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A CLASS OF BISIMPLE INVERSE SEMIGROUPS[1]

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The purpose of this paper is to study a certain generalization of the bicyclic semigroup and to determine the structure of some classes of bisimple (inverse) semigroups mod groups.

Let S be a bisimple semigroup and let E_S denote the collection of idempotents of S. E_S is said to be integrally ordered if under its natural order it is order isomorphic to I^0, the nonnegative integers, under the reverse of their usual order. E_S is lexicographically ordered if it is order isomorphic to $I^0 \times I^0$ under the order $(n, m) < (k, s)$ if $k < n$ or $k = n$ and $s < m$. If $\mathscr{H}$ is Green's relation and E_S is lexicographically ordered, $S/\mathscr{H} \cong (I^0)^4$ under a simple multiplication. A generalization of this result is given to the case where E_S is n-lexicographically ordered. The structure of S such that E_S is integrally ordered and the structure of a class of S such that E_S is lexicographically ordered are determimed mod groups. These constructions are special cases of a construction previously given by the author. This paper initiates a series of papers which take a first step beyond the Rees theorem in the structure theory of bisimple semigroups.

The theory of bisimple inverse semigroups has been investigated by Clifford [2] and Warne [7], [8], and [9].

If S is a bisimple semigroup such that E_S is lexicographically ordered, $S/\mathscr{H}$ is shown to be isomorphic to the semigroup obtained by embedding the bicyclic semigroup C in a simple semigroup with identity by means of the Bruck construction [1]. We denote this semigroup by CoC. An interpretation of this construction introduced by the author in [10] is used.

In [2, p. 548, main theorem], Clifford showed that S is a bisimple inverse semigroup with identity if and only if $S \cong \{(a, b): a, b \in P\}$, where P is a certain right cancellative semigroup with identity isomorphic to the right unit subsemigroup of S, under a suitable multiplication and definition of equality. In the special case $\mathscr{L}$ (Green's relation) is a congruence on P(equivalently, $\mathscr{H}$ is a congruence on S), Warne showed [8, p. 1117, Theorem 2.1; p. 1118, Theorem 2.2 and first remark] that $P \cong U \times P/\mathscr{L}$, where U is the group of units of P (of S), under a Schreier multiplication or equivalently, $S \cong \{((a, b), (c, d)): a, c \in U,\ b, d \in P/\mathscr{L}\}$. Warne also notes [8, p. 1118, second remark and p. 1121, Example 2] that a class of semigroups

[1] Some of the results given here have been stated in a research announcement in the Bull. Amer. Math. Soc. [12].

studied by Rees [6, p. 108, Theorem 3.3] may be substituted as a class of P in the above construction (here, $P/\mathscr{L} \cong (I^0, +)$, [8, p. 1118, Equation 2.9]). By [2, p. 553, Theorem 3.1], this substitution will yield the multiplication for the class of bisimple (inverse with identity) semigroups such that E_S is integrally ordered in terms of ordered quadruples. We carry out the indicated calculations, which are routine, in detail here to yield Equation 3.4, which with the equality definition $((g, n), (h, m)) = ((g_1, n_1), (h_1, m_1))$ if $gg_1^{-1} = hh_1^{-1}$, $n = n_1$ and $m = m_1$, is the structure theorem in terms of ordered quadruples. (The author was aware of this result in the spring of 1963.)

N. R. Reilly informed us he had a multiplication for these semigroups (*, p. 572) in terms of ordered triples. His elegant formulation follows from our quadruple formulation by an application of [2, p. 548, Equation 1.2]. A still more convenient formulation is $S \cong U \times C$ with a suitable multiplication.[2]

Next, it is shown that for a class of bisimple semigroups S such that E_S is lexicographically ordered, $S \cong GX(CoC)$, where G is a certain group, under a suitable multiplication. The above techniques of [8] are again utilized here. The greater generality achieved in the integrally ordered case appears to arise from the fact that in this case P is a splitting extension of U by I^0 (i.e., in notation of [8, p. 1117], $a^b = e$, the identity of U for all $a, b \in I^0$).[3]

These structure theorems resemble the Rees theorem for completely simple semigroups [3] in that they completely describe the structure or certain classes of bisimple semigroups mod groups.

$\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's relations [3, p. 47]. R_a denotes the equivalence class containing the element a. Unless otherwise stated, the definitions and terminology of [3] will be used.

1. Preliminary discussion. We first summarize the construction of Clifford referred to in the introduction.

Let S be a bisimple inverse semigroup with identity. Such semigroups are characterized by the following conditions [8, p. 1111; 3, 4, 2 are used].

A1: S is bisimple.

A2: S has an identity element.

A3: Any two idempotents of S commute.

It is shown by Clifford [2] that the structure of S is determined by that of its right unit semigroup P and that P has the following properties:

B1: The right cancellation law holds in P.

B2: P has an identity element

[2] See p. 576, (2).

[3] See p. 576, (3), (5).

B3: The intersection of two principal left ideals of P is a principal left ideal of P.

Let P be any semigroup satisfying B1, B2 and B3. From each class of $\mathscr{L}$-equivalent elements of P, let us pick a fixed representative. B3 states that if a and b are elements of P, there exists c in P such that $Pa \cap Pb = Pc$. c is determined by a and b to within $\mathscr{L}$-equivalence. We define avb to be the representative of the class to which c belongs. We observe also that

$$a \vee b = b \vee a \; . \tag{1.1}$$

We define a binary operation $*$ by

$$(a * b)b = a \vee b \tag{1.2}$$

for each pair of elements a, b of P.

Now let $P^{-1}oP$ denote the set of ordered pairs (a, b) of elements of P with quality defined by

(1.3) $(a, b) = (a', b')$ if $a' = ua$ and $b' = ub$ where u is a unit in P (u has a two sided inverse with respect to 1, the identity of P) .

We define product in $P^{-1}oP$ by

$$(a, b)(c, d) = ((c * b)a, (b * c)d) \; . \tag{1.4}$$

Clifford's main theorem states: Starting with a semigroup P satisfying B1, 2, 3, Equations (1.2), (1.3), and (1.4) define a semigroup $P^{-1}oP$ satisfying A1, 2, 3. P is isomorphic with the right unit subsemigroup of $P^{-1}oP$ (the right unit subsemigroup of $P^{-1}oP$ is the set of elements of $P^{-1}oP$ having a right inverse with respect to 1; this set is easily shown to be a semigroup). Conversely, if S is a semigroup satisfying A1, 2, 3, its right unit subsemigroup P satisfies B1, 2, 3 and S is isomorphic to $P^{-1}oP$.

The following results are also obtained:

LEMMA 1.1 [2]. *For a, b in P and u, v in U, the group of units of P, we have $(ua * vb)v = a * b$. The unit group of P is equal to the unit group of S. $a\mathscr{H}b$ (in S) if and only if $a\mathscr{L}b$ (in P). $a\mathscr{L}b$ (in P) if and only if $a = ub$ for some u in U.*

THEOREM 1.1 [2]. *Let S be a semigroup satisfying* A1, 2, 3, *and let P be its right unit subsemigroup. Then P satisfies* B3 *(as well as* B1 *and* B2*), and the semi-lattice of principal left ideals of P under intersection is isomorphic with the semi-lattice of idempotent* *elements of* S.

We now briefly review the work of Rédei [5] on the Schreier extension theory for semigroups (we actually give the right-left dual of his construction) and we also present some pertinent material from [8]. Let G be a semigroup with identity e. We consider a congruence relation ρ on G and call the corresponding division of G into congruence classes a compatible class division of G. The class H containing the identity is said to be the main class of the division. H is easily shown to be a subsemigroup of G. The division is called right normal if and only if the classes are of the form,

$$Ha_1, Ha_2, \cdots (a_1 = e) \tag{1.5}$$

and $h_1 a_i = h_2 a_i$ with h_1, h_2 in H implies $h_1 = h_2$. The system (1.5) is shown to be uniquely determined by H. H is then called a right normal divisor of G and G/ρ is denoted by G/H.

Let G, H, and S be semigroups with identity. Then, if there exists a right normal divisor H' of G such that $H \cong H'$ and $S \cong G/H'$, G is said to be a Schreier extension of H by S.

Now, let H and S be semigroups with identities E and e respectively. Consider HXS under the following multiplication:

$$(A, a)(B, b) = (AB^a a^b, ab)(A, B \text{ in } H;\ a, b \text{ in } S) \tag{1.6}$$

$$a^b, B^a (\text{in } H)$$

designate functions of the arguments a, b and B, a respectively, and are subject to the conditions

$$a^e = E, e^a = E, B^e = B, E^a = E\,. \tag{1.7}$$

We call $H \times S$ under this multiplication a Schreier product of H and S and denote it by HoS.

Rédei's main theorem states:

THEOREM 1.2 (Redei). *A Schreier product* $G = HoS$ *is a semigroup if and only if*

$$(AB)^c = A^c B^c (A, B \textit{ in } H\colon c \textit{ in } S) \tag{1.8}$$

$$(B^a)^c c^a = c^a B^{ca} (B \textit{ in } H;\ a, c \textit{ in } S) \tag{1.9}$$

$$(a^b)^c c^{ab} = c^a (ca)^b (a, b, c \textit{ in } S) \tag{1.10}$$

are valid. These semigroups (up to an isomorphism) are all the Schreier extensions of H *by* S *and indeed the elements* (A, e) *form a right normal divisor* H' *of* G *for which*

(1.11) $$G/H' \cong S(H'(E, a) \rightarrow a)$$
$$H' \cong H((A, e) \rightarrow A)$$

are valid.

THEOREM 1.3 [8]. *Let U be a group with identity E and let S be a semigroup satisfying* B1 *and* B2 *(denote its identity by e) and suppose S has a trivial group of units. Then every Schreier extension $P = UoS$ of U by S satisfied* B1 *and* B2 *(the identity is (E, e)) and the group of units of P is $U' \cong \{(A, e) : A \text{ in } U\} \cong U$ Furthermore $\mathscr{L}$ is a congruence relation on P and $P/\mathscr{L} \cong S$. P satisfies* B3 *if and only if S satisfies* B3.

Conversely, let P be a semigroup satisfying B1 *and* B2 *on which $\mathscr{L}$ is a congruence relation. Let U be the group of units of P. Then U is a right normal divisor of P and $P/U \cong P/\mathscr{L}$. Thus, P is a Schreier extension of U by $P/\mathscr{L}$. $P/\mathscr{L}$ satisfies* B1 *and* B2 *and has a trivial group of units.*

The following statements are valid for any semigroup obeying the conditions of Theorem 1.3 (i.e. semigroups satisfying B1, B2 on which $\mathscr{L}$ is a congruence).

(1.12) $$P(A, a) = \{(C, ba): C \text{ in } U, b \text{ in } S\}\,.$$

(1.13) $$(A, a)L(B, b) \text{ if and only if } a = b\,.$$

As remarked in [8], the semigroups considered by Rees (Theorem 1.5 below) fall into this category.

Now, Rees defines a *right normal divisor* in a different manner than Rédei. He says that V is a right normal divisor of a semigroup P satisfying B1 and B2 if V is a subgroup of the unit group U of P and $aU \subseteqq Ua$ for all a in P. However, let us show that the Rees definition is just a specialization of the Rédei definition to the case where the main class is a group and the semigroup we are dealing with satisfies B1 and B2. In this case, suppose that V is a right normal divisor in the sense of Rédei. Then, clearly, V is a subgroup of U. The congruence class containing a is just Va. Let u in V. Then, $u\rho 1$. Thus, $au\rho a$, i.e., au in Va. Conversely, suppose V is a right normal divisor in the sense of Rees. Let us define $a\rho b$ if and only if $Va = Vb$. It is easily seen that ρ is a congruence on P with main class V, i.e., V is a right normal divisor in the sense of Rédei.

Let us now briefly review the theory of Rees [6]. Let P be a semigroup satisfying B1 and B2. The partially ordered system of principal left ideals of P, ordered by inclusion, will be denoted by $O(P)$ and termed the ideal structure of P. If $(O, \geqq)$ is a partially ordered set, we denote the set of all elements x of O satisfying $x \leqq a$ by O_a and term such a set a section of O. Then we take as $P(O)$ the set of all order isomorphic mappings γ of $O(P)$ onto sections of $O(P)$. If U is the group of units of S, $M = (g$ in U/xg in Ux for all x in $P)$ is the greatest right normal divisor of P.

The following theorems are established.

THEOREM 1.4 [6]. *If P has an ideal structure $O(P)$ and M is the right normal divisor just described, then there is a subsemigroup P' of $P(O)$ isomorphic to P/M. Further, every principal left ideal of $P(O)$ has a generator in P'.*

THEOREM 1.5 [6]. *A semigroup P satisfying* B1 *and* B2 *whose ideal structure is isomorphic with ϑ (the ideal structure of $(I^0, +)$) and whose group of units is isomorphic with a given group G is isomorphic with a semigroup $T = G \times I^0$ under the following multiplication* (1.14) $(g, m)(h, n) = (g(h\alpha^m), m + n)$, *$g, h$ in G, m, n in I^0, α being an endomorphism of G, α^0 being interpreted as the identity transformation of G and conversely T has the above properties.*

LEMMA 1.2. *Let S be a bisimple inverse semigroup with identity with right unit subsemigroup P. U, the group of units of P, is a right normal divisor of P if and only if $\mathscr{H}$ is a congruence on S.*

Proof. Let U be a right normal divisor of P. Let $(a, b), (c, d)$ be in S and suppose that $(a, b)\mathscr{H}(c, d)$. Now $(a, b)\mathscr{R}(c, d)$ if and only if $a = uc$ where u in U and $(a, b)\mathscr{L}(c, d)$ if and only if $b = vd$ where v in U. I will prove the first. Suppose that $(a, b)\mathscr{R}(c, d)$. Then there exists $(x, y), (w, z)$ in S such that $(a, b) = (c, d)(x, y)$ and $(c, d) = (a, b)(w, z)$. Thus, by 1.3 and 1.4 $a = p(x * d)c$ and $c = q(w * b)a$ where p, q in U. Thus, by B1 and B2 $a = uc$ for some u in U by B1 and B2. Now suppose that $a = u'c$ for some u' in U. We note first that $(b * b)b = b \vee b = ub$ for some u in U by 1.2, the definition of $\vee$, and Lemma 1.1. Thus, $b^* b = u$ by B1.

Now $(a, b)(b, u'd) = (ua, uu'd) = (u'^{-1}a, d) = (c, d)$ by (1.3). Similarly $(c, d)(d, u'^{-1}b) = (a, b)$, i.e., $(a, b)\mathscr{R}(cd)$.

Let (p, q) be in S. Then by (1.4),

$$(a, b)(p, q) = ((p * b)a, (b * p)q)$$
$$(c, d)(p, q) = ((p * d)c, (d * p)q)$$

Since $(a, b)\mathscr{H}(c, d)$ there exists u, v in U such that $a = uc, b = vd$. Thus, by Lemma 1.1 and the fact that U is a right normal divisor

$$(p * b)a = (p * vd)uc = (1p * vd)vv^{-1}uc = (p * d)v^{-1}uc = t(p * d)c ,$$

where t is in U.

Thus, $(a, b)(p, q)\mathscr{R}(c, d)(p, q)$ and $\mathscr{R}$ is a right congruence. Since $\mathscr{R}$ is always a left congruence, it is a congruence. One shows similarly that $\mathscr{L}$ is a congruence. Thus, $\mathscr{H}$ is a congruence relation on S.

Suppose $\mathscr{H}$ is a congruence on S. Let a, b in P and suppose $a\mathscr{L}b$ (in P). By Lemma 1.1 $a\mathscr{H}b$ (in S). Thus c in P implies $ca\mathscr{H}cb$ (in S) and $ca\mathscr{L}cb$ (in P) by Lemma 1.1. Hence $\mathscr{L}$ is a congruence on P and U is a right normal divisor of P by Theorem 1.3.

2. **The Bruck product.** Let S be an arbitrary semigroup and C be the bicyclic semigroup ([3], p. 43), i.e., C is the set of all pairs of nonnegative integers with multiplication given by $(m, n)(m', n') = (m + m' - \min(n, m'), n + n' - \min(n, m'))$. Consider $W = C \times S$ with multiplication given by $((m, n), s)((m', n'), s') = ((m, n)(m', n'), f(n, m'))$ where $f(n, m') = s, ss'$, or s' according to whether $n > m'$, $n = m'$, or $n < m'$. We call W the Bruck product of C and S and write $W = CoS$. I used a special case of this product in [10]. CoC is easily shown to be a bisimple inverse semigroup with identity for which E_S is lexicographically ordered. If S is an arbitrary semigroup, let S^1 be S with an appended identity [3, p. 4]. One can show that CoS^1 is a simple semigroup with identity containing S as a subsemigroup. Since this is equivalent to the construction of R. H. Bruck [1] for embedding an arbitrary semigroup in a simple semigroup with identity, we call o a Bruck product.

THEOREM 2.1 [8]. *Let S and S^* be bisimple inverse semigroups with identity with right unit subsemigroups P and P^* respectively. $S \cong S^*$ if and only if $P \cong P^*$.*

THEOREM 2.2. *Let S be a bisimple (inverse) semigroup. E_S is lexicographically ordered if and only if $\mathscr{H}$ is a congruence on S and $S/\mathscr{H} \cong CoC$ where CoC denotes the Bruck product of C by C.*

Proof. First we suppose that E_S is lexicographically ordered. Clearly S has an identity. For let e be the largest element of the lexicographic chain. If a in S, a is in R_f for some f in E_S since S is regular. Then, $ea = efa = fa = a$. Similarly, $ae = a$. Let P be the right unit subsemigroup of S. Then by Theorem 1.1, we may write the ideal structure of P, $O(P)$ as follows:

$$\begin{array}{l}(0,0) > (0,1) > (0,2) > (0,3) > \\ (1,0) > (1,1) > (1,2) > (1,3) > \\ (2,0) > (2,1) > (2,2) > (2,3) >\end{array}$$

$$(3, 0) > (3, 1) > (3, 2) > (3, 3) >$$
$$(4, 0) > (4, 1) > (4, 2) > (4, 3) >$$

If we define for (m, k) in $O(P)$

$$(n, s)t_{(m,k)} = (n + m, s) \text{ if } n > 0$$
$$(m, s + k) \text{ if } n = 0$$

we easily see that $t_{(m,k)}$ is an order isomorphism of $O(P)$ onto the section of $O(P)$ determined by (m, k). In fact all order isomorphisms of $O(P)$ onto sections of $O(P)$ are of this form.

Clearly $P(O) \cong I^0 X I^0$ under the multiplication

$$(n, s)(m, k) = (n + m, s) \text{ if } n > 0$$
$$(m, s + k) \text{ if } n = 0 .$$

Thus, the only subsemigroup of $P(O)$ containing a generator of every principal left ideal of $P(O)$ is $P(O)$ itself. This follows since $P(O)(n, k) = ((u + n, v)\colon u, v \text{ in } I^0, u > 0) U ((n, v + k)\colon v \text{ in } I^0)$. The unit group of $P(O)$ is trivial (note the identity of $P(O)$ is $(0, 0)$).

By Theorem 1.4, $P/M \cong P(O)$. Since the unit group of $P(O)$ is trivial, $M = U$. Thus, again by Theorem 1.4, U is a right normal divisor of P. Thus, $\mathscr{H}$ is a congruence on S by Lemma 1.2. Since ([8], p. 1111) any homomorphic image of a bisimple inverse semigroup with identity is a bisimple inverse semigroup with identity, $S/\mathscr{H}$ is such a semigroup.

Let $a \to \bar{a}$ denote the natural homomorphism of S onto $S/\mathscr{H}$. If $\bar{a}$ is a right unit of $S/\mathscr{H}$ there exists $\bar{x}$ in $S/\mathscr{H}$ such that $\bar{a}\bar{x} = \bar{1}$, where 1 is the identity of S. Thus, $ax\mathscr{H}1$ and there exists y in S such that $axy = 1$, i.e., a in P. Now, if a in P, $ax = 1$ for some x in S. Thus, $\bar{a}\bar{x} = 1$ and $\bar{a}$ is in the right unit subsemigroup of $S/\mathscr{H}$. Hence the right unit subsemigroup of $S/\mathscr{H}$ is $P/\mathscr{H} = P/\mathscr{L} \cong P(O)$ by Lemma 1.1. Now, as noted above CoC is a bisimple inverse semigroup with identity. It is easily seen that the right unit subsemigroup of CoC is isomorphic to $P(O)$. Thus, by Theorem 2.1 $S/\mathscr{H} \cong CoC$. The converse is clear.

COROLLARY 2.1. *S is a bisimple (inverse) semigroup with trivial unit group and E_S is lexicographically ordered if and only if S is isomorphic to CoC.*

Proof. This follows from Theorem 2.3 of [3].

LEMMA 2.1. *Let S be a bisimple (inverse) semigroup. E_S is integrally ordered if and only if $\mathscr{H}$ is a congruence on S and $S/\mathscr{H} \cong C$.*

Proof. $\mathscr{H}$ is a congruence on S by ([8], p. 1118) and Lemma 1.2. By Theorem 1.1, Theorem 1.5, 1.14, and 1.13, $P/\mathscr{L} \cong I^0$, where I^0 is the nonnegative integers under addition. But, as above, $P/\mathscr{L}$ is the right unit subsemigroup of $S/\mathscr{H}$. Hence $S/\mathscr{H} \cong C$ by Theorem 2.1. The converse is clear.

LEMMA 2.2. *S is a bisimple (inverse) semigroup with trivial unit group and E_S integrally ordered if and only if $S \cong C$.*

Let S be a semigroup. We say E_S is n-lexicographically ordered if and only if E_S is order isomorphic to $\underbrace{I^0 \times I^0 \times xI^0}_{n \text{ times}}$ under the order

$$(k_1, k_2, \cdots, k_n) < (s_1, s_2, \cdots, s_n)$$

if $k_1 > s_1$ or $k_1 = s_1, k_2 > s_2$ or $k_i = s_i (i = 1, 2, j-1), k_j > s_j$ or $k_i = s_i (i = 1, 2, n-1), k_n > s_n$. E_S is 2-lexicographically ordered if and only if E_S is lexicographically ordered. E_S is 1-lexicographically ordered if and only if E_S is integrally ordered.

We will define the n-dimensional bicyclic semigroup C_n as follows: $C_1 = C$ and $C_n = (Co \cdots o(Co(Co(CoC)))\ \ )$ for $n > 1$ where o is the Bruck product (there are $n-1$ o's).

C_n is a bisimple inverse semigroup with E_{C_n} n-lexicographically ordered. The 1-dimensional bicyclic semigroup is the bicyclic semigroup. The 2-dimensional bicyclic semigroup is the Bruck product CoC of C and C.

The following theorem and corollary are obtained by employing the techniques used in the proofs of Theorem 2.1 and Corollary 2.1 respectively.

THEOREM 2.3. *S is a bisimple (inverse) semigroup with E_S n-lexicographically ordered if and only if $\mathscr{H}$ is a congruence on S and $S/\mathscr{H} \cong C_n$.*

COROLLARY 2.2. *S is a bisimple (inverse) semigroup with E_S n-lexicographically ordered and trivial unit group if and only if $S \cong C_n$.*

3. Multiplications on two classes of bisimple inverse semigroups.

THEOREM 3.1. *S is a bisimple (inverse) semigroup such that E_S is integrally ordered if and only if $S \cong G \times C$ where G is a group and C is the bicyclic semigroup under the multiplication:*

$$(z, n, m)(z', n_1, m_1) = (z\alpha^{n_1-r}z'\alpha^{m-r}, (n, m)(n_1, m_1)) \tag{3.1}$$

where $r = \min(m, n_1)$, α an endomorphism of G, α^0 is the identity transformation of G and juxtaposition is multiplication in G and C.

Proof. As in the proof of Theorem 1.7, S is a bisimple inverse semigroup with identity. By Theorem 1.1, Cliffords's main theorem, and Theorem 1.5, $P \cong U \times I^0$ where U is the group of units of S under the multiplication 1.14 if and only if E_S is integrally ordered. The $\mathscr{L}$-classes of P are $L_0, L_1, L_2 \cdots L_n \cdots$ where $L_n = ((g, n)\colon g \text{ in } U)$ by 1.13. Let (e, n) where e is the identity of U be a representative element of L_n. Thus, $(e, n) \vee (e, m) = (e, \max(n, m))$ by 1.12 and the definition of $\vee$. Using (1.2) by a routine calculation, we have

$$\begin{aligned}(e, n)*(e, m) &= (e, n-m) && \text{if } n \geqq m \\ &= (e, o) && \text{if } m \geqq n\end{aligned} \tag{3.2}$$

Using Lemma 1.1, (1.14), and Theorem 1.3, we obtain

$$\begin{aligned}(g, n)*(h, m) &= (h^{-1}\alpha^{n-m}, n-m) && \text{if } n \geqq m \\ &(h^{-1}, o) && \text{if } m \geqq n\end{aligned} \tag{3.3}$$

Now using (1.14) (1.4), and (3.3), we obtain

$$\begin{aligned}&((g, n), (h, m))((g_1, n_1), (h_1, m_1)) \\ &\quad = ((h^{-1}g)\alpha^{n_1-r}, n_1 + n - r, (g_1^{-1}h_1)\alpha^{m-r}, m + m_1 - r)\,.\end{aligned} \tag{3.4}$$

Now, by (1.3) and (3.4), we have

$$\begin{aligned}&(e, n, g^{-1}h, m)(e, n_1, g_1^{-1}h_1, m_1) \\ &\quad = (e, n_1 + n - r, (g^{-1}h)\alpha^{n_1-r}(g_1^{-1}h_1)\alpha^{m-r}, m + m_1 - r)\end{aligned}$$

Let $z = g^{-1}h$ and $z' = g_1^{-1}h_1$. Then

$$*(n, z, m)(n_1, z', m_1) = (n + n_1 - r\; z\alpha^{n_1-r}, z'\alpha^{m-r}, m + m_1 - r)$$

or

$$(z, n, m)(z', n_1, m_1) = (z\alpha^{n_1-r}z'\alpha^{m-r}, (n, m)(n_1, m_1))\,.$$

The converse follows by Cliffords theorem.

To actually determine the multiplication on S, one determines P (we are actually given P here) and then places P in the Clifford construction. However, after one ascertains the multiplication, a very short proof of the fact can be given by the use of Theorem 1.6.

Alternative proof of Theorem 2.1. Let $S^* = G \times C$ be a groupoid with multiplication (3.1). We can show that S^* is a bisimple inverse semigroup with identity by routine calculation (we must go through this to prove the converse anyway). It is easily seen that the right unit subsemigroup P^* of S^* is isomorphic to P. Thus, $S \cong S^*$ by Theorem 2.1.

A semigroup with zero, 0, is said to be 0-right cancellative if a, b, c in S, $c \neq 0$, $ac = bc$ implies that $a = b$. If G is a group, let $\varepsilon(G)$ denote the semigroup of endomorphisms of G.

A nontrivial group G is said to be a $*$-group if

(1) Every nontrivial endomorphism of G maps G onto G.

(2) $\varepsilon(G)$ is 0-right cancellative. ((1) $\longrightarrow$ (2) if G is an abelian group).[4] The *-groups include all cyclic groups of prime order, all groups of type p^∞, and the additive group of rational numbers.[5]

If S is a semigroup with identity 1 and a, x in S with $ax = 1$, we write $x = a^{-1}$.

THEOREM 3.2. *S is a bisimple (inverse) semigroup such that* (1) *E_S is lexicographically ordered,* (2) *U is a *-group,* (3) *$aa^{-1} = 1$ implies that $Ua \subseteqq aU$, if and only if $S \cong GX(CoC)$ where G is a *-group, C is the bicyclic semigroup, o is the Bruck product, with the multiplication,*

$$(g, (n, k), (m, l))(h, (n_1, k_1), (m_1, l_1))$$
$$= (g\alpha^{n_1 - r}h\alpha^{k-r}, ((n, k), (m, l))((n_1, k_1), (m_1, l_1))$$

where $r = \min(n_1, k)$ and α is a nontrivial endomorphism of G α^0 denotes the identity transformation, and juxtaposition denotes multiplication in G and CoC.

Proof. Let P be the right unit subsemigroup of S. If U is a right normal divisor of P, then clearly $\mathscr{L}$ is a congruence on P. Thus by Theorem 2.2 Lemma 1.2, and Theorem 1.3, P is a Schreier extension of U by $P/U(=P/\mathscr{L})$. Now, the semigroup of right units P^* of CoC is easily seen to be isomorphic to $I^0 \times I^0$ under the multiplication

$$(n, m)(p, q) = (n + p, m) \text{ if } n > 0$$
$$(n + p, m + q) \text{ if } n = 0$$

Now $a = (1, o)$ and $b = (o, 1)$ are generators of P^* and $ab = a$. Now, as remarked in the proof of Theorem 2.2 the right unit subsemigroup

[4] (1) $\rightarrow$ (2) also if G is simple or finite.

[5] The *-groups also include all nontrivial finite simple groups.

of $S/\mathscr{H} \cong CoC$ (Theorem 2.2) is $P/\mathscr{L}$. Thus, we may label the $\mathscr{L}$-classes of P as $\{L_{(n,k)} : n, k \text{ in } I^0\}$. Now let a' in $L_{(1,0)}$ and b^* in $L_{(0,1)}$. Thus, $a'b^* = ua'$ for some u in U. Thus by (3) $ua' = a'v$ for some v in U. Hence, $a'b^* = a'v$, $a'b^*v^{-1} = a'$. Let $b^*v^{-1} = b'$. Now, since U is a right normal divisor of P, $b^*v^{-1} = wb^*$ for some w in U and b' in $L_{(0,1)}$. Thus, $\{b'^k a'^s, k, s \text{ in } I^0\}$ form a complete system of representative elements (5) which is also a semigroup. Thus the factors c^d of (1.6) are all equal to E, the identity of U. Thus, (1.6) becomes

$$(3.5) \qquad (A, n, k)(B, m, l) = (AB^{(n,k)}, (n, k)(m, l))$$

where A, B in U, (n, k), $(m, 1)$ in $P/\mathscr{L}$ and juxtaposition is multiplication in U and $P/\mathscr{L}$. Now let $a = (1, 0)$ and $b = (0, 1)$, and let $e = (0, 0)$, the identity of $P/\mathscr{L}$. Then $(E, a)(g, e) = (g\alpha, e)(E, a)$ (a, fixed), α a transformation of U, since U is a right normal divisor of P and $\{(g, e)\colon g \text{ in } U\}$ is isomorphic to U (Theorem 1.3). Now $(E, a)(g, e) = (g^a, a)$ by 1.6. Hence $g^a = g\alpha$. Similarly, $g^b = g\beta$. By (1.8) α and β are endomorphisms of U. By (1.9), $(g^b)^a = g^{ab} = g^a (g \text{ in } U)$. Thus $g\alpha = g\beta\alpha$, g in U, i.e., $\alpha = \beta\alpha$. Let us first suppose that $\alpha \neq 0$ in $\varepsilon(U)$. Then since $\varepsilon(U)$ is 0-right cancellative β is the identity automorphism of U. Now, by 1.9, $g^{(n,k)} = g^{(0,1)^k(1,0)^n} = (g^{(1,0)^n})^{(0,1)^k} = g\alpha^n\beta^k = g\alpha^n$ and (3.5) becomes

$$(A, n, k)(B, m, l) = (A(B\alpha^n), (n, k)(m, l))$$

By routine calculation, we can show that $S^* = Ux(CoC)$ under the multiplication

$$\begin{aligned}(g, (n, k), (m, l))&(h, (n_1, k_1), (m_1, l_1)) \\ &= (g\alpha^{n_1 - r}h\alpha^{k-r}, (((n, k), (m, l))((n_1, k_1), (m_1, l_1)))),\end{aligned}$$

where $r = \min(n_1, k)$ and α is an endomorphism of U, is a bisimple inverse semigroup with identity. To show associativity is straight forward, but tedious. Now,

$(g, (n, k), (m, l))\mathscr{R}(h, (n_1, k_1), (m_1, l_1))$ if and only if $n = n_1$ and $m = m_1$

and

$(g, (n, k), (m, l))\mathscr{L}(h, (n_1, k_1), (m_1, l_1))$ if and only if $k = k_1$ and $l = l_1$

Thus, if

$$\begin{aligned}&(g, (n, k), (m, l)), (h, (u, v), (r, s)) \text{ in } S^*, \\ &(g, (n, k), (m, l))\mathscr{R}(g, (n, v), (m, s))\mathscr{L}(h, (u, v), (r, s))\end{aligned}$$

and S^* is bisimple. $(E, (0, 0), (0, 0))$ where E is the identity of U is the identity of S^*.

The idempotents of S^* are $\{(E, (n, n), (k, k)), n, k \text{ in } I^0\}$. It is easily seen that these commute.

Thus, S^* is a bisimple inverse semigroup with identity, [8, p. 1111].

The right unit subsemigroup P^* of S^* is $\{(g, 0, n, 0, k) : n, k \text{ in } I^0, g \in G\}$. It is seen immediately that P^* is isomorphic to P and hence $S \cong S^*$ by Theorem 2.1. Let us give the converse of this case. Now it is quite easily seen that the unit group of S is $\{g, (0, 0), (0, 0)\} \cong G$. (the unit group is $H_{((0,0),0,0))}$). Thus, U is a *-group.

The right unit subsemigroup P of S is $\{(g, n, k)\colon n, k \text{ in } I^0\}$ under the multiplication

$$(g, n, k)(h, m, s) = (g(h\alpha^n), n + m, k) \text{ if } n > 0$$
$$(g, 0, k)(h, m, s) = (gh, m, k + s)$$

Let $(g, 0, 0) \varepsilon U$ and $(h, m, s), m > 0$ be in P. Since G is a *-group, there exists g' in G such that $h^{-1}gh = g'\alpha^m$ (since α is nontrivial, α^m is nontrivial) as $\varepsilon(G)$ is 0-right cancellative). Thus

$$(g, 0, 0)(h, m, s) = (gh, m, s) = (h(g'\alpha^m), m, s) = (h, m, s)(g', 0, 0).$$

Next, we consider $(h, 0, m)$. Now, let $g' = h^{-1}gh$. Then,

$$(g, 0, 0)(h, 0, m) = (gh, 0, m) = (hg', 0, m) = (h, 0, m)(g', 0, 0)$$

Hence, U satisfies (3).

$$E_S = \{E, (n, n), (k, k)\colon n, k \text{ in } I^0\}$$

and multiplication in E_S is given by

$$\begin{aligned}(n, k)(m, l) &= (n, k) \text{ if } n > m \\ &= (n, k) \text{ if } n = m \text{ and } k > l\,.\end{aligned}$$

Thus (1) is satisfied.

Next, suppose α is the zero of $\varepsilon(U)$, i.e., $g\alpha = E, g$ in U. This means $g^a = E, g$ in U. Now $g^{(n,k)} = g^{(0,1)^k(1,0)^n} = (g^{(1,0)^n})^{(0,1)^k} = (E)^{(0,1)^k} = E$ if $n \neq 0$. If $n = 0$, $g^{(n,k)} = g^{(0,k)} = g\beta^k$. Thus, our multiplication (3.5) becomes $(A, n, k)(B, m, s) = (A, n + m, k)$ if $n \neq 0$,

$$(A, 0, k)(B, m, s) = (A(B\beta^k), m, k + s))\,.$$

Now, by (3), if $(g, 0, 0)$ in U, there exists $(g', 0, 0)$ in U such that if $m \neq 0$

$$(g, 0, 0)(B, m, s) = (gB, m, s) = (B, m, s)(g', 0, 0) = (B, m, s)\,.$$

Hence, $gB = B$ and $g = E$. Since g was arbitrary, U is a trivial group and we have a contradiction. Thus α cannot be a trivial endomorphism.

EXAMPLE. Let G be a *-group, C be the bicyclic semigroup, and o be the Bruck product. If we let α be the trivial endomorphism of G in the 1-dimensional (2-dimensional) case, S is a bisimple inverse semigroup with E_S integrally (lexicographically) ordered and with group of units a *-group. However (3) of Theorem (3.2) is not satisfied. $S = CoG$ is the 1-dimensional case.

Added in proof. (1) A nontrivial group is called an e-group if every nontrivial endomorphism of G is an epimorphism. The following theorem has a proof similar to that of Theorem 3.2.

THEOREM. *In Theorem* 3.2, *replace* *-*group by e-group and the multiplication given there by*

$$\begin{aligned}&(g, (n, k), (m, 1))(h, (r, s), (u, v))\\&\quad = (g\alpha^{r-\delta}\beta^{u-\gamma_1(r,k)}h\alpha^{k-\delta}\beta^{1-\gamma_2(r,k)}, ((n, k), (m, 1))((r, s), (u, v)))\end{aligned}$$

where if $r>k$, $\gamma_1(r, k) = 0$, $\gamma_2(r, k) = 1$; *if* $k>r$, $\gamma_1(r, k) = u$, $\gamma_2(r, k) = 0$; *if* $k = r$, $\gamma_1(r, k) = \gamma_2(r, k) = \min(u, 1)$, $\delta = \min(k, r)$ *and* α, β *are nontrivial endomorphisms of* G *such that* $\beta\alpha = \alpha$.

(2) N. R. Reilly [11] has determined a structure theorem equivalent to Theorem 3.1 by different methods. According to his terminology, a bisimple semigroup S is called a bisimple ω-semigroup if E_S is integrally ordered. If E_S is lexicographically ordered we will call S an L-bisimple semigroup.

(3) A bisimple semigroup S is L_n-bisimple (I-bisimple, I-ω-bisimple) if E_S is n-lexicographically ordered (is order isomorphic to Z under the reverse of the usual order, is order isomorphic to ZXI^0 under the usual lexicographic order [Van der Waerden, Vol. 1, p. 81]). We describe the structure of these classes of semigroups completely mod groups in [12], [13], and [16]. The structure theorem for L-bisimple semigroups generalizes Theorem 3.2. We investigate several of the properties of L-bisimple, I-bisimple and I-ω-bisimple semigroups, such as homomorphisms, congruences, and (ideal) extensions in [12], [13], [14], [17], and [18]. The method of attack- initiated here- which readily allows applications of results of [7]-[9] is used throughout.

(4) We will also call the n-dimensional bicyclic semigroup the $2n$-cyclic semigroup in future papers.

(5) We have also studied some of the properties of the semigroups whose structure has been given here in [13] and [15].

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