REPRESENTATION OF A POINT OF A SET AS SUM OF TRANSFORMS OF BOUNDARY POINTS

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In a previous paper [1] we established a condition (Theorem I) for real numbers such that, in a linear space of dimension at least 2, every point of a 2-bounded set can always be represented as a sum of boundary points of the set, multiplied by these numbers. It is natural to ask for the corresponding condition in the case of complex numbers. Multiplication of a point by a real or complex number can be regarded as a special similarity. A more general theorem in which these similarities are replaced by linear transformations, or operators, will be proved in the present paper.

DEFINITION. Let B be a real Banach space with conjugate space B'. Let $S \subset B$ and $x' \in B'$, $\|x'\| = 1$. The *x'-width* of S is

$$w_{x'}(S) = \sup_{x,y \in S} (x - y)x' , \qquad w_{x'}(\phi) = -\infty .$$

The *width* of S is $w(S) = \inf w_{x'}(S)$.

Let $\mathfrak{A}$ be a linear transformation of B and $\mathfrak{A}^*$ the adjoint operation on B' defined by $x(x'\mathfrak{A}^*) = (x\mathfrak{A})x'$. Then $x'\mathfrak{A}^* = 0$ or we can define $x'_{\mathfrak{A}} = x'\mathfrak{A}^*/\|x'\mathfrak{A}^*\|$.

In the following all sets are assumed to be in a real Banach space.

LEMMA 1. (1) *If S is bounded then $w_{x'}(S)$ is a continuous function of x'.*

(2) $w_{x'}(S + T) = w_{x'}(S) + w_x(T)$ (*with the proviso that* $-\infty$ *added to anything—even* $+\infty$*—is* $-\infty$).

(3) *If S has interior points then* $w(S) > 0$.

(4) $$w_{x'}(S\mathfrak{A}) = \begin{cases} 0 \quad \textit{if} \quad x'\mathfrak{A}^* = 0 ; \\ w_{x'_{\mathfrak{A}}}(S) \cdot \|x'\mathfrak{A}^*\| \qquad \textit{if } x'\mathfrak{A}^* \neq 0 . \end{cases}$$

The proofs are all obvious.

LEMMA 2. *Let T be a connected set so that no translate of $-T$ is contained in the interior of S, then $S + T \subset T + \operatorname{bd} S$.*

Proof. Let $s \in S$, $t \in T$; then $s + t - T$ contains $s \in S$ but is not contained in the interior of S. Hence $(s + t - T) \cap \operatorname{bd} S$ is not empty and $s + T \subset T + \operatorname{bd} S$.

Received January 1963. Presented to the American Mathematical Society (by title) October 26, 1962. Sponsored (in part) by the Office of Naval Research, U.S.A.

LEMMA 3. *If S is bounded and* $-\mathrm{cl} S \subset \mathrm{int}\, T$ *then no translate of* $-\mathrm{cl}\, T$ *is contained in* $\mathrm{int}\, S$.

Proof. For one-dimensional spaces this is obvious since the hypothesis implies $\mathrm{diam}\, S < \mathrm{diam}\, T$. If the lemma were false then $a - \mathrm{cl}\, T \subset \mathrm{int}\, S$ for some point a. The mapping $x \to a - x$ leaves the lines through $a/2$ invariant and the contradiction follows from the fact that the inclusion is false for the intersection of the sets with such lines l for which $l \cap \mathrm{int}\, S \neq \phi$.

LEMMA 4. *Let* $w_{x'}(S) < \infty$, *let T be a connected set, and let* $U = (S + T)\backslash(T + \mathrm{bd}\, S)$, *then*

$$w_{x'}(U) \leqq w_{x'}(S) - w_{x'}(T) .$$

Proof. If $w_{x'}(T) = \infty$ then $S + T \subset T + \mathrm{bd}\, S$ by Lemma 2. If $w_{x'}(T) < \infty$ let $a = \inf_{s \in S} sx'$, $b = \sup_{s \in S} sx'$, $c = \inf_{t \in T} tx'$, $d = \sup_{t \in T} tx'$. If $s \in S$, $t \in T$ so that $(s + t)x' < a + d$ then $s + t - T$ contains s in S and $\inf_{t_1 \in T} (s + t - t_1)x' < a$ so that $s + t - T$ contains points in the complement of S. Since $s + t - T$ is connected it follows that $(s + t - T) \cap \mathrm{bd}\, S \neq \phi$ or $s + t \in T + \mathrm{bd}\, S$. Thus $\inf_{u \in U} ux' \geqq a + d$.

Similarly, if $s \in S$, $t \in T$ and $(s + t)x' > b + c$ then $s + t - T$ contains $s \in S$ while $\sup_{t_1 \in T} (s + t - t_1)x' > b$ so that $s + t - T$ contains points in the complement of S. Hence $(s + t - T) \cap \mathrm{bd}\, S \neq \phi$ and $s + t \in T + \mathrm{bd}\, S$. Thus $\sup_{u \in U} ux' \leqq b + c$, and hence

$$\begin{aligned} w_{x'}(U) &= \sup_{u \in U} ux' - \inf_{u \in U} ux' \leqq (b + c) - (a + d) = (b - a) - (d - c) \\ &= w_{x'}(S) - w_x(T) . \end{aligned}$$

DEFINITION. Let S be a bounded connected set in B. The *outer set, oS, of S* is the complement of the unbounded component of the complement of S and the *outer boundary*, $\mathrm{obd}\, S$, of S is the boundary of oS. Clearly $\mathrm{obd}\, S \subset \mathrm{bd}\, S$ and if $\dim B \geqq 2$ then $\mathrm{obd}\, S$ is connected.

THEOREM 1. *Let $S_1, S_2, \cdots, S_n$ be bounded connected sets in B with* $\dim B \geqq 2$ *so that no translate of* $-\mathrm{cl}\ oS_1$ *is contained in* $\mathrm{int}\, oS_i$ $(i = 2, \cdots, n)$. *Then*

$$\begin{aligned} & w_{x'}((S_1 + S_2 + \cdots + S_n)\backslash(\mathrm{obd}\, S_1 + \mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n)) \\ & \qquad \leqq w_{x'}(S_1) - w_{x'}(S_2) - \cdots - w_{x'}(S_n) . \end{aligned}$$

Proof. By repeated application of Lemma 2 we have $S_1 + \cdots + S_n \subset oS_1 + \cdots + oS_n \subset oS_1 + \mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n$ and the theorem follows from Lemma 4 where oS_1 plays the role of S and $\mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n$ that of T.

COROLLARY. *If $S_1, \cdots, S_n$ satisfy the conditions of Theorem* 1 *and in addition for each i there is an x'_i so that $w_{x'_i}(S_i) < \sum_{j \neq i} w_{x'_i}(S_j)$ then $S_1 + \cdots + S_n \subset \operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n$.*

DEFINITION. Let B be a real Banach space with $\dim B \geqq 2$. A set of bounded linear operators $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is *admissible* if for every bounded set $S \subset B$ and every point $p \in S$ there exist outer boundary points $x_1, \cdots, x_n \in \operatorname{obd} S$ such that

$$p = x_1\mathfrak{A}_1 + \cdots + x_n\mathfrak{A}_n .$$

THEOREM 2. *If a set $\mathfrak{A}$ of operators $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is admissible then*
(i) $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathscr{I}$, *the identity.*
(ii) *For each i there exists an $x' \in B'$, $x' \neq 0$ such that*

$$\| x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x'\mathfrak{A}_j^* \| .$$

If B is finite dimensional, $\dim B \geqq 2$, *and* $\mathfrak{A}$ *satisfies* (i) *and*

$$\text{(ii}'\text{)} \qquad \| x'\mathfrak{A}_i^* \| \leqq \sum \| x'\mathfrak{A}_j^* \| , \qquad i = 1, \cdots, n$$

for all $x' \in B'$ then $\mathfrak{A}$ is admissible.

Proof. The necessity of (i) and (ii) is nearly obvious. If $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n \neq \mathscr{I}$, let $p \in B$ be a point which is not invariant under $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n$ and let $S = \{p\}$.

If S is the unit ball of B and

$$0 = x_1\mathfrak{A}_1 + \cdots + x_n\mathfrak{A}_n , \qquad \| x_1 \| = \cdots = \| x_n \| = 1$$

then

$$\| x_i\mathfrak{A}_i x' \| \leqq \sum_{j \neq i} \| x_j\mathfrak{A}_j x' \|$$

or

$$\| x_i x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x_j x'\mathfrak{A}_j^* \| .$$

Now if $\inf_{\|x\|=1} \| x\mathfrak{A}_i \| = 0$, then for every $\varepsilon > 0$ there exists an x' with $\| x' \| = 1$ and $\| x'\mathfrak{A}_i^* \| < \varepsilon$ and (ii) is trivial. If $\inf_{\|x\|=1} \| x\mathfrak{A}_i \| > 0$ then $\mathfrak{A}_i^*$ is onto and we can pick x' so that $\| x_i x'\mathfrak{A}_i^* \| = \| x'\mathfrak{A}_i^* \|$ and hence $\| x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x_j x'\mathfrak{A}_j^* \| \leqq \sum_{j \neq i} \| x'\mathfrak{A}_j^* \|$.

To prove the sufficiency of (i) and (ii′) we may restrict attention to connected sets since we may consider the component of p in S. Let $S_i = S\mathfrak{A}_i$. If for each S_i there is an S_j so that $j \neq i$ and no translate of $-\operatorname{cl} S_j$ is contained in $\operatorname{int} S_i$ then according to Lemma 2 we have

$$\begin{aligned} S \subset S_1 + \cdots + S_n \subset oS_1 + \cdots + oS_n \\ \subset \operatorname{obd} S_1 + (oS_2 + \cdots + oS_n) \\ \subset \operatorname{obd} S_1 + \operatorname{obd} S_2 + (oS_3 + \cdots + oS_n) \subset \cdots \\ \subset \operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n . \end{aligned}$$

Since B is finite dimensional we have $\operatorname{obd} S_i = (\operatorname{obd} S)\mathfrak{A}_i$ so that

$$S \subset (\operatorname{obd} S)\mathfrak{A}_1 + \cdots + (\operatorname{obd} S)\mathfrak{A}_n$$

which was to be proved. We may therefore assume that $-\operatorname{cl} S_j$ has a translate in $\operatorname{int} S_1$ for each $j = 2, \cdots, n$. Then according to Lemma 3 and Theorem 1

$$\begin{aligned} (1) \qquad & w_{x'}((S_1 + \cdots + S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)) \\ & \leqq w_{x'}(S_1) - w_{x'}(S_2) - \cdots - w_{x'}(S_n) . \end{aligned}$$

Since S_1 has an interior $\mathfrak{A}_1$, and hence $\mathfrak{A}_1^*$, are regular and we can choose x' so that $w_{x'_1}(S) = w(S)$ where $x'_1 = x'\mathfrak{A}_1^*/||x'\mathfrak{A}_1^*||$. By part (4) of Lemma 1 we have $w_{x'}(S_j) \geqq w(S) \cdot ||x'\mathfrak{A}_j||$. Thus (1) becomes

$$\begin{aligned} w_{x'}((S_1 + \cdots S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)) \leqq w(S)(||x'\mathfrak{A}_1^*|| - \sum_{j \neq 1} ||x'\mathfrak{A}_j^*||) \\ \leqq 0 \end{aligned}$$

so that $(S_1 + \cdots + S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)$ has no interior points and is therefore empty since $\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n$ is closed. So we have again

$$\begin{aligned} S \subset S_1 + \cdots + S_n \subset \operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n \\ = (\operatorname{obd} S)\mathfrak{A}_1 + \cdots + (\operatorname{obd} S)\mathfrak{A}_n . \end{aligned}$$

REMARK. The hypothesis that B is finite dimensional can be dropped if we assume that the mappings $\mathfrak{A}_i$ are onto. If the $\mathfrak{A}_i$ are similarities of B onto itself then (ii) and (ii′) have the same simple form

$$\text{(ii}''\text{)} \qquad ||\mathfrak{A}_i|| \leqq \sum_{j \neq i} ||\mathfrak{A}_j|| \qquad i = 1, \cdots, n .$$

We thus have the following:

THEOREM 2′. *A set of similarities $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ of a Banach space B of dimension at least 2 onto itself is admissible if and only if it satisfies conditions* (i) *and* (ii″).

In the manner analogous to that used in [1] we can generalize the validity of Theorem 2 to a class of linear spaces which we define as follows.

DEFINITIONS. Let B be a linear space and let $\mathscr{F}$ be a family of linear transformations of B onto itself so that $\mathscr{F}$ is transitive on the nonzero elements of B. A *B-space* S is a linear subspace of a (finite or infinite) direct product of copies of B that is closed under simultaneous application of $\mathscr{F}$ to the components of a point. If x, $y \in S$ and $y \neq 0$ then $\{x + yF \mid F \in \mathscr{F}\}$ is a *B-subspace* of S. The B-subspaces can be given the topology of B by the association $x+yF \leftrightarrow zF$, $z \in B$, $z \neq 0$ where the choice of z is arbitrary due to the transitivity of $\mathscr{F}$. We can therefore define boundedness in B-subspaces (if boundedness is defined in B) and a set in S is *B-bounded* if through every point of the set there is a B-subspace whose intersection with the set is bounded.

THEOREM 3. *Theorem 2 remains valid for B-bounded sets in a B-space where B satisfies the conditions stated in Theorem 2. If B is one-dimensional then the same theorem holds for sets which are 2-bounded (in the sense of* [1]) *and satisfy the other conditions of Theorem 2.*

This is an immediate consequence of Theorem 2 if we consider the bounded intersection of S with a B-subspace through a point p of S.

Theorem 3 applied to the conditions of Theorem 2′ subsums the results of [1]. As one application we give the following:

THEOREM 4. *Let $f(z)$ be analytic in a proper subdomain D of the Riemann sphere and continuous in* $\operatorname{cl} D$. *Let $\alpha_1, \cdots, \alpha_n$ be complex numbers satisfying*

(i) $$\alpha_1 + \cdots + \alpha_n = 1$$

and

(ii) $$|\alpha_i| \leqq \sum_{i \neq j} |\alpha_j| .$$

Then for every $z_0 \in D$ there exist $z_1, \cdots, z_n$ in $\operatorname{bd} D$ *such that*

$$f(z_0) = \alpha_1 f(z_1) + \cdots + \alpha_n f(z_n) .$$

REFERENCE

1. T. S. Motzkin and E. G. Straus, *Representation of a point of a set as a linear combination of boundary points*, Proceedings of the Symposium on Convexity, Seattle 1961.

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