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SEMI-ALGEBRAS THAT ARE LOWER SEMI-LATTICES

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This paper is concerned with uniformly closed sets of continuous real-valued functions defined on a compact Hausdorff space that are at the same time semi-algebras (wedges closed under multiplication) and lower semi-lattices. The principal result is that any such set can be represented as an intersection of lower semi-lattice semi-algebras of three elementary types. This is an adaptation of a similar theorem of Choquet and Deny for lower semi-lattice wedges. A modified form of the theorem is also given for the case that the lower semi-lattice semi-algebra is in fact a lattice.

Throughout, E denotes a compact Hausdorff space and $C(E)$ the family of all continuous real-valued functions defined on E. For two functions f and g in $C(E)$, the functions $f \cap g$ and $f \cup g$, defined respectively for any point η of E by

$$(f \cap g)(\eta) \equiv \min\{f(\eta), g(\eta)\}; \ (f \cup g)(\eta) \equiv \max\{f(\eta), g(\eta)\} ,$$

are also in $C(E)$. A subset P of $C(E)$ is

(a) a *lower semi-lattice* if and only if $f, g \in P \Rightarrow f \cap g \in P$,

(b) an *upper semi-lattice* if and only if $f, g \in P \Rightarrow f \cup g \in P$,

(c) a *lattice* if and only if P is both an upper and a lower semi-lattice,

(d) a *wedge* if and only if $f, g \in P \Rightarrow \alpha f + \beta g \in P$, for any nonnegative real numbers α, β,

(e) a *semi-algebra* if and only if P is a wedge and $f, g \in P \Rightarrow fg \in P$,

(f) *closed under squaring* if and only if $f \in P \Rightarrow f^2 \in P$.

Choquet and Deny [4] determined those uniformly closed wedges contained in $C(E)$ which are semi-lattices in terms of certain classes of Radon measures which generate the dual wedge. The theorem for the lower semi-lattice case can be formulated as follows. Let σ be a positive Radon measure and ξ be a point of E. Define the sets

$$L_{\sigma,\xi} \equiv \{f: f \in C(E), \sigma(f) \leqq f(\xi)\} ,$$
$$L_{\sigma} \equiv \{f: f \in C(E), \sigma(f) \leqq 0\} .$$

Each of these is a uniformly closed wedge which is a lower semi-lattice. We use W' to denote the dual wedge of all those Radon measures which take nonnegative values on the wedge W, and δ_ξ to denote the Radon measure with unit mass all concentrated at the point ξ of E.

THEOREM 1 (Choquet-Deny). *Let W be a uniformly closed wedge which is a lower semi-lattice contained in $C(E)$. Suppose that $\mathscr{L}_1$ is the family of all pairs (σ, ξ), with ξ a point of E and σ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $\delta_\xi - \sigma \in W'$; suppose that $\mathscr{L}_2$ is the family of all positive Radon measures σ such that $-\sigma \in W'$. Then*

$$W = [\cap \{L_{\sigma,\xi}: (\sigma, \xi) \in \mathscr{L}_1\}] \cap [\cap \{L_\sigma: \sigma \in \mathscr{L}_2\}] .$$

For the proof, see [4]; the result is valid even if $\mathscr{L}_1$ is void or $\mathscr{L}_2$ consists of the zero measure alone. (The convention that a void intersection is the whole of the space is adopted.) An analagous theorem holds for upper semi-lattices. These results were originally given in a more general setting with the underlying space not necessarily compact, but with the function space given the topology of uniform convergence on compacta.

F. F. Bonsall, [1], [2], considered the relationship between lattice and algebraic properties of a function wedge. He showed that any uniformly closed semi-algebra A containing the function 1 and contained in $C^+(E)$ (the set of all nonnegative functions in $C(E)$) is a lattice if and only if it has the "type 1 property", i.e.,

$$f \in A \Rightarrow f/(1 + f) \in A .$$

In addition, he gave an interesting characterization of such semi-algebras as sets of functions monotone with respect to certain quasi-orderings on E. In [2], Bonsall gave intersection theorems for certain closed wedges and semi-algebras contained in $C^+(E)$ which were upper semi-lattices and permitted reduction by constants. (A subset K of $C(E)$ permits reduction by constants if and only if $f \in K$, $\lambda \geqq 0 \Rightarrow (f - \lambda) \cup 0 \in K$.)

The main purpose of the present paper is to show that any uniformly closed lower semi-lattice semi-algebra contained in $C(E)$ is an intersection of ones of certain elementary types. The result obtained does not require the full force of the multiplication property of a semi-algebra, but only closure under squaring; its proof depends heavily on Theorem 1. In the final section, a similar intersection theorem for lattices is deduced. Unlike earlier results for semi-algebras, the theorems here are not restricted to nonnegative functions.

Because of the asymmetry introduced into the situation by the multiplication, one cannot trivially obtain a corresponding result for upper semi-lattice semi-algebras. It seems that the class of these semi-algebras is much more extensive and varied than the class of lower semi-lattices, so that a complete determination is still in the future.

By abuse of notation, we use, for any Radon measure σ, the symbol σ to refer both to the continuous linear functional defined on $C(E)$ and to the corresponding regular measure defined on the Borel subsets of E, but no confusion will result from this. The support of σ in E is denoted by $S(\sigma)$.

2. The Principal Result. Let σ be a positive Radon measure with support $S(\sigma)$, ξ and ζ be two points of E and N a closed subset of E. Then it is clear that each of the sets

$$\begin{aligned} A_{\sigma,\xi} &\equiv \{f: f \in C(E),\ \sigma(f) \leqq f(\xi),\ 0 \leqq f(\eta) \leqq f(\xi)\ (\forall \eta \in S(\sigma))\} \\ B_{\xi,\zeta} &\equiv \{f: f \in C(E),\ f(\xi) = f(\zeta)\} \\ C_N &\equiv \{f: f \in C(E),\ f(\eta) = 0\ (\forall \eta \in N)\} \end{aligned}$$

is a uniformly closed semi-algebra which is a lower semi-lattice, and that any intersection of sets of these forms is such a semi-algebra. It will be shown that every uniformly closed lower semi-lattice semi-algebra is an intersection of sets of the forms $A_{\sigma,\xi}$, $B_{\xi,\zeta}$ and C_N.

LEMMA 1. *Let A be a closed subwedge of $C(E)$ closed under squaring, and suppose that $\delta_\xi - \sigma \in A'$ where σ is a positive Radon measure on E and ξ is a point of E. Then, for $f \in A$,*

$$|f(\eta)| \leqq |f(\xi)|$$

whenever $\eta \in S(\sigma)$.

Proof. If $f \in A$, then $f^2 \in A$. Suppose $f(\xi) = 0$. Then $(-\sigma)(f^2) = 0$, so that f^2 vanishes almost everywhere (σ). Hence $f(\eta) = f^2(\eta) = 0$ whenever $\eta \in S(\sigma)$.

On the other hand, if $f \in A$ and $f(\xi) = \lambda \neq 0$, then $g \equiv \lambda^{-2} f^2 \in A \cap C^+(E)$ and $g(\xi) = 1$. Define

$$G \equiv \{\eta: \eta \in E,\ g(\eta) > 1\} = \{\eta: \eta \in E,\ |f(\eta)| > |f(\xi)|\}\ .$$

If G is void, then $|f(\eta)| \leqq |f(\xi)|$ for η belonging to E, and, a fortiori, to $S(\sigma)$. If G is nonvoid, then G, being open, is σ-integrable. Let K be any compact subset of G, and let $\lambda_K \equiv \inf\{g(\eta): \eta \in K\}$. Since g attains its minimum on K, $\lambda_K > 1$. For m any power of 2, g^m belongs to A, and ϕ_K, the characteristic function of K, satisfies $\phi_K \leqq \lambda_K^{-m} g^m$, so that

$$\sigma(K) \leqq \lambda_K^{-m}\sigma(g^m) \leqq \lambda_K^{-m} g^m(\xi) = \lambda_K^{-m}\ .$$

Hence $\sigma(K) = 0$. Since $\sigma(G) = \sup\{\sigma(K): K \text{ compact}, K \subseteqq G\} = 0$, then $G \cap S(\sigma) = \phi$ and the result follows.

LEMMA 2. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be the families defined as in Theorem 1 with respect to the uniformly closed wedge A, and suppose that A is closed under squaring. Let*
$N \equiv \{\eta: \eta \in E, f(\eta) = 0\ (\forall f \in A)\}$. *Then*:

(a) *$\sigma \in \mathscr{L}_2$ if and only if $S(\sigma) \subseteqq N$;*

(b) *if $(\sigma, \xi) \in \mathscr{L}_1$ and every function in A takes a nonnegative value at the point ξ, then $A \subseteqq A_{\sigma,\xi}$;*

(c) *if $(\sigma, \xi) \in \mathscr{L}_1$ and some function in A takes a negative value at the point ξ, then there exists closed disjoint subsets M_0 and M_1 of $S(\sigma)$ (either possibly void) such that*

(i) $M_0 \cup M_1 = S(\sigma)$,

(ii) $M_0 \subseteqq N$,

(iii) $\eta \in M_1 \Rightarrow A \subseteqq B_{\xi,\eta}$,

(iv) $\sigma(M_1) = 1$.

Proof. (a) If $A \subseteqq L_\sigma$ and $f \in A$, then $(-\sigma)(f^2) = 0$, from which f^2, and hence f, vanishes on $S(\sigma)$. This part is now clear.

(b) It must be shown that whenever $\eta \in S(\sigma)$ and $f \in A$, then $0 \leqq f(\eta) \leqq f(\xi)$. By Lemma 1, we know that $|f(\eta)| \leqq f(\xi)$. Suppose, if possible, that, for some point ζ in $S(\sigma)$, some positive ε and some $f \in A$, $f(\zeta) = -\varepsilon$. Choose a positive integer m such that $f(\xi) < m\varepsilon$, and let $h = f \cap mf$. Then $h \in A$ and $|h(\zeta)| = -h(\zeta) = m\varepsilon > f(\xi) = h(\xi)$, so that Lemma 1 is contradicted.

(c) Let $f, g \in A$ and suppose that $f(\xi) = -1$, $g(\xi) = +1$. Then $f + g \in A$ and $(f + g)(\xi) = 0$, so that, by Lemma 1, $(f + g)(\eta) = 0$ for every point η in $S(\sigma)$. In particular, $(f + f^2)(\eta) = 0$ for $\eta \in S(\sigma)$, with the consequence that f takes only the values 0 and -1 on $S(\sigma)$. Define $M_0 \equiv S(\sigma) \cap \{\eta: f(\eta) = 0\}$ and $M_1 \equiv S(\sigma) \cap \{\eta: f(\eta) = -1\}$. Evidently M_0 and M_1 are closed, disjoint sets satisfying (i).

Let $h \in A$. If $h(\xi) = 0$, then, by Lemma 1, h vanishes everywhere on $S(\sigma)$. If $h(\xi) > 0$, then the argument of the last paragraph with g replaced by $(h(\xi))^{-1}h$ yields $f(\eta) + (h(\xi))^{-1}h(\eta) = 0$ for each point η in $S(\sigma)$. If $h(\xi) < 0$, then $(-h(\xi))^{-1}h \in A$, and the argument of the last paragraph applied to $(-h(\xi))^{-1}h$ and f^2 yields $(-h(\xi))^{-1}h(\eta) + f^2(\eta) = 0$ for each point η in $S(\sigma)$. In any case

$$h(\eta) = \begin{cases} 0 & (\eta \in M_0) \\ h(\xi) & (\eta \in M_1)\ , \end{cases}$$

so that (ii) and (iii) are true. Part (iv) may be seen by noting that, for $h \in A$, $h(\xi)\sigma(M_1) = \sigma(h) \leqq h(\xi)$ and both positive and negative values are possible for h at ξ.

THEOREM 2. *Let A be a uniformly closed subwedge of $C(E)$ such that* (i) *A is a lower semi-lattice,*

(ii) *A is closed under squaring.*

Let $\mathscr{F}_1$ be the family of all pairs (σ, ξ), with ξ a point of E and σ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $A \subseteqq A_{\sigma,\xi}$; let $\mathscr{F}_2$ be the family of all pairs (ξ, ζ) of distinct points of E such that $A \subseteqq B_{\xi,\zeta}$; let $N \equiv \{\eta: \eta \in E,\ f(\eta) = 0\ (\forall f \in A)\}$.
Then:

$$\text{(I)} \qquad A = [\cap \{A_{\sigma,\xi}: (\sigma, \xi) \in \mathscr{F}_1\}] \cap [\cap \{B_{\xi,\zeta}: (\xi, \zeta) \in \mathscr{F}_2\}] \cap C_N .$$

Proof. By Theorem 1, with $\mathscr{L}_1$ and $\mathscr{L}_2$ defined with reference to A, we have that A is the intersection of all sets of the form $L_{\sigma,\xi}$ with $(\sigma, \xi) \in \mathscr{L}_1$ and of the form L_σ with $\sigma \in \mathscr{L}_2$. Denote by F the set on the right hand side of (I). Clearly, $A \subseteqq F$. On the other hand, if $f \in F$, then, by Lemma 2(a), $f \in L_\sigma$ for each $\sigma \in \mathscr{L}_2$. Let $(\sigma, \xi) \in \mathscr{L}_1$. If every function in A is nonnegative at ξ, then, by Lemma 2(b), $A \subseteqq A_{\sigma,\xi}$, so that $(\sigma, \xi) \in \mathscr{F}_1$ and $F \subseteqq A_{\sigma,\xi} \subseteqq L_{\sigma,\xi}$. If some function in A is negative at ξ, then there is a decomposition $\{M_0, M_1\}$ of $S(\sigma)$ satisfying the conditions of Lemma 2(c). Let $f \in F$. Then f belongs to C_N and so vanishes on $M_0 \subseteqq N$. Also, if $\eta \in M_1$, then $A \subseteqq B_{\xi,\eta}$, so that $(\xi, \eta) \in \mathscr{F}_2$ and $f \in B_{\xi,\eta}$, i.e., $f(\eta) = f(\xi)$. Since $\sigma(M_1) = 1$, this yields $f(\xi) = f(\xi)\,\sigma(M_1) = \sigma(f)$, so that $f \in L_{\sigma,\xi}$. In either case, $F \subseteqq L_{\sigma,\xi}$. Hence

$$A \subseteqq F \subseteqq [\cap \{L_{\sigma,\xi}:\ (\sigma, \xi) \in \mathscr{L}_1\}] \cap [\cap \{L_\sigma:\ \sigma \in \mathscr{L}_2\}] = A .$$

REMARK. The result is valid if any of $\mathscr{F}_1$, $\mathscr{F}_2$ and N are void. If $\mathscr{F}_1$ is void, A is a lattice, so that the property of being a lower semi-lattice but not a lattice forces all the functions in A to be nonnegative at least on a nonvoid subset of E.

CONSEQUENCES OF THEOREM 2. (a) Since all sets of the forms $A_{\sigma,\xi}$, $B_{\xi,\zeta}$ and C_N are semi-algebras, the wedge A satisfying the conditions of Theorem 2 is automatically a semi-algebra.

(b) Theorem 2 holds if the condition (ii) is strengthened to "A is a semi-algebra".

(c) Any wedge A contained in $C^+(E)$ which satisfies the conditions of Theorem 2 is an ideal of some semi-algebra T which has the type 1 property. For $(\sigma, \xi) \in \mathscr{F}_1$, let

$$T_{\sigma,\xi} \equiv \{f: f \in C^+(E),\ f(\eta) \leqq f(\xi)\ (\forall \eta \in S(\sigma))\} .$$

Then $A_{\sigma,\xi} \cap C^+(E)$ is an ideal of $T_{\sigma,\xi}$, so that T may be taken to be

$$T \equiv [\cap \{T_{\sigma,\xi}: (\sigma, \xi) \in \mathscr{F}_1\}] \cap [\cap \{B_{\xi,\zeta}: (\xi, \zeta) \in \mathscr{F}_2\}] \cap C_N .$$

(d) Any wedge A contained in $C^+(E)$ which satisfies the conditions of Theorem 2 and in addition contains the function 1 has the type 1 property, and hence is a lattice.

3. Semi-algebras which are lattices. In this section, let A be a uniformly closed semi-algebra contained in E which is a lattice. Since A is in particular a lower semi-lattice, the representation (I) given in Theorem 2 is valid, and, in fact, when $\mathscr{F}_1$ is void, expresses A as an intersection of lattices. However if $\mathscr{F}_1$ is nonvoid, then (I) is unsatisfactory since semi-algebras of the form $A_{\sigma,\xi}$ are not lattices unless σ is either the zero measure or has all of its mass concentrated at one point. This section will be concerned with modifying the family $\mathscr{F}_1$ so that A is given as the intersection of certain elementary lattices.

Suppose $\mathscr{F}_1$ contains the pair (σ, ξ) with $S(\sigma)$ containing at least two points. For $\eta \in S(\sigma)$, define the function $p \equiv p_{\sigma,\xi}$ by

$$p(\eta) \equiv \sup \{f(\eta): f \in A,\ f(\xi) = 1\} .$$

(There is no loss of generality in supposing that the supremum is taken over a nonvoid set, for otherwise $S(\sigma) \cup \{\xi\}$ would be a subset of N, defined as in Theorem 2.) Note that $0 \leqq p(\eta) \leqq 1$ for each point η of $S(\sigma)$ and that $p(\eta) = 0$ if and only if $\eta \in N$. The set

$$\begin{aligned} P_{\sigma,\xi} &\equiv \{f: f \in C(E),\ p(\eta)f(\xi) \geqq f(\eta) \geqq 0\ (\forall \eta \in S(\sigma))\} \\ &= C_{N \cap S(\sigma)} \cap [\cap \{A_{\rho,\xi}: \rho = p(\eta)^{-1}\delta_\eta,\ \eta \in S(\sigma)\backslash N\}] \end{aligned}$$

is a uniformly closed lattice semi-algebra which contains A. We show that $P_{\sigma,\xi} \subseteqq A_{\sigma,\xi}$, so that

$$\begin{aligned} A &\subseteqq [\cap\{P_{\sigma,\xi}: (\sigma, \xi) \in \mathscr{F}_1\}] \cap [\cap \{B_{\xi,\zeta}: (\xi, \zeta) \in \mathscr{F}_2\}] \cap C_N \\ &\subseteqq [\cap\{A_{\sigma,\xi}: (\sigma, \xi) \in \mathscr{F}_1\}] \cap [\cap \{B_{\xi,\zeta}: (\xi, \zeta) \in \mathscr{F}_2\}] \cap C_N = A . \end{aligned}$$

Let $u \in P_{\sigma,\xi}$. If $u(\xi) = 0$, then $u(\eta) = 0$ for each η belonging to $S(\sigma)$ so that $\sigma(u) = 0 = u(\xi)$ and $u \in A_{\sigma,\xi}$. If $u(\xi) \neq 0$, suppose, with no loss of generality, that $u(\xi) = 1$. Since $u(\eta) \leqq p(\eta)$ for $\eta \in S(\sigma)$ and since u is continuous, for given positive ε and given point $\zeta \in S(\sigma)$, there exists a function $f_\zeta \in A$ and an open subset V_ζ of E such that $\zeta \in V_\zeta$, $f_\zeta(\xi) = 1$ and $f_\zeta(\eta) > u(\eta) - \varepsilon$ for each point η of $V_\zeta \cap S(\sigma)$. Because $S(\sigma)$ is compact, there exists a finite set $\zeta_1, \zeta_2, \cdots, \zeta_k$ of points of $S(\sigma)$ such that

$$S(\sigma) \subseteqq \cup \{V_{\zeta_i}: i = 1, 2, \cdots, k\} .$$

The function $f \equiv f_{\zeta_1} \cup f_{\zeta_2} \cup \cdots \cup f_{\zeta_k}$ belongs to A and $f(\xi) = 1$, $f(\eta) > u(\eta) - \varepsilon$ for each point η of $S(\sigma)$. Hence

$$\begin{aligned}\sigma(u) &\leqq \sigma(f+\varepsilon) = \sigma(f) + \sigma(\varepsilon) \\ &\leqq f(\xi) + \sigma(\varepsilon) = 1 + \sigma(\varepsilon) \,.\end{aligned}$$

Since $\sigma(u) \leqq 1 + \sigma(\varepsilon)$ for each positive ε, $\sigma(u) \leqq 1$. It is deduced that for any function u in $P_{\sigma,\xi}$, u belongs to $A_{\sigma,\xi}$.

We can now obtain the following result.

THEOREM 3. *Let A be a uniformly closed subwedge of $C(E)$ which is a lattice and closed under squaring. Let $\mathscr{F}_1^1$ be the family of all pairs (σ, ξ), with ξ a point of E and σ a positive Radon measure which either is the zero measure or has total mass at least unity all concentrated at a point distinct from ξ, such that $A \subseteqq A_{\sigma,\xi}$; let $\mathscr{F}_2$ and N be defined as in Theorem 2. Then the equation*

$$A = [\cap \{A_{\sigma,\xi}\colon (\sigma, \xi) \in \mathscr{F}_1^1\}] \cap [\cap \{B_{\xi,\zeta}\colon (\xi, \zeta) \in \mathscr{F}_2\}] \cap C_N$$

expresses A as an intersection of uniformly closed lattice semi-algebras.

REMARK. If the wedge A is contained in $C^+(E)$, then a simpler representation for A is possible. Define for $0 \leqq \alpha \leqq 1$ and points ξ, η of E the set

$$Q_{\alpha,\xi,\eta} \equiv \{f\colon f \in C^+(E),\ \alpha f(\xi) \geqq f(\eta)\} \,.$$

Then A can be expressed as an intersection of semi-algebras of the form $Q_{\alpha,\xi,\eta}$. (Observe that $C^+(E) \subseteqq A_{0,\xi}$, that $B_{\xi,\zeta} \cap C^+(E) = Q_{1,\xi,\zeta} \cap Q_{1,\zeta,\xi}$ and that $C_N \cap C^+(E) = \cap \{Q_{0,\xi,\eta}\colon \xi \in E, \eta \in N\}$.)

BIBLIOGRAPHY

1. F. F. Bonsall, *Semi-algebras of continuous functions*, Proc. Lon. Math. Soc. (3) **10** (1960), 122-140.

2. ———, *Algebraic properties of some convex cones of functions*, Quart. J. Math. (Oxford) (2) **14** (1963), 225-230.

3. N. Bourbaki, *Intégration-Chapitres* I-IV, Hermann (ASI 1175), 1952.

4. G. Choquet and J. Deny, *Ensembles semi-réticulés et ensembles réticulés de fonctions continues*, Jour. de Math. Pures et App. **36** (1957), 179-189.

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