nu_3$. In this way, F^*_n is defined as the union of F^*_{n-1}, $E^0_n \cup \gamma^0_{n-1}$, every component of $F^0_{m+1} - F^0_m$ $(m \geqq n)$ which is adjacent to F^*_{n-1}, and $\bigcup_{\nu=1}^{k_n} H^\nu_{n+1}$. By construction, $E^*_n = E^0_n \cup \bigcup_{\nu=1}^{k_n} H^\nu_{n+1}$.

The desired exhaustion $\{F_n\}$ is obtained by taking a refinement of $\{F^*_n\}$ as follows: Consider E^0_n and the function U^0_n for the exhaustion $\{F^0_n\}$. Let $\partial E^0_n - \gamma^0_n - \gamma^0_{n-1} = \beta_{n1} \cup \beta_{n2} \cup \cdots \cup \beta_{nk_n}$ be the decomposition into components and let $U^0_n \equiv a_\nu$ on $\beta_{n\nu}$ $(\nu = 1, 2, \cdots, k_n)$. We may assume, without loss of generality, that the a_ν's are different by pairs. We suppose that

$$0 \equiv a_0 < a_1 < \cdots < a_{k_n} < a_{k_n+1} \equiv \log M^0_n.$$

Take a'_ν $(a_{\nu-1} < a'_\nu < a_\nu\,;\ \nu = 1, 2, \cdots, k_n, a'_{k_n+1} \equiv \log M^0_n)$ and a''_ν $(a_\nu < a''_\nu < a_{\nu+1}\,;\ \nu = 1, \cdots, k_n, a''_0 \equiv 0)$ so close to a_ν that

$$\sum_{\nu=1}^{k_n+1} (a'_\nu - a''_{\nu-1}) \geqq \log M^0_n - 2^{-n} \,. \tag{1}$$

Consider the sets

$$D^\nu_n = \{p\,;\, a''_{\nu-1} < U^0_n(p) < a''_\nu\},\ \nu = 1, 2, \cdots, k_n + 1,\ (a''_{k_n+1} \equiv \log M^0_n)$$
$$D'^\nu_n = \{p\,;a''_{\nu-1} < U^0_n(p) < a'_\nu\},\ \nu = 1, 2, \cdots, k_n + 1\,.$$

The modulus $\log \mu'^{(\nu)}$ of D'^ν_n with respect to $\beta^\nu = \{p\,;\, U^0_n(p) = a''_{\nu-1}\}$ and $\partial D'^\nu_n - \beta^\nu$ is equal to $a'_\nu - a''_{\nu-1}$, since the function $U^0_n(p) - a''_{\nu-1}$ plays the role of $u_n(p)$ introduced in No. 3. Let $\log \mu^{(\nu)}$ be the modulus of D^ν_n

with respect to β^ν and $\partial D_n^\nu - \beta^\nu$. Since $\mu^{(\nu)} \geqq \mu'^{(\nu)}$, we obtain, by (1),

$$\sum_{\nu=1}^{k_n+1} \log \mu^{(\nu)} \geqq \log M_n^0 - 2^{-n} . \tag{2}$$

We have decomposed E_n^0 into $k_n + 1$ subsets D_n^ν. $E_n^* - E_n^0$ consists of components H_{n+1}^ν such that $\beta_{n\nu} = \partial H_{n+1}^\nu \cap \partial E_n^0 (\nu = 1, 2, \cdots, k_n)$. By decomposing H_{n+1}^ν into $k_n - \nu + 1$ slices, we obtain a decomposition of E_n^* into $k_n + 1$ parts. It is possible to divide each of the other components of $F_n^* - \overline{F}_{n-1}^*$ into $k_n + 1$ pieces so that we get an exhaustion $\{F_n\}$ which is a refinement of $\{F_n^*\}$. D_n^ν plays the role of E_n with respect to this exhaustion. Therefore, by (2), we get

$$\sum_{n=2}^{\infty} \log \mu_n \geqq \sum_{n=2}^{\infty} \log M_n^0 - 1 = \infty .$$

6. **Remark.** On a "schlichtartig" surface, every exhaustion is canonical. If F is an arbitrary Riemann surface, the question arises whether or not Savage's criterion is still necessary under the restriction that $\{F_n\}$ is canonical. The answer is given by

THEOREM 4. *There exist a γ of an F which is weak and such that $\prod_{n=2}^{\infty} \mu_n < \infty$ for every canonical exhaustion.*

Construction of F: In the plane $|z| < \infty$, consider the closed intervals

$$I_k : [2^{k^2}, 2^{k^2} + 1] \qquad (k = 2, 3, \cdots)$$

on the positive real axis, and the circular arcs

$$\alpha_\nu : |z| = \nu, |\arg z| \leqq \frac{\pi}{2}$$

$$(\nu = 2^{k^2} + 2, 2^{k^2} + 3, \cdots, 2^{(k+1)^2} - 1 ; k = 2, 3, \cdots) .$$

Take two replicas of the slit plane $(|z| < \infty) - \bigcup_{k=2}^{\infty} I_k$ and connect them crosswise across I_k $(k = 2, 3, \cdots)$. From the resulting surface, delete all the α_ν's on both sheets. This is a Riemann surface F of infinite genus.

F has an ideal boundary component γ over $z = \infty$, which is evidently weak.

Let $\{F_n\}$ be an arbitrary canonical exhaustion. Consider E_n corresponding to γ (No. 3). The interval I_k determines a closed analytic curve C_k on F. Since $\gamma_{n-1} = \partial E_n \cap \overline{F}_{n-1}$ is a dividing cycle, the intersection number $\gamma_{n-1} \times C_k$ vanishes and, therefore, $\gamma_{n-1} \cap C_k$ consists of an even number of points whenever it is not void.* Take two consecutive points

* *Added in proof.* We should have mentioned the case where γ_{n-1} tangents C_k. The following discussion covers this case if the number of the points of $\gamma_{n-1} \cap C_k$ is counted with the multiplicity of tangency and case $p=q$ is not excluded.

p and q in $\gamma_{n-1} \cap C_k$. There are two possibilities according as the arc $\widehat{pq} \subset \gamma_{n-1}$ is homotopic to $\widehat{pq} \subset C_k$ or not. If the latter case happens for at least one pair of p and q, we shall say that γ_{n-1} intersects C_k *properly*.

Since γ_{n-1} is a closed curve separating γ from F_{n-1}, there exists a number k such that γ_{n-1} intersects C_k properly. If there is more than one k, we take the greatest one and denote it by $k(n)$.

To estimate μ_n, let $\{c\}_n$ be the family of all cycles in E_n separating γ_{n-1} from $\partial E_n - \gamma_{n-1}$. We have mentioned that $\log \mu_n = 2\pi/\lambda\{c\}_n$. Let C_k be a curve for which there are numbers n with $k(n) = k$. Evidently these n are finite in number and consecutive. Let n_k be the greatest.

I. If $k(n) = k$ and $n < n_k$ then γ_{n-1} and γ_n intersect C_k properly. Since every $c \in \{c\}_n$ separates γ_{n-1} from γ_n, it has a component which intersects C_k and is not completely contained in the doubly connected region Δ_k consisting of all points that lie over $\{z\,;\ 2^{k^2} - 1 < |z| < 2^{k^2} + 2, |\arg z| < \pi/2\}$. Therefore, every c contains a curve in $\{c'\}^{(k)}$ which is the family of all curves in the right half-plane connecting I_k with the imaginary axis. Consequently

$$(3) \qquad \sum_{\substack{k(n)=k \\ n\neq n_k}} \frac{1}{\lambda\{c\}_n} \leqq \frac{1}{\lambda\{c'\}^{(k)}}\,.$$

II. $k(n) = k$ and $n = n_k$. Consider all the α_ν ($\nu \geqq 2^{k^2} + 2$) on the upper sheet. Let G_{n-1} be the component of $F - \overline{F}_{n-1}$ such that $\partial G_{n-1} = \gamma_{n-1}$. For a sufficiently large ν, α_ν is an ideal boundary component of G_{n-1}. Let $\nu(k)$ be the least ν with this property. If $\nu(k) = 2^{k^2} + 2$, then every $c \in \{c\}_n$ separates γ_{n-1} from $\alpha_{\nu(k)}$ and, therefore, it has a component intersects either C_k or one of four line segments over $[2^{k^2} - 1, 2^{k^2}]$ or $[2^{k^2} + 1, 2^{k^2} + 2]$. When $\nu(k) = 2^{l^2} + 2$ for some $l > k$, then γ_{n-1} separates $\alpha_{\nu(k)-3}$ from $\alpha_{\nu(k)}$ and every $c \in \{c\}_n$ separates γ_{n-1} from $\alpha_{\nu(k)}$, so that c has a component with the above property. If $\nu(k)$ is not of the form $2^{l^2} + 2$, then, for the same reason, every $c \in \{c\}_n$ has a component which intersects the line segment on the upper sheet lying over $[\nu(k) - 1, \nu(k)]$, and is not contained in the simply connected region on the upper sheet consisting of all points over $\{z\,;\ \nu(k) - 1 < |z| < \nu(k), |\arg z| < \pi/2\}$. In any case, every $c \in \{c\}_n$ contains a curve in $\{c''\}^{(k)}$ which is the family of all curves in the right half-plane connecting $[\nu(k) - 3, \nu(k)]$ with the imaginary axis. Therefore,

$$(4) \qquad \frac{1}{\lambda\{c\}_n} \leqq \frac{1}{\lambda\{c''\}^{(k)}}\,.$$

By (3) and (4), we obtain

$$(5) \qquad \sum_{n=2}^{\infty} \log \mu_n = 2\pi \sum_{n=2}^{\infty} \frac{1}{\lambda\{c\}_n} \leqq 2\pi \sum_{k=2}^{\infty} \left(\frac{1}{\lambda\{c'\}^{(k)}} + \frac{1}{\lambda\{c''\}^{(k)}} \right).$$

To show the convergence of $\sum_{k=2}^{\infty} 1/\lambda\{c'\}^{(k)}$, we make use of the transformation $z \rightarrow z^2$. It is immediately seen that $\lambda\{c'\}^{(k)}$ is equal to the extremal distance between $[-\infty, 0]$ and $I'_k = [2^{2k^2}, (2^{k^2}+1)^2]$ with respect to the region $A = \{[-\infty, 0] \cup I'_k\}^c$. Since A is conformally equivalent to Teichmüller's extremal region $\{[-1, 0] \cup [P, \infty]\}^c$ where

$$P = \frac{2^{2k^2}}{(2^{k^2}+1)^2 - 2^{2k^2}} ,$$

we have (Teichmüller [8])

$$\begin{aligned} \lambda\{c'\}^{(k)} &\sim \frac{\log P}{2\pi} \qquad (P \rightarrow \infty) \\ &\sim \frac{k^2 \log 2}{2\pi} \qquad (k \rightarrow \infty) , \end{aligned}$$

and, therefore, $\sum_{k=2}^{\infty} 1/\lambda\{c'\}^{(k)} < \infty$. Similarly $\sum_{k=2}^{\infty} 1/\lambda\{c''\}^{(k)} < \infty$ because $\nu(k) \geqq 2^{k^2} + 2$. We conclude that

$$\sum_{n=2}^{\infty} \log \mu_n < \infty .$$

References

1. H. Grötzsch, *Eine Bemerkung zum Koebeschen Kreisnormierungsprinzip*, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl., **87** (1935), 319-324.
2. M. Jurchescu, *Modulus of a boundary component*, Pacific J. Math., **8** (1958), 791-809.
3. K. Noshiro, *Open Riemann surface with null-boundary*, Nagoya Math. J., **3** (1951), 73-79.
4. L. Sario, *Über Riemannsche Flächen mit hebbarem Rand*, Ann. Acad. Sci. Fenn. Ser. A. I, **50** (1948), pp. 79.
5. ———, *Modular criteria on Rieman surfaces*, Duke Math. J., **20** (1953), 279-286.
6. ———, *Capacity of the boundary and of a boundary component*, Ann. Math., **59** (1954), 135-144.
7. N. Savage, *Weak boundary components of an open Riemann surface*, Duke Math. J., **24** (1957), 79-96.
8. O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbibldung*, Deutsche Math., **3** (1938), 621-678.

University of California, Los Angeles,
and
Tokyo Institute of Technology