COMMUTING BOOLEAN ALGEBRAS OF PROJECTIONS

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0. Introduction. One of the more important problems in the theory of spectral operators is to decide when the sum and product of two bounded commuting spectral operators is again spectral. J. Wermer [7] has shown that the sum and product of two bounded commuting spectral operators on Hilbert space is again spectral. N. Dunford [4, Theorem 19] and S. R. Foguel [5, Theorem 7] have shown that if the Boolean algebra of projections generated by the resolutions of the identity of two bounded commuting spectral operators on a weakly complete Banach space is bounded, then the sum and product of these operators are spectral. We therefore wish to determine conditions that insure the boundedness of the Boolean algebra of projections generated by two bounded commuting algebras of projections on a Banach space. We shall show that it suffices that one of the original algebras be strongly complete, countably decomposable, and contains no projection of infinite multiplicity. The example of S. Kakutani [6] shows that the Boolean algebra of projections generated by two commuting, strongly complete, algebras of bound 1, but both of infinite multiplicity on a non weakly complete space, need not be bounded. By slightly reworking his example, we shall show that the order of magnitude of our estimates is sharp, even for spaces of finite dimension. By taking a suitable direct sum of these examples, we obtain a separable reflexive Banach space on which we have two commuting, strongly complete, Boolean algebras of projections, both of bound 1, neither having a projection of infinite uniform multiplicity, but such that the algebra of projections they generate is unbounded. On this same Banach space we also show that the sum and product of two bounded commuting spectral operators need not be spectral.

This paper is divided into four sections: the first is devoted to the proof of a combinatorial inequality, the second contains our main theorem on the boundedness of projections, the third section consists of examples. The last section is an appendix to section two.

1. A combinatorial inequality. The required inequality is the

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assertion of the following theorem.

THEOREM 1.1. *Let $\alpha_1, \cdots, \alpha_N$ be any N complex numbers, and let $\mathscr{S}$ be the collection of all subsets S of the set $1, \cdots, N$ of indices. Then for any S_0 in $\mathscr{S}$,*

$$\left|\sum_{s\in S_0} \alpha_s\right| \leqq 2\sqrt{N\pi}\cdot 2^{-N} \sum_{S\in\mathscr{S}} \left|\sum_{s\in S} \alpha_s\right| . \tag{1.1}$$

That is, the sum of any particular subset of the α's cannot exceed in absolute value the average of the absolute values of sums taken over all subsets by more than a factor which has order of magnitude $N^{1/2}$.

It suffices to prove the slightly stronger

THEOREM 1.1. a. *Let $\beta_1, \cdots, \beta_{2N}$ be any $2N$ complex numbers, and $\mathscr{R}$ the collection of all subsets R of $\{1, \cdots, 2N\}$. Then*

$$\left|\sum_{r=1}^{n} \beta_r\right| \leqq 2\sqrt{N\pi}\cdot 4^{-N} \sum_{R\in\mathscr{R}} \left|\sum_{r\in R} \beta_r\right| . \tag{1.2}$$

This implies Theorem 1.1, for suppose that N, S_0, and the α's of that theorem are given, with $S_0 = \{s_1, \cdots, s_n\}$. Define

$$\begin{aligned} &\beta_r = \alpha_{s_r}\,, \quad 1 \leqq r \leqq n\,; \quad \beta_{N+r} = 0\,, 1 \leqq r \leqq N\,; \\ &\beta_r = 0\,, \quad n+1 \leqq r \leqq N\,; \quad \beta_{N+r} = \alpha_{s_r}\,, \quad n+1 \leqq r \leqq N\,; \end{aligned}$$

where $s_{n+1}, \cdots, s_N$ are those integers between 1 and N which are not in S_0. Then we have

$$\left|\sum_{s\in S_0} \alpha_s\right| = \left|\sum_{r=1}^{n} \beta_r\right| .$$

Also, every S in $\mathscr{S}$ determines 2^N R's in $\mathscr{R}$: namely

$$\{r \mid 1 \leqq r \leqq n \text{ and } s_r \in S\} \cup \{r \mid n+1 \leqq r \leqq N \text{ and } s_r \in S\}$$

together with any of the 2^N subsets of $\{n+1, \cdots, N+n\}$, such that

$$\left|\sum_{s\in S} \alpha_s\right| = \left|\sum_{r\in R} \beta_r\right| ,$$

so that

$$2^N \sum_{S\in\mathscr{S}} \left|\sum_{s\in S} \alpha_s\right| = \sum_{R\in\mathscr{R}} \left|\sum_{r\in R} \beta_r\right| .$$

Now if (1.2) holds, then we have

$$\left|\sum_{s\in S_0} \alpha_s\right| \leqq 2\sqrt{N\pi}\cdot 2^{-N} \sum_{S\in\mathscr{S}} \left|\sum_{s\in S} \alpha_s\right|$$

which is (1.1).

We will now show that it suffices to prove Theorem 1.1.a in the special case

$$\beta_1 = \cdots = \beta_N = 1 \ , \quad \beta_{N+1} = \cdots = \beta_{2N} = -1 \ .$$

We will first show that if we replace both β_i and β_j, $1 \leqq i, j \leqq N$, by their common average $\frac{1}{2}(\beta_i + \beta_j)$ and we have (1.2) for this new set of β's, then we necessarily had (1.2) for our original β's (Lemma 1.2 below). We then show that we can perform these two-at-a-time averagings in such a way as to eventually make the resulting β_i's, $1 \leqq i \leqq N$, all arbitrarily close to their common average (Lemma 1.3 below). By the continuity of both sides of (1.2) in the β_i's, it then suffices to prove (1.2) in the case $\beta_1 = \cdots = \beta_N$. Similarly, we may assume $\beta_{N+1} = \cdots = \beta_{2N}$. By re-indexing the β's if necessary, we may suppose

$$\left|\sum_{r=1}^{N} \beta_r\right| \geqq \left|\sum_{r=N+1}^{2N} \beta_r\right| \ ;$$

and by the homogenity of both sides of (1.2), it suffices to prove Theorem 1.2 in the case $\beta_1 = \cdots = \beta_N = 1$, $\beta_{N+1} = \cdots = \beta_{2N} = \gamma$ where γ is some complex number, $|\gamma| \leqq 1$. We will then show that we need only consider $\gamma = -1$ (Lemma 1.4 below).

LEMMA 1.2. *Suppose we set* $\beta_1' = \beta_2' = \frac{1}{2}(\beta_1 + \beta_2)$, $\beta_r' = \beta_r$, $3 \leqq r \leqq N$. *Then if* (1.2) *holds for the* β'*'s, then it holds for the* β*'s.*

Proof. Partition $\mathscr{R}$ into four disjoint classes:

$$\mathscr{R}_1 = \{R \mid 1 \in R, 2 \in R\} \ , \quad \mathscr{R}_3 = \{R \mid 1 \notin R, 2 \in R\} \ ,$$
$$\mathscr{R}_2 = \{R \mid 1 \in R, 2 \notin R\} \ , \quad \mathscr{R}_4 = \{R \mid 1 \notin R, 2 \notin R\} \ .$$

If R is in $\mathscr{R}_1$ or $\mathscr{R}_4$, then $\sum_{r\in R}\beta_r = \sum_{r\in R}\beta_r'$. Now note that there is a one-to-one correspondence between $\mathscr{R}_2$ and $\mathscr{R}_3$: R is in $\mathscr{R}_2$ if and only if $R' = R \cup \{2\} - \{1\}$ is in $\mathscr{R}_3$. Then we have

$$\begin{aligned}\left|\sum_{r\in R}\beta_r'\right| + \left|\sum_{r\in R'}\beta_r'\right| &= \left|\beta_1 + \beta_2 + 2\sum_{r\in R\cap R'}\beta_r\right| \\ &= \left|\sum_{r\in R}\beta_r + \sum_{r\in R'}\beta_r\right| \\ &\leqq \left|\sum_{r\in R}\beta_r\right| + \left|\sum_{r\in R'}\beta_r\right| \ .\end{aligned}$$

Summing over all R in $\mathscr{R}_2$, we have

$$\sum_{R\in\mathscr{R}_2\cup\mathscr{R}_3}\left|\sum_{r\in R}\beta_r'\right| \leqq \sum_{R\in\mathscr{R}_2\cup\mathscr{R}_3}\left|\sum_{r\in R}\beta_r\right| \ ;$$

together with equality for R in $\mathscr{R}_1$ and $\mathscr{R}_4$, this proves the lemma. Note that the use of the particular indices 1 and 2 is irrelevant for our purposes; we only need that both indices are no greater than N or that both exceed N, so that $\sum_{r=1}^{N} \beta'_r = \sum_{r=1}^{N} \beta_r$.

LEMMA 1.3. *Let $\beta_1, \cdots, \beta_N$ be any N complex numbers. Then by a finite sequence of two-at-a-time averagings, we may obtain new numbers $\beta'_1, \cdots, \beta'_N$ such that $\max_{i,j} |\beta'_i - \beta'_j|$ is arbitrarily small.*

Proof. Suppose that all the β's are real and let β be their average. Let $\theta = \max_r |\beta - \beta_r|$. Partition $\{1, \cdots, N\}$ into three disjoint classes:

$$R_1 = \{r \mid \beta - \theta \leqq \beta_r < \beta - \theta/3\} ,$$
$$R_2 = \{r \mid \beta - \theta/3 \leqq \beta_r \leqq \beta + \theta/3\} ,$$
$$R_3 = \{r \mid \beta + \theta/3 < \beta_r \leqq \beta + \theta\} .$$

By averaging a β_i, i in R_1 with a β_j, j in R_3, we obtain numbers between $\beta - \theta/3$ and $\beta + \theta/3$; by doing this, we may exhaust either R_1 or R_3, so that we may initially assume that one of these, say R_3, is empty. In this case the cardinality of R_2 must exceed that of R_1, for otherwise the sum of the β's would be less than $N\beta$. Now we may average each β_i, i in R_1, with a distinct β_j, j in R_2, and obtain numbers between $\beta - 2\theta/3$ and β. Then if β'_r are the resultant set of numbers, $\max_r |\beta - \beta'_r| \leqq 2\theta/3$. By repeating this process, we may arrive at numbers differing arbitrarily little from β. For complex β's, we first perform two-at-a-time averagings to make the real parts of the β's as nearly equal as desired, and then do the same for the imaginary parts. Notice that when we perform any averagings, neither the maximum difference of the real parts nor of the imaginary parts can increase, so that when we average to make the imaginary parts nearly equal, we do not increase the maximum difference of the real parts.

We therefore assume $\beta_1 = \cdots = \beta_N = 1$ and $\beta_{N+1} = \cdots = \beta_{2N} = \gamma$, $|\gamma| \leqq 1$. Now each set R of $\mathscr{R}$ determines two integers k and p which are respectively the numbers of indices of R which do not, resp. do, exceed N. For such an R, $|\sum_{r \in R} \beta_r| = |k + p\gamma|$. Since there are $\binom{N}{k}$ subsets of $\{1, \cdots, N\}$ of cardinality k, and $\binom{N}{p}$ subsets of $\{N+1, \cdots, 2N\}$ of cardinality p, the number of R's for which $|\sum_{r \in R} \beta_r| = |k + p\gamma|$ is $\binom{N}{k}\binom{N}{p}$. Thus in this case (1.2) becomes

$$N \leqq A_N(\gamma) = 2 \cdot \sqrt{N\pi} 4^{-N} \sum_{k=0}^{N} \sum_{p-0}^{N} |k + p\gamma| \binom{N}{k}\binom{N}{p} .$$

Since $|k + p\gamma| \geqq |k - p|\gamma||$, it suffices to prove that $A_N(-1) \leqq A_N(\gamma)$, $-1 \leqq \gamma \leqq 0$, and then that $A_N(-1) \geqq N$.

LEMMA 1.4. *$A_N(-1) \leqq A_N(\gamma)$ for all $|\gamma| \leqq 1$*

Proof. We have just seen that it suffices to consider real negative γ; to see that it suffices to consider $\gamma = -1$, note that for fixed N,

$$\frac{1}{2\sqrt{N\pi}} \cdot 4^N A_N = (\gamma) \sum_{k=1}^{N} \sum_{p=0}^{N} |k + p\gamma| \binom{N}{k}\binom{N}{p} = G_N(\gamma)$$

is a piecewise linear continuous function of γ. Where it exists, its derivative with respect to γ is

$$\begin{aligned} &\sum_{k=0}^{N} \sum_{p=0}^{[k/|\gamma|]} p\binom{N}{k}\binom{N}{p} - \sum_{k=0}^{N} \sum_{p=[k/|\gamma|]+1}^{N} p\binom{N}{k}\binom{N}{p} \\ &\geqq \sum_{k=0}^{N} \sum_{p=0}^{k} p\binom{N}{k}\binom{N}{p} - \sum_{k=0}^{N} \sum_{p=k+1}^{N} p\binom{N}{k}\binom{N}{p} \\ &= \sum_{k=0}^{N} \sum_{p=0}^{k} p\binom{N}{k}\binom{N}{p} - \sum_{N-k=0}^{N} \sum_{N-p+1=1}^{N-k} p\binom{N}{k}\binom{N}{p} \\ &= \sum_{k=0}^{N} \sum_{p=0}^{k} \left[p\binom{N}{k}\binom{N}{p} - (N-p+1)\binom{N}{N-k}\binom{N}{N-p+1}\right] \\ &= 0 . \end{aligned}$$

Thus $G_N(\gamma)$ is a non-decreasing function of γ and so obtains its minimum at $\gamma = -1$.

Finally, we compute $G_N = G_N(-1)$. We have

$$\begin{aligned} G_{N+1} &= \sum_{k=0}^{N+1} \sum_{p=0}^{N+1} |k-p| \binom{N+1}{p}\binom{N+1}{k} \\ &= \sum_{k=0}^{N+1} \sum_{p=0}^{N+1} |k-p| \left[\binom{N}{p}\binom{N}{k} + \binom{N}{p}\binom{N}{k-1} + \binom{N}{p-1}\binom{N}{k} + \binom{N}{p-}\binom{N}{k-1}\right] \\ &= \sum_{k=0}^{N} \sum_{p=0}^{N} |k-p| \binom{N}{p}\binom{N}{k} + \sum_{k=0}^{N} \sum_{p=0}^{N} |k+1-p| \binom{N}{p}\binom{N}{k} \\ &\quad + \sum_{k=0}^{N} \sum_{p=0}^{N} |k-p-1| \binom{N}{p}\binom{N}{k} + \sum_{k=0}^{N} \sum_{p=0}^{N} |k-p| \binom{N}{p}\binom{N}{k} \\ &= 4G_N + 2\sum_{k=0}^{N} \binom{N}{k}^2 = 4G_N + 2\binom{2N}{N} . \end{aligned}$$

We have used the convention $\binom{N}{n} = 0$ if $n < 0$ or $n > N$. The third equality is a simple change of index of summation. The next-to-last equality comes from noting that

$$|k-p-1| + |k-p+1| - 2|k-p| = \begin{cases} 0 \text{ if } k \neq p \\ 2 \text{ if } k = p . \end{cases}$$

We then have by an easy induction

$$G_N = 4^N \frac{\Gamma(N+1/2)}{\sqrt{\pi}\,\Gamma(N)} ,$$

whence by Stirling's formula, and the crudest sort of estimates,

$$G_N \geqq 4^N \cdot \frac{1}{2}\sqrt{\frac{N}{\pi}}\,,$$

so that $A_N \geqq N$.

2. The boundedness theorem. Let X be a Banach space, X^* its adjoint, $\mathscr{E}$ and $\mathscr{F}$ bounded Boolean algebras of projections on X, with bounds M_1 and M_2 respectively, such that $EF = FE$ for all E in $\mathscr{E}$ and F in $\mathscr{F}$; E will be assumed to be strongly complete [1, Definition 2.1]. I is the identity operator on X and will be assumed to belong to both $\mathscr{E}$ and $\mathscr{F}$; we denote $I - E$ $(I - F)$ by E' (F'). The operator $\sum_\iota a_\iota E_\iota$, where the E_ι are mutually disjoint projections from $\mathscr{E}$ and $\sup|a_\iota| < \infty$, is a bounded operator on X with norm at most $4M_1 \cdot \sup|a_\iota|$ [4, p. 341]. We use the usual lattice supremum, infimum, and comparison signs for our projections as well as for closed subspaces of X: $E_1 \vee E_2 = E_1 + E_2 - E_1E_2$, $E_1 \wedge E_2 = E_1E_2$, $E_1 \leqq E_2$ if and only if $E_1E_2 = E_1$; $\mathfrak{M}_1 \vee \mathfrak{M}_2$ is the smallest closed manifold in X containing both of the closed manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{M}_1 \wedge \mathfrak{M}_2$ is the intersection of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and $\mathfrak{M}_1 \leqq \mathfrak{M}_2$ means that $\mathfrak{M}_1$ is contained in $\mathfrak{M}_2$. $\mathfrak{M}(x)$ denotes the least closed manifold of X containing Ex for all E in $\mathscr{E}$. If x is in X, we call the projection in $\mathscr{E}$, $C(x) = \wedge\{Ex \mid Ex = x\}$ the *carrier projection* of x; x is *full over* E if $C(x) \geqq E$.

We assume that there is an integer N for which the following condition $(*_N)$ holds:

$(*_N)$ *Let x be in X, and suppose that $\mathfrak{M}(F_ix) \wedge \bigwedge_{j\neq i} \mathfrak{M}(F_jx) = 0$ for all i, $1 \leqq i \leqq n$, for some choice of $F_1, \cdots, F_n$. Then either $\bigwedge_{i=1}^n C(F_ix) = 0$, or else $n \leqq N$.*

This condition holds, for example, if $\mathscr{E}$ is countably decomposable and has no projection of infinite multiplicity. The proof requires rather extensive background material which we will have no other occasion to use, and so is deferred to an appendix.

We wish to obtain a bound for the norm of $\bigvee_{m=1}^M E_mF_m$ which is independent of M and the particular E_m's in $\mathscr{E}$ and F_m's in $\mathscr{F}$. Accordingly, fix $E_m \in \mathscr{E}$, and $F_m \in \mathscr{F}$, $m = 1, \cdots, M$; $x \in X$ and $x^* \in X^*$ with $|x| \leqq 1$, $|x^*| \leqq 1$. We will estimate $x^*\sum_{m=1}^M E_mF_mx$.

First notice that, without loss of generality, we may assume that the F_m's are all disjoint: let L be an index running over all subsets of $\{1, \cdots, M\}$, and define

$$E_L = \bigvee_{l\in L} E_l\,, \qquad F_L = \bigwedge_{l\in L} F_l \wedge \bigwedge_{l\notin L} F_l'\,.$$

It is well known that the non-zero F_L are the atoms of the Boolean algebra of projections generated by the F_m's, and are mutually disjoint with sum I. Now we have

$$\begin{aligned}\bigvee_\iota E_\iota F_\iota, &\leqq \bigvee_\iota E_\iota(\bigvee_{\{L|\iota\in L\}} F_L) \leqq \bigvee_\iota(\bigvee_{\{L|\iota\in L\}} E_L F_L)\\ &\leqq \bigvee_L E_L F_L \leqq \bigvee_L(\bigvee_{\iota\in L} E_\iota)F_L \leqq \bigvee_L(\bigvee_{\iota\in L} E_\iota F_\iota) \leqq \bigvee_\iota E_\iota F_\iota\ ;\end{aligned}$$

thus we have found a way of expressing $\bigvee_{m=1}^M E_m F_m$ with the F's disjoint.

Now let J and K be two indices running over all subsets of $\{1, \cdots, M\}$, and define

$$E_J = \bigwedge_{j\in J} E_j \wedge \bigwedge_{j\notin J} E_j'\ , \qquad G_K = \bigwedge_{k\in K} C(F_k x) \wedge \bigwedge_{k\notin K} C(F_k x)'\ .$$

$\{E_J\}$ and $\{G_K)$ are both disjoint families of projections with sum I.

LEMMA 2.1. 1. *$C(F_k x) = \bigvee_{\{K|k\in K\}} G_K$,*
2. *$G_K F_k x = 0$ if $k \notin K$,*
3. *If $k \in K$ and $G_K \neq 0$, then $G_K F_k x \neq 0$,*
4. *$F_k x = \sum_J \sum_K E_J G_K F_k x$,*
5. *$\sum_{m=1}^M E_m F_m x = \sum_J \sum_K \sum_{\{m\in J\cap K\}} E_J G_K F_m x$,*
6. *For a fixed K, there are most N integers m for which $G_k F_m x \neq 0$.*

Proof. $1-4$ are clear. 5 follows from the fact that the E_j's and G_K's have sum I, and if $m \notin J$, then $E_J E_m = 0$; if $m \notin K$, then $G_K F_m z = 0$; while if $m \in J \cap K$, then $E_J G_K E_m F_m x = E_J G_K F_m x$.

6. Suppose that $G_K F_m x \neq 0$ for $m = m_1, \cdots, m_{N+1}$. Then by 2, $\{m_1, \cdots, m_{N+1}\} \subseteqq K$, and by 1, each $F_{m_n} x$ is full over G_K. Since $F_m z = z$ for every z in $\mathfrak{M}(F_m x)$, the disjointness of the F_m's gives

$$\mathfrak{M}(F_{m_i} x) \wedge \bigvee_{j\neq 1} \mathfrak{M}(F_{m_j} x) = 0$$

for $1 \leqq i \leqq N+1$, which contradicts $(*_N)$.

Now define

$$\alpha(m, J, K) = x^* E_J G_K F_m x\ .$$

As a corollary to Lemma 2.1, parts 5 and 6, we have

5a. $x^* \sum_{m=1}^M E_m F_m x = \sum_J \sum_K \sum_{\{m\in J\cap K\}} \alpha(m, J, K)$,

6a. *For a fixed K, there are at most N integers m for which $\alpha(m, J, K) \neq 0$.*

Let P be any subset of $\{1, \cdots, M\}$ and define

$$\beta(P, J, K) = \sum_{p\in P} \alpha(p, J, K) = x^*\Big(\sum_{p\in P} F_p\Big) E_J G_K\ .$$

Let T_P be the operator $\sum_J \sum_K \overline{\operatorname{sgn}}\, \beta(P, J, K) E_J G_K$, where $\overline{\operatorname{sgn}}\, re^{i\theta} = e^{-i\theta}$ if $r \neq 0$, and 0 if $r = 0$. T_P is an operator on X of norm at most $4M_1$.

Thus we have

$$\left|x^*\Big(\sum_{p\in P} F_p\Big)T_p x\right| \leqq |x^*|\left|\sum_{p\in P} F_p\right| |T_p||x| \leqq 4M_1M_2\,;$$

but on the other hand

$$x^*\Big(\sum_{p\in P} F_p\Big)T_P x = \sum_J\sum_K\Big[\overline{\operatorname{sgn}}\,\beta(P, J, K)\cdot x^*\Big(\sum_{p\in P} F_p\Big)E_J G_K x\Big] \tag{2.1}$$
$$= \sum_J\sum_K|\beta(P, J, K)| \leqq 4M_1M_2\,.$$

We are now in a position to prove the principal theorem of this paper.

THEOREM 2.2. *Let $\mathscr{E}$ and $\mathscr{F}$ be commuting bounded Boolean algebras of projections on a Banach space with bounds M_1 and M_2 respectively, $\mathscr{E}$ strongly complete. Suppose condition ($*N$) is satisfied for some N. Then the Boolean algebra of projections generated by $\mathscr{E}$ and $\mathscr{F}$ is bounded, with bound $8\sqrt{N\pi}M_1M_2$.*

Proof. For each J, K, there are at most N integers $m_1, \cdots, m_N$ for which $\alpha(m, J, K) \neq 0$. Let

$$\alpha_s = \alpha(m_s, J, K)\,,\quad 1 \leqq s \leqq N\,,$$
$$S_0 = \{s \mid m_s \in J\cap K\}\,,$$

and apply Theorem 1.1. We obtain

$$\left|\sum_{m\in J\cap K}\alpha(m, J, K)\right| \leqq 2\sqrt{N\pi}\cdot 2^{-N}\sum_{S\in\mathscr{S}}\left|\sum_{s\in S}\alpha(m_s, J, K)\right|\,.$$

Now for any S, there are 2^{M-N} distinct sets P of $\{1, \cdots, M\}$ for which $\sum_{s\in S}\alpha(m_s, J, K) = \sum_{p\in P}\alpha(p, J, K)$; namely, $\{m_s \mid s \in S\}$ together with any of the 2^{M-N} subsets of integers between 1 and M which are not one of $m_1, \cdots, m_N$. Thus

$$2^{M-N}\sum_{S\in\mathscr{S}}\left|\sum_{s\in S}\alpha(m_s, J, K)\right| = \sum_P\left|\sum_{p\in P}\alpha(p, J, K)\right|\,,$$

and

$$\left|\sum_{m\in J\cap K}\alpha(m, J, K)\right| \leqq 2\sqrt{N\pi}\cdot 2^{-M}\sum_P\left|\sum_{p\in P}\alpha(p, J, K)\right|\,,$$

Summing over all J, K, we have for arbitrary x, x^* of norm 1, E_m's and F_m's,

$$\begin{aligned}\left|x^* \sum_{m=1}^{M} E_m F_m x\right| &\leqq \sum_J \sum_K \left| \sum_{m \in J \cap K} \alpha(m, J, K)\right| \\ &\leqq 2\sqrt{N\pi}\cdot 2^{-M} \sum_P \sum_J \sum_K \left| \sum_{p \in P} \alpha(p, J, K)\right| \\ &\leqq 2\sqrt{N\pi}\cdot 2^{-M} \sum_P (4M_1M_2) = 2\sqrt{N\pi}\cdot 4M_1M_2 ,\end{aligned}$$

which is exactly our theorem.

3. Examples. Inspired by the example of S. Kakutani [6], we construct an example in a finite dimensional space to show that the order of magnitude of our bound is sharp. We imitate his paper in the construction of algebras of projections as much as possible and omit proofs which essentially appear in his paper.

Let N be a power of 2, $N = 2^n$, and let S and S' be the set of integers $\{1, \cdots, N\}$; $C(S)$, the continuous functions on S with the sup norm, is simply the N dimensional vector space of N-tuples. Let $S^* = S \times S'$, and let our Banach space X be $C(S^*)$, but with the minimal cross product norm induced from $C(S)$ and $C(S')$. Our X corresponds to the space $C(S) \circledast C(S')$ of Kakutani, and has dimension N^2. The elements of X may be thought of in a natural way as $N \times N$ matrices $x(s, s')$. Let $\mathscr{E}_N$ and $\mathscr{F}_N$ be the commuting Boolean algebras of projections of bound 1 generated respectively by E_i and F_i, $1 \leqq i \leqq N$, both of multiplicity N:

$$E_i x(s, s') = \begin{cases} x(s, s') \text{ if } s = i , \\ 0 \text{ if } s \neq i , \end{cases} \qquad E_i x(s, x') = \begin{cases} x(s, s') \text{ if } s' = i , \\ 0 \text{ if } s' \neq i . \end{cases}$$

Then there is a projection G in the Boolean algebra of projections generated by $\mathscr{E}_N$ and $\mathscr{F}_N$ such that $2G - I$ takes the element of X defined by $x(s, s') \equiv 1$ into the element $\rho(s, s')$ defined by

$$\rho(s, s') = (-1) \sum_{i=1}^{n} \varepsilon_i(s)\varepsilon_i(s')$$

where s has the unique representation

$$s = \varepsilon_1(s)2^{n-1} + \delta_2(s)2^{n-2} + \cdots + \varepsilon_{n-1}(s)2 + \varepsilon_n(s) + 1, \varepsilon_i(s) = 0 \quad \text{or} \quad 1 .$$

If we put a measure μ on S which assigns to each point the measure $1/N$, then the N functions on S, $\rho(s, i)$, $1 \leqq i \leqq N$, form an orthonormal base for $L^2(S, \mu)$, and the computations on pp. 368 and 369 of [6] carry over exactly to show that the norm of $\rho(s, s')$ in X is no less than $\sqrt{N}$. Since the element of X, $x(s, s')$, has norm 1, this says that the norm of $2G - I$ is at least $\sqrt{N}$, or that the norm of G is at least $\frac{1}{2}(\sqrt{N} - 1)$.

Let us now take one copy X_N of the above example for each N, and form the l_2 direct sum of the X_N, which we call X. Elements of X are sequences $\{x_N\}$ where $x_N \in X_N$ and

$$|| \{x_N\} || = \left[\sum_{N=1}^{\infty} || x_N ||_{X_N}^2 \right]^{1/2} < \infty .$$

The algebras $\mathscr{E}_N$ and $\mathscr{F}_N$ on X_N have a natural extension to all of X by defining $\mathscr{E}_N(X_M) = \mathscr{F}_N(X_M) = 0$, $M \neq N$. Let $\mathscr{E}$ and $\mathscr{F}$ be respectively the commuting Boolean algebras of bound 1 of projections on X generated by all the $\mathscr{E}_N$, resp. $\mathscr{F}_N$, and note that the generated algebra contains a projection of norm at least $\frac{1}{2}(\sqrt{N} - 1)$ on the subspace X_N; we thus see that the algebra generated by $\mathscr{E}$ and $\mathscr{F}$ is not bounded. Since X is an l_2 direct sum of finite dimensional (hence reflexive) spaces, X must be itself reflexive and also separable.

Now let T and T' be operators on X, defined by

$$T\left(\sum_{N=1}^{\infty} \circledast\, x_N(s, s') \right) = \sum_{N=1}^{\infty} \circledast\, 2^{-N} 3^{-s} x(s, s')$$

$$T'\left(\sum_{N=1}^{\infty} \circledast\, x_N(s, s') \right) = \sum_{N=1}^{\infty} \circledast\, 5^{-s'} x_N(s, s') .$$

Then T and T' are bounded commuting scalar-type spectral operators on X. The operator TT' has simple eigenvalues at the distinct points $2^{-M}3^{-i}5^{-j}$, $1 \leqq i, j \leqq M < \infty$. The projection $E_{M,i,j}$ corresponding to the eigenvalue $2^{-M}3^{-i}5^{-j}$ satisfies

$$E_{M,i,j}\left(\sum_{N=1}^{\infty} \circledast\, x_N(s, s') \right) = \sum_{N=1}^{\infty} \circledast\, \delta_{MN} \delta_{is} \delta_{js'} x_N(s, s') ,$$

where δ_{ij} is the Kronecker delta. Thus the Boolean algebra of projections generated by the $E_{M,i,j}$ contains both $\mathscr{E}$ and $\mathscr{F}$, and therefore is unbounded. TT' cannot be spectral. Also the sum of two spectral operators on X need not always be spectral. For if this were so, $T + T'$ would be spectral, hence $(T + T')^2$; also $(T + T')^2 - T'^2 = 2TT'$.

4. Appendix. We show that $(*_N)$ is satisfied if the Boolean algebra $\mathscr{E}$ is countably decomposable and has no projection of infinite multiplicity. We will make use of the representation theory of such algebras of projections originally given by J. Dieudonné [3] but used here in the form due to W. G. Bade [2]:

There is a compact Hausdorff space Ω, the Stone space for $\mathscr{E}$, and a natural correspondence between $\mathscr{E}$ and the Boolean algebra of Borel sets of Ω. We will allow ourselves to confuse the set $\sigma \subset \Omega$ with the corresponding projection $E(\sigma)$ in $\mathscr{E}$. A projection E *has multiplicity* N if there exist N elements $x_1, \cdots, x_N$ of X such that $EX = \bigvee_{n=1}^{N} \mathfrak{M}(x_n)$, and if for every $N-1$ elements $y_1, \cdots, y_{N-1}$ of X, $EX \neq \bigvee_{n=1}^{N-1} \mathfrak{M}(y_n)$. *E has uniform multiplicity* N if E has multiplity N, and $0 < E_1 \leqq E$ implies that E_1 has multiplicity N. By using theorem of Bade [2, Theorem 3.4], and assuming that $\mathscr{E}$ contains no projection of infinite multiplicity, we can decompose Ω into a finite union of disjoint sets, $\Omega = e_1 \cup \cdots \cup e_N$ for some N, where e_n has uniform multiplicity n. It will suffice to consider the case $\Omega = e_N$. In this case, we can find an $\mathscr{E}$-basis $x_1, \cdots, x_N$ for X and a dual basis $x_1^*, \cdots, x_N^*$ such that $X = \bigvee_{n=1}^{N} \mathfrak{M}(x_n)$ and $x_m^* E(\sigma) x_n = 0$ if $m \neq n$ and is > 0 if $m = n$ and $E(\sigma)x_n \neq 0$. Let us write $\mu(x^*, x)$ for the measure $x^*E(\cdot)x$. Then each x in X determines, essentially uniquely, N scalar functions $f_n(\omega)$ on Ω, $f_n(\omega)$ being the Radon-Nikodým derivative of $\mu(x_n^*, x)$ with respect to $\mu(x_n^*, x_n)$. Also each x^* in X^* determines, essentially uniquely, N scalar functions $g_n(\omega)$ on Ω, $g_n(\omega)$ being the Radon-Nikodým derivative of $\mu(x^*, x_n)$ with respect to $\mu(x_n^*, x_n)$. The product $f_n g_n$ is in $L^1(\Omega, \mu(x_n^*, x_n))$ for each n, and $x^*x = \sum_{n=1}^{N} \int f_n(\omega) g_n(\omega) d\mu(x_n^*, x_n)$.

Note that the measures $\mu(x_n^*, x_n)$ are all absolutely continuous with respect to one another, and every measure $\mu(x^*, x)$ is absolutely continuous with respect to all of the $\mu(x_n^*, x_n)$. When we say measurable, we mean with respect to any, hence all, $\mu(x_n^*, x)$.

Now suppose that $F_1, \cdots, F_{N+1}$ are disjoint projections, commuting with each $E \in \mathscr{E}$, and such that for some x and some $\sigma \subset \Omega$, $\sigma \neq 0$, each $F_n x$ is full over σ. We can assume for simplicity that $\sigma = \Omega$. The fact that each F is a bounded projection commuting with every E in $\mathscr{E}$, insures that $F_z = z$ for every z in $\mathfrak{M}(Fx)$. The disjointness of the F_n's then gives us $\mathfrak{M}(F_n x) \wedge \bigvee_{i \neq n} \mathfrak{M}(F_i x)$ for $n = 1, \cdots, N+1$.

The following two lemmas will allow us to reach a contradiction.

LEMMA 4.1. *Let $A(\omega)$ be a matrix of measurable functions on Ω. Then if $M(\omega)$ is a fixed minor of $A(\omega)$, det $M(\omega)$ is a measurable function. If $r(A, \omega)$ denotes the rank of $A(\omega)$, then $r(A, \omega)$ is a measurable function.*

Proof. If $M(\omega)$ is a fixed minor of $A(\omega)$, det $M(\omega)$ is a sum of products of measurable functions, hence is measurable. Also the set on which $\det M(\omega) \neq 0$ is measureable, and so the Boolean algebra of sets generated by the supports of $M(\omega)$ for all minors M of A, is an algebra of measurable sets. $r(A, \omega)$ is a simple function on this algebra, and so is measurable.

$\sigma(r_0, A)$ will denote the set of ω for which $r(A, \omega) = r_0$. $\sigma(r_0, A, M)$ will denote the subset of $\sigma(r_0, A)$ for which the r_0-rowed minor M has non-zero determinant. The $\sigma(r_0, A, M)$ mutually exhaust $\sigma(r_0, A)$. Let $\{\sigma\}$ be a finite collection of mutually disjoint Borel sets such that each σ is contained in some $\sigma(r, A, M)$, and mutually exhaust $\sigma(r, A)$ and hence exhaust Ω.

For the moment, fix σ. Let M be a r-rowed minor of $A(\omega)$ for which $\sigma \subset \sigma(r, A, M)$. Let $p_1, \cdots, p_r$ be the row indices of M and $q_1, \cdots, q_r$ the column indices.

LEMMA 4.2. *Let $g_1(\omega), \cdots, g_N(\omega)$ be N measurable functions such that on σ, the column N-tuple $(g_1(\omega), \cdots, g_N(\omega)$ is pointwise linearly dependent upon the r columns $(a_{1,q_j}(\omega), \cdots, a_{N,q_j}(\omega))$ of $A(\omega)$. Then there exist r measurable functions $u_j(\omega)$ such that on σ,*

$$g_n(\omega) = \sum_{j=1}^{r} u_j(\omega) a_{n,q_j}(\omega) \quad \text{for } n = 1, \cdots, N .$$

Proof. The minor $M(\omega)$ has non-zero determinant on. Let $M^{-1}(\omega) = (w_{p_i, q_j}(\omega))$, the w's being measurable functions on Ω. We have

$$\sum_{j=1}^{r} a_{p_i, q_j}(\omega) w_{p_j, q_k}(\omega) \equiv \delta_{ik} .$$

Define

$$u_j(\omega) = \sum_{i=1}^{r} w_{p_j, q_i}(\omega) \cdot g_{p_i}(\omega)$$

Then, if n is one of the p_i, we have

$$\begin{aligned} \sum_{j=1}^{r} u_j(\omega) a_{n,q_j}(\omega) &= \sum_{j=1}^{r} \sum_{i=1}^{r} w_{p_j, q_i}(\omega) a_{n, q_j}(\omega) g_{p_i}(\omega) \\ &= \sum_{i=1}^{r} \delta_{n, p_i} g_{p_i}(\omega) = g_n(\omega) . \end{aligned}$$

And if for some ω_0 and some n_0 not a p_i,

$$\sum_{j=1}^{r} u_j(\omega_0) a_{n_0, q_j}(\omega_0) \neq g_{n_0}(\omega_0) ,$$

then the matrix, evaluated at ω_0,

$$\begin{pmatrix} a_{p_1, q_1} \cdots a_{p_1, q_r} & g_{p_1} \\ \cdots \quad\quad \cdots & \cdots \\ a_{p_r, q_1} \cdots a_{p_r, q_r} & g_{p_r} \\ a_{n_0, q_1} \cdots a_{n_0, q_r} & g_{n_0} \end{pmatrix} ,$$

has rank $r + 1$, contrary to the assumption that the g_n are linearly dependent upon the r columns of A with indices q_j.

Now let the matrix A have its entires defined by

$$a_{ij}(\omega) = \frac{d\mu(x_i * F_j x)}{d\mu(x_i * x_i)} , \quad 1 \leqq i \leqq N, \quad 1 \leqq j \leqq N + 1 .$$

Then the $N + 1$st column is pointwise linearly dependent upon the first N columns. Selecting one of the non-zero sets σ and applying Lemma 6.2, we have the existence of N measurable functions $u_j(\omega)$ on σ for which we have

$$a_{i,N+1}(\omega) = \sum_{j=1}^{N} u_j(\omega) a_{ij}(\omega) , \quad 1 \leqq i \leqq N .$$

Let now $\tau \neq 0$ be a subset of σ on which each of the functions $u_j(\omega)$ is bounded. We then have

$$x_i{*}E(\tau)F_{N+1}x = x_i{*}\sum_{j=1}^{N} \int_\tau u_j(\omega)E(d\omega)F_j x$$

which implies

$$E(\sigma)F_{N+1}x = \sum_{j=1}^{N} \left(\int_\tau u_j(\omega)E(d\omega)\right)F_j x$$

(this makes sense all the u_j's are bounded on τ); that is,

$$E(\tau)F_{N+1}x \in \bigvee_{j=1}^{N} \mathfrak{M}(F_j x) ,$$

which is the desired contradiction.

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