PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

ON SIMPLE ALGEBRAS OBTAINED FROM HOMOGENEOUS GENERAL LIE TRIPLE SYSTEMS

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We continue the investigation of the simple anti-commutative algebras obtained from a homogeneous general L.t.s. In particular we consider the algebra which satisfies

$$(1) \qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) .$$

The usual process of analyzing a nonassociative algebra is to decompose it relative to elements whose right and left multiplications are diagonalizable linear transformations e.g. idempotents or Cartan subalgebras. In this paper we show that such a process yields only Lie algebras and indicates the difficulty in finding any non-Lie multiplication table for a simple anticommutative algebra satisfying (1).

A general Lie triple system [2] is an extension of a Lie triple system used in differential geometry and Jordan algebras. A general L.t.s. may be regarded as an anti-commutative algebra A with a trilinear operation $[x, y, z]$ so that the mappings $D(x, y): z \to [x, y, z]$ are derivations of A which generate a Lie algebra, $I(A)$, under commutation satisfying certain natural identities. A homogeneous general L.t.s. is a general L.t.s. for which the operation $[x, y, z]$ is a homogeneous expression in the products of x, y and z; that is, using anti-commutativity, $[x, y, z] = \alpha xy \cdot z + \beta yz \cdot x + \gamma zx \cdot y$ for some fixed α, β, γ in the base field. From [1] we see that if A is a homogeneous general L.t.s. over a field of characteristic zero which is either an irreducible general L.t.s. or $I(A)$-irreducible or a simple algebra, then A is a Lie or Malcev algebra or satisfies

$$(1) \qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$$

where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. The main result of this paper is the following theorem.

THEOREM. *If A is a simple finite dimensional anti-commutative algebra over a field F of characteristic zero which satisfies* (1) *and if A contains a nonzero element u so that right multiplication by u, R_u, is a diagonalizable linear transformation, then A is a Lie algebra.*

Received August 3, 1964. Sponsored in part by NSF Grant GP-1453.

2. Proof of theorem. For any anti-commutative algebra we have the identity

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned}$$

But using (1) we also have

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = -2[J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y)] . \end{aligned}$$

Thus using the two preceding identities we have

$$\begin{aligned} J(w, xy, z) + J(w, yz, x) + J(w, zx, y) \\ = J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned} \tag{2}$$

Now let $u \neq 0$ be an element of A so that $R_u: x \to xu$ is a diagonalizable linear transformation. Then $R_u \neq 0$, for this implies that the one dimensional subspace uF is an ideal of A and therefore equals A. Thus $A^2 = 0$, a contradiction to the simplicity of A. Since R_u acts diagonally in A we may write

$$A = A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$$

where

$$A_\lambda = \{x \in A : x(R_u - \lambda I) = 0\} .$$

We shall now prove

$$A_\alpha A_\beta \subset A_{\alpha+\beta} . \tag{3}$$

For let $x \in A_\alpha$, $y \in A_\beta$, then from (1)

$$\begin{aligned} J(u, x, y)R_u &= J(u, u, xy) + J(u, x, yu) + J(u, y, ux) \\ &= \beta J(u, x, y) - \alpha J(u, y, x) \\ &= (\alpha + \beta)J(u, x, y) . \end{aligned}$$

Thus $J(u, x, y) \in A_{\alpha+\beta}$ and therefore

$$xy(R_u - (\alpha + \beta)I) = xy \cdot u + yu \cdot x + ux \cdot y \in A_{\alpha+\beta} .$$

From this $xy(R_u - (\alpha + \beta)I)^2 = 0$ and setting $xy = \Sigma z_\gamma \in A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$ we see by the diagonal action of R_u that $xy \in A_{\alpha+\beta}$. In particular (3) shows A_0 is a subalgebra of A.

Next we shall show

$$(4) \qquad J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{or} \quad \alpha + \beta + \gamma = 0$$

for any characteristic roots α, β, γ of R_u. Let $x \in A_\alpha, y \in A_\beta, z \in A_\gamma$, then from (3) $J(x, y, z) \in A_{\alpha+\beta+\gamma}$ and therefore

$$\begin{aligned}(\alpha + \beta + \gamma)J(x, y, z) &= J(x, y, z)R_u \\ &= J(u, x, yz) + J(u, y, zx) + J(u, z, xy) \\ &= -\alpha x \cdot yz + (\alpha + \beta + \gamma)x \cdot yz + (\beta + \gamma)yz \cdot x \\ &\quad - \beta y \cdot zx + (\alpha + \beta + \gamma)y \cdot zx + (\alpha + \gamma)zx \cdot y \\ &\quad - \gamma z \cdot xy + (\alpha + \beta + \gamma)z \cdot xy + (\alpha + \beta)xy \cdot z \\ &= 0 .\end{aligned}$$

and this equation proves (4).

from (1) and (3) we have

$$J(A_0, A_0, A_0)A_0 \subset J(A_0, A_0, A_0)$$

and for $\alpha \neq 0$ we have from (1), (3) and (4),

$$\begin{aligned}J(A_0, A_0, A_0)A_\alpha &\subset J(A_\alpha, A_0, A_0) \\ &= 0 .\end{aligned}$$

Thus $J(A_0, A_0, A_0)A \subset J(A_0, A_0, A_0)$ and therefore $J(A_0, A_0, A_0)$ is an ideal of A whicn is contained in $A_0 \neq A$. Since A is a simple algebra this yields

$$(5) \qquad J(A_0, A_0, A_0) = 0 .$$

Next we shall prove that if α is a nonzero characteristic root so that $-\alpha$ is also a characteristic root, then

$$(6) \qquad J(A_\alpha, A_{-\alpha}, A_0) = 0 .$$

For using (1), (3) and (5) we obtain

$$J(A_\alpha, A_{-\alpha}, A_0)A_0 \subset J(A_\alpha, A_{-\alpha}, A_0)$$

and for any $\beta \neq 0$ we also obtain

$$\begin{aligned}J(A_\alpha, A_{-\alpha}, A_0)A_\beta &\subset J(A_\beta, A_\alpha, A_{-\alpha}A_0) \\ &\quad + J(A_\beta, A_{-\alpha}, A_0A_\alpha) \\ &\quad + J(A_\beta, A_0, A_\alpha A_{-\alpha}) \\ &\subset J(A_\beta, A_\alpha, A_{-\alpha}) + J(A_\beta, A_0, A_0) \\ &= 0 ,\end{aligned}$$

also using (4). Thus as in the proof of (5), $J(A_\alpha, A_{-\alpha}, A_0)$ is an ideal of A which must be zero. Adopting the usual convention that if α is a characteristic root but $-\alpha$ is not, then $A_{-\alpha} = 0$ we see that (6) holds for any characteristic root α.

Next let

$$B = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha ,$$

then if $\beta \neq 0$ we see from (3) that $BA_\beta \subset B$. If $\beta = 0$, then from (6) we obtain $(A_\alpha A_{-\alpha})A_0 \subset A_\alpha A_{-\alpha}$ and therefore $BA_0 \subset B$. Thus B is an ideal of A and therefore $B = 0$ or $B = A$. If $B = 0$, then $R_u = 0$, a contradiction. Therefore we have

$$A = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha . \tag{7}$$

Now from (4) and (6) we have for any characteristic roots β and $\alpha \neq 0$, $J(A_\alpha, A_{-\alpha}, A_\beta) = 0$ and therefore

$$J(A_\alpha, A_{-\alpha}, A) = 0 \qquad (\alpha \neq 0) . \tag{8}$$

We shall use (7) and (8) together with the following lemma to prove A is a Lie algebra.

LEMMA. *Let* $N = \{x \in A : J(x, A, A) = 0\}$, *then*
(i) $J(a, b, A) = 0$ *implies* $ab \in N$;
(ii) N *is an ideal of* A *which is a Lie algebra.*

Proof. Clearly (ii) follows from (i). So let $a, b \in A$ be such that $J(a, b, A) = 0$ and let $w, z \in A$. Then from (1) and (2) we have

$$\begin{aligned} 0 &= wJ(a, b, z) \\ &= J(w, ab, z) + J(w, bz, a) + J(w, za, b) \\ &= J(wa, b, z) + J(wb, z, a), \text{ using (2)} . \end{aligned} \tag{9}$$

Now interchanging z and w in this last equation we obtain $0 = J(za, b, w) + J(zb, w, a) = J(w, bz, a) + J(w, za, b)$ and using this in (9) yields $J(ab, w, z) = 0$; that is, $ab \in N$.

To show that A is a Lie algebra, suppose it is not. Then from the lemma $N = 0$ and from (8) $A_\alpha A_{-\alpha} \subset N = 0$. Thus from (7) $A = \sum_{\alpha \neq 0} A_\alpha$ and therefore $A_0 = 0$; this contradicts $0 \neq u \in A_0$.

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