PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

SOME LOWER BOUNDS FOR LEBESGUE AREA

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It is well known in area theory that a continuous map f of the unit square Q^2 into Euclidean space E^2 can have zero Lebesgue area even though its range has a nonempty interior. This cannot happen if f is suitably well-behaved, for example, if f is light, Lipschitzian, or as we shall see below, if f satisfies a certain interiority condition. The purpose of this paper is to determine conditions under which an arbitrary measurable set $A \subset Q^2$ will support the Lebesgue area of f. The results imply that if $f \mid A$ is Lipschitz and if one of the coordinate functions of f is BVT (and continuous), then the Lebesgue area of f is no less than the integral of the multiplicity function $N(f, A, y)$, where $N(f, A, y)$ is the number (possibly ∞) of points in $f^{-1}(y) \cap A$. We show that the BVT condition cannot be omitted. The proofs of theorems involving Lebesgue area depend upon a new co-area formula for real valued BVT functions.

2. Preliminaries. Our proofs rely heavily upon the following topological theorem [3, p. 513] which was first proved by Federer in the 2-dimensional case [8, p. 358]. We believe that this result is yet to be fully exploited in area theory.

THEOREM 2.1. *If X is a k-dimensional finitely triangulable space and $u: X \to E^1$, $v: X \to E^{k-1}$, $f: X \to E^1 \times E^{k-1}$ are continuous maps such that $f(x) = (u(x), v(x))$ for $x \in X$, then there is a countable set $D \subset E^1$ such that*

$$S[f, (s, t)] = S[v \mid u^{-1}(s), t] \quad \text{for} \quad (s, t) \in (E^1 - D) \times E^{k-1} .$$

Here $S[f, (s, t)]$ denotes the stable multiplicity of f at (s, t) [9, (3.10)].

In the case $X = Q^2$, the unit square, (and this will be the only case of interest to us throughout the remainder of this paper) this theorem provides a very simple criterion to determine the stability of f at a point (s, t); for t is a positive stable value of $v \mid u^{-1}(s)$ if and only if there is a nondegenerate continuum $C \subset u^{-1}(s)$ such that $t \in$ interior $v(C)$. Thus, the stable multiplicity function is positive at almost all points in the range of a monotone map and in the case of a light map, it is positive on an open dense set. In view of the following proposition, we see that mappings which are similar to Whyburn's quasi-open maps [19, p. 110], [22, (3.9)] also have positive stable values.

PROPOSITION 2.2. Suppose $f: Q^2 \to E^2$ is a continuous map such that for each $y \in f(Q^2)$, there is a component K of $f^{-1}(y)$ with the property that for each sufficiently small open connected set U containing y, there is a component V of $f^{-1}(U)$ containing K which maps onto U by f. Then, for all but countably many $y \in f(Q^2)$, $S(f, y) > 0$.

Proof. Select a point $y \in f(Q^2)$ whose first coordinate is not contained in the set D of (2.1). Let U_i be a sequence of sufficiently small open connected sets such that $U_i \supset$ closure U_{i+1} and whose intersection is a closed vertical line segment λ containing y in its interior. Then the intersection of the corresponding V_i will be a continuum $C \supset K$ that will be mapped onto λ. By (2.1), $S(f, y) > 0$. Now by repeating this argument with horizontal line segments instead of vertical ones, the result follows.

It is easy to verify that if $S(f, y) > 0$, then the converse of (2.2) holds, c.f. [21, (2.4)].

The notion of stability is crucial in area theory since

$$\mathfrak{L}(f) = \int_{Q^2} S(f, y)\,dL_2(y) \,, \tag{2.2.1}$$

where $\mathfrak{L}(f)$ is the Lebesgue area of f and L_2 is 2-dimensional Lebesgue measure. By a result of Cesari [1], (2.2.1) is a special case of a more general theorem due to Federer [9, (7.9)].

DEFINITIONS 2.3. H_n^k will denote k-dimensional Hausdorff measure in E^n, F_n^k k-dimensional Favard measure [7, (2.18)], L_n n-dimensional Lebesgue measure, and dim (A, x) will denote the topological dimension of a set A at a point x. A real valued map f on a topological space is called *almost light* if $f^{-1}(y)$ is totally disconnected for L_1 almost all $y \in E^1$. A map $f: Q^2 \to E^1$ is said to satisfy *condition N_1 on a set A* if it maps sets of H_2^1 measure zero of A into sets of L_1 measure zero.

We will use the following notion which was first introduced in [6, p. 48]. An L_n measurable set $E \subset E^n$ has the unit vector $n(x)$ as the *exterior normal* to E at x if, letting

$$\begin{aligned} S(x, r) &= \{y: |y - x| < r\} \,, \\ S_+(x, r) &= S(x, r) \cap \{y: (y - x)\cdot n(x) \geqq 0\} \,, \\ S_-(x, r) &= S(x, r) \cap \{y: (y - x)\cdot n(x) \leqq 0\} \,, \\ \alpha(n) &= L_n[S(x, 1)] \,, \end{aligned} \tag{2.3.1}$$

we have

$$2 \lim_{r \to 0+} L_n[S_-(x, r) \cap E]/\alpha(n)r^n = 1 \,, \quad 2 \lim_{r \to 0+} L_n[S_+(x, r) \cap E]/\alpha(n)r^n = 0 \,.$$

Let BV denote the class of all locally integrable functions $u: Q^n \to E^1$

such that the ith partial derivative of u in the sense of distributions is a totally finite measure μ_i. This class contains those functions which are BVT. For $u \in BV$ and B any Borel subset of Q^n let $I(u, E) = |\mu|(E)$ where $|\mu|$ is the total variation of the vector-valued measure $(\mu_1, \mu_2, \cdots, \mu_n)$. In the case that u is ACT observe that for any Borel set $B \subset Q^n$,

$$I(u, B) = \int_B |\operatorname{grad} u(x)| dL_2(x) \tag{2.3.2}$$

where grad u is the ordinary gradient of u. Thus, in this case $I(u, \cdot)$ can be extended to all Lebesgue measurable sets.

If $B \subset E^n$ is a Borel set then $P(B)$ will denote the *perimeter of* B. If F is the set of x for which the exterior normal to B exists at x and if $P(B) < \infty$, then we see from [2] and [10] that

$$P(B) = H_n^{n-1}(F) . \tag{2.3.3}$$

F is called the *reduced boundary of* B and note that $F \subset \operatorname{bdry} B$. For $u: Q^n \to E^1$ in BV and $E(s) = \{x: u(x) > s\}$, Fleming and Rishel [14] proved that

$$I(u, Q^n) = \int_{E^1} P[E(s)] dL_1(s) . \tag{2.3.4}$$

In the case that u is Lipschitzian, theorems obtained independently by Federer [11, (3.1)] and Young [20, Th. 4] imply that

$$I(u, A) = \int_{E^1} H_n^{n-1}[u^{-1}(s) \cap A] dL_1(s) \tag{2.3.5}$$

whenever $A \subset Q^n$ is a Lebesgue measurable set.

3. Metric theorems.

The following co-area formula is an extension of (2.3.5) and although the proof is only given for functions defined on E^2, it is clear that it will generalize to E^n without any essential change. The author is indebted to Casper Goffman for his suggestion that this co-area formula might be valid.

The following notation will be used throughout the proof. Let (q, r, s) be coordinates in E^3 and define $\delta: E^3 \to E^1$, $\Pi_2: E^3 \to E^2$, $\Pi_1: E^2 \to E^1$ by $\delta(q, r, s) = s$, $\Pi_2(q, r, s) = (r, s)$ and $\Pi_1(q, r) = r$. If $u: Q^2 \to E^1$ then $u': Q^2 \to E^3$ is defined by $u'(q, r) = (q, r, u(q, r))$. G^2 will denote the group of orthogonal transformations on E^2 and φ the unique Haar measure on G^2 for which $\varphi(G^2) = 1$. For $R \in G^2$ let $R^*: E^3 \to E^3$ be defined by $R^*(q, r, s) = (q', r', s)$ where $R(q, r) = (q', r')$.

THEOREM 3.1. *If* $u: Q^2 \to E^1$ *is* $BVT(ACT)$, *then*

$$I(u, D) = \int_{E^1} H_2^1[u^{-1}(s) \cap D] dL_1(s)$$

whenever $D \subset Q^2$ *is a Borel* (L_2 *measurable*) *set.*

Proof. Let

$$g(s) = H_2^1[u^{-1}(s) \cap D] = H_3^1[\delta^{-1}(s) \cap u'(D)] .$$

If u is BVT and D a Borel set, then $A = u'(D)$ is an analytic set and therefore it is the union of an increasing sequence of compact sets and a set N of H_3^2 measure zero. Using the Eilenberg inequality [4] we see that

$$H_3^1[\delta^{-1}(s) \cap N] = 0$$

for L_1 almost all $s \in E^1$. Thus, in order to show that g is L_1 measurable it is sufficient to consider the case when A is compact; but then, it can be shown as in [11, (3.1)] that g is the limit of upper semi-continuous functions.

If $u: Q^2 \to E^1$ is ACT and $N \subset Q^2$ a set for which $L_2(N) = 0$, then [18, (3.17)] and [12] imply that $H_3^2[u'(N)] = 0$. Thus, $u'(D)$ is H_3^2 measurable whenever $D \subset Q^2$ is L^2 measurable and the measurability of g follows as it did above.

Let

$$\alpha(D) = \int_{E^1} H_2^1[u^{-1}(s) \cap D] dL_1(s) .$$

It is now clear that α is a measure on Borel (L_2 measurable) sets if u is $BVT(ACT)$. Moreover, from [18, (3.17)], [12], and [4] we see that α is absolutely continuous with respect to L_2 if u is ACT. Hence, it is only necessary to prove the theorem in case u is BVT. For this purpose we only need to show that $I(u, W) = \alpha(W)$ for rectangles $W \subset Q^2$ because both $I(u, \cdot)$ and α are measures over the Borel sets. We may as well assume that $W = Q^2$.

In view of (2.3.4) and (2.3.3) it is obvious that $I(u, Q^2) \leqq \alpha(Q^2)$. The opposite inequality will follow from the last of four parts into which the remainder of the proof is divided.

PART 1. *For* L_1 *almost all* $s \in E^1$, $u^{-1}(s)$ *is* $(H_2^1, 1)$ *rectifiable.*

Proof. Since u is BVT, $\mathfrak{L}(u') < \infty$ [16, p. 516]. If $A = u'(Q)$ then it follows from [12] that $H_3^2(A) < \infty$ and that A is $(H_3^2, 2)$ rectifiable. Now apply [13, (8.16)] to obtain a countable number of 2-dimensional proper regular submanifolds M_i of class C^1 for which

$$H_3^2\left[A - \bigcup_{i=1}^{\infty} M_i\right] = 0 .$$

Letting $M = \bigcup_{i=1}^{\infty} M_i$ the Eilenberg inequality [4] implies

$$H_3^1[\delta^{-1}(s) \cap (A - M)] = 0$$

and

$$H_3^1[\delta^{-1}(s) \cap A] < \infty$$

for L_1 almost all s. In view of (2.3.5) one can easily verify that for each i, $\delta^{-1}(s) \cap M_i$ is $(H_2^1, 1)$ rectifiable and therefore that $\delta^{-1}(s) \cap M_i \cap A$ is $(H_2^1, 1)$ rectifiable for L_1 almost all $s \in E^1$. But the union of $\delta^{-1}(s) \cap M_i \cap A$ occupies H_2^1 almost all of $\delta^{-1}(s) \cap A$ and thus the result follows.

PART 2. *For L_1 almost all $s \in E^1$, $F_2^1[u^{-1}(s)] = H_2^1[u^{-1}(s)]$.*

Proof. This follows from Part 1 and [7, (5.14)].

PART 3.

$$\int_{E^1} H_2^1[u^{-1}(s)]dL_1(s) = \Pi 2^{-1}\int_{G^2}\int_{E^1} N[\Pi_2 R^* u', Q^2, y]dL_2(y)d\varphi(R) .$$

Proof. For each $s \in E^1$ apply [7, (5.11)] to obtain

$$\begin{aligned} F_2^1[u^{-1}(s)] &= \Pi 2^{-1}\int_{G^2}\int_{E^1} N[\Pi_1 R, u^{-1}(s), r]dL_1(r)d\varphi(R) \\ &= \Pi 2^{-1}\int_{G^2}\int_{E^1} N[\Pi_2 R^* u', Q^2, (r, s)]dL_1(r)d\varphi(R) . \end{aligned}$$

By integrating with respect to s, the result follows from Part 2 and Fubini's theorem.

PART 4.

$$I(u, Q^2) \geqq \int_{E^1} H_2^1[u^{-1}(s)]dL_1(s) .$$

Proof. Select a sequence of Lipschitz functions $u_k: Q^2 \to E^1$ which converge uniformly to u and for which $I(u_k, Q^2) \to I(u, Q^2)$ as $k \to \infty$. A result of [18, (3.5)] states that for each $R \in G^2$ and continuous $v: Q^2 \to E^1$,

$$N[\Pi_2 R^* v', Q^2, y] = S[\Pi_2 R^* v', y] \tag{1}$$

for L_2 almost all $y \in E^2$. Recall that the stable multiplicity function is lower semi-continuous with respect to uniform convergence. Thus, from Part 3, (1), Fatou's lemma, and (2.3.5)

$$\begin{aligned}
\int_{E^1} H_2^1[u^{-1}(s)]dL_1(s) &= \Pi 2^{-1}\int_{G^2}\int_{E^2} N[\Pi_2 R^* u', Q^2, y]dL_2(y)d\varphi(R) \\
&= \Pi 2^{-1}\int_{G^2}\int_{E^2} S[\Pi_2 R^* u', y]dL_2(y)d\varphi(R) \\
&\leqq \liminf_{k\to\infty} \Pi 2^{-1}\int_{G^2}\int_{E^2} S[\Pi_2 R^* u'_k, y]dL_2(y)d\varphi(R) \\
&= \liminf_{k\to\infty} \Pi 2^{-1}\int_{G2}\int_{E^3} N[\Pi_2 R^* u'_k, Q^2, y]dL_2(y)d\varphi(R) \\
&= \lim_{k\to\infty}\int_{E^1} H_2^1[u_k^{-1}(s)]dL_1(s) \\
&= \lim_{k\to\infty} I(u_k, Q^2) = I(u, Q^2)\ .
\end{aligned}$$

COROLLARY 3.2. *If $u\colon Q^2 \to E^1$ is BVT, then the following hold for L_1 almost all $s \in E^1$:*

(i) *$H_2^1[u^{-1}(s)] < \infty$ and $u^{-1}(s)$ is $(H_2^1, 1)$ rectifiable,*

(ii) *the exterior normal to $E(s)$ exists at H_2^1 almost all $x \in u^{-1}(s)$.*

Proof. The first statement follows from the proof of Part 1 in (3.1) and the second from (3.1), (2.3.4), and (2.3.3).

LEMMA 3.3. *If $u\colon Q^2 \to E^1$ is BVT, then for L_1 almost all $s \in E^1$, $\dim[u^{-1}(s), x] > 0$ for H_2^1 almost all $x \in u^{-1}(s)$.*

Proof. If $B \subset E^2$, $x \in E^2$, denote by $W(x)$ the set of all straight lines passing through x and by $U(B, x)$ those $\lambda \in W(x)$ for which x is not a cluster point of $\lambda \cap B$. Since we may identify $W(x)$ with the unit semi-circle S_+^1, we can regard the restriction of H_2^1 to S_+^1 as defining a measure μ on $W(x)$. In the same manner, we can define a measure ν on the homogeneous space of all orthogonal projections $p\colon E^2 \to E^1$.

Suppose, for some $s \in E^1$, that $H_2^1[u^{-1}(s)] < \infty$ and that $u^{-1}(s)$ is $(H_2^1, 1)$ rectifiable. Letting

$$D_s = u^{-1}(s) \cap \{x\colon \mu[U(u^{-1}(s), x)] = 0\}\ ,$$

it follows from [7, (8.3)] that $L_1[p(D_s)] = 0$ for ν almost all p. But D_s is also $(H_2^1, 1)$ rectifiable and therefore, from [7, (5.14)] it follows that $H_2^1(D_s) = 0$. Thus, in view of (3.2), for L_1 almost all $s \in E^1$ the following two conditions hold at H_2^1 almost all $x \in u^{-1}(s)$:

(i) the exterior normal to $E(s)$ exists at x,

(ii) $\mu[U^{-1}(s), x] > 0$.

We will conclude the proof by showing that for all such s and x, $\dim[u^{-1}(s), x] > 0$. For if we assume that $\dim[u^{-1}(s), x] = 0$, this means that there exist arbitrarily small open sets G containing x whose boundaries do not intersect $u^{-1}(s)$. By the Phragmen-Brouwer theorem, it can be assumed that bdry G is connected. For every $r > 0$, let

$$U_r[u^{-1}(s), x] = W(x) \cap \{\lambda: S(x, r) \cap u^{-1}(s) \cap (\lambda - \{x\}) = 0\} .$$

From (ii) we know that there exists $\alpha > 0$ and $r_0 > 0$ such that $\mu[U_{r_0}(u^{-1}(s), x)] = \alpha$. Choose $G \subset S(x, r_0/2)$. Since bdry G is connected and bdry $G \cap u^{-1}(s) = 0$, either bdry $G \subset E(s)$ or bdry $G \subset F(s) = \{x: u(x) < s\}$. Suppose bdry $G \subset E(s)$ and because of (i), r_0 may be assumed to have been chosen so small that (see (2.3.1)),

$$2L_2[S_+(r_0, x) \cap E(s)]/\Pi r_0^2 < \alpha/\Pi . \tag{3}$$

Now, for each $\lambda \in U_{r_0}(u^{-1}(s), x)$, $S(x, r_0) \cap u^{-1}(s) \cap (\lambda - \{x\}) = 0$ and $\lambda \cap \text{bdry}\, G \neq 0$. Therefore, since bdry $G \subset E(s)$, the union of all such λ in $S(x, r_0) - \{x\}$ is contained in $E(s)$ and its L_2 measure is no less than αr_0^2, which contradicts (3). The case of bdry $G \subset F(s)$ is treated in a similar way and thus the proof is concluded.

LEMMA 3.4. *Suppose $f: Q^2 \to E^2$ is continuous and $f = (u, v)$ where u is BVT. Then $f^{-1}(y)$ is totally disconnected for L_2 almost all $y \in E^2$.*

Proof. Let λ be a horizontal (or vertical) line segment in Q^2 on which u as a function of one variable is of bounded variation. Thus, if λ is the line $r = r_0$, the function $u(\cdot, r_0)$ is of bounded variation and consequently, $N[u(\cdot, r_0), \lambda, s] < \infty$ for L_1 almost all $s \in E^1$. This implies that $f(\lambda)$ intersects almost all vertical lines in a finite number of points and therefore, by Fubini's theorem, $L_2[f(\lambda)] = 0$. Since u is BVT, there exist a countable dense set of vertical lines and a countable dense set of horizontal lines such that the image of each line is a set of L_2 measure zero. If Λ denotes the union of these vertical and horizontal lines, then $L_2[f(\Lambda)] = 0$. Now if C is a nondegenerate continuum of $f^{-1}(y)$, for some $y \in E^2$, then clearly C must intersect Λ. Thus $y \in f(\Lambda)$ and the result follows.

COROLLARY 3.5. *With the same hypotheses as in 3.4, for L_1 almost all $s \in E^1$, $v \mid u^{-1}(s)$ is almost light.*

THEOREM 3.6. *Suppose $f: Q^2 \to E^2$ is continuous, $f = (u, v)$, u is BVT and v satisfies condition N_1 on an analytic set $A \subset Q^2$. Then*

$$\mathfrak{L}(f) \geqq \int_{E^2} N(f, A, y)\, dL_2(y) .$$

Proof. Let $W_s = u^{-1}(s) \cap \{x: \dim [u^{-1}(s), x] > 0\}$. It follows from (2.1), (3.3), (3.5) and [9, (3.3), (3.5), (3.12)] that for L_1 almost all $s \in E^1$

$$\begin{aligned}\int_{E^1} S[f, (s, t)]dL_1(t) &= \int_{E^1} S[v \mid u^{-1}(s), t]dL_1(t)\\ &\geqq \int_{E^1} N[v, W_s, t]dL_1(t)\\ &\geqq \int_{E^1} N[v, u^{-1}(s) \cap A, t]dL_1(t)\\ &= \int_{E^1} N[f, A, (s, t)]dL_1(t) .\end{aligned}$$

Now by integrating with respect to s the result follows from Fubini's theorem and (2.2.1). The analyticity of A is needed only to assure the L_2 measurability of the last integrand.

COROLLARY 3.7. *If $f: Q^2 \to E^2$ is continuous, if f is Lipschitzian on an L_2 measurable set $A \subset Q^2$, and $f = (u, v)$ where u is BVT, then*

$$\mathfrak{L}(f) \geqq \int_{E^2} N(f, A, y)dL_2(y) .$$

REMARK 3.8. It is easy to see that if neither of the coordinate functions of f is BVT, then the conclusion of (3.7) may not hold. For this purpose let $A \subset Q^2$ be a dendrite for which $L_2(A) > 0$. Then a result from [15, p. 290] implies that A is a retract of Q^2. If $r: Q^2 \to A$ is the retraction and $i: A \to A$ the identity map, then $f = ir$ is clearly Lipschitzian on A and $\mathfrak{L}(f) = 0$ since the range of f has no interior.

THEOREM 3.9. *Suppose $f: Q^2 \to E^2$ is continuous, $f = (u, v)$, u is ACT, v satisfies condition N_1 on Q^2, the approximate partial derivatives of v exist L_2 almost everywhere on Q^2, and Jf, the approximate Jacobian of f, is integrable. Then*

$$\mathfrak{L}(f) = \int_{Q^2} | Jf(x) | dL_2(x) = \int_{E^2} N(f, Q_2, y)dL_2(y) .$$

Proof. Referring to [5, (5.4)] and (3.6) we see that we only need to prove that f carries sets of L_2 measure zero into sets of L_2 measure zero. If this were not the case, then there would exist an L_2 null set $N \subset Q^2$ for which $L_2[f(N)] > 0$. We may assume that $f(N)$ is measurable since N can be taken as a G_δ set. Thus, $L_1[v(u^{-1}(s) \cap N)] > 0$ and therefore $H_2^1[u^{-1}(s) \cap N] > 0$ for all s in some set of positive L_1 measure. But, from (2.3.2) and (3.1)

$$0 = \int_N | \operatorname{grad} u(x) | dL_2(x) = \int_{E^1} H_2^1[u^{-1}(s) \cap N]dL_1(s) > 0$$

a contradiction.

COROLLARY 3.10. *If u is ACT and v Lipschitzian on Q^2, then*

$$\mathfrak{L}(f) = \int_{Q^2} | Jf(x) | dL_2(x) = \int_{E^2} N(f, Q^2, y) dL_2(y) .$$

REMARK 3.11. The above corollary is an extension of a theorem proved in [17, p. 437], where only the first part of the equality is established. Both (3.8) and (3.9) are related to the following unsolved problem c.f. [16, p. 380], [17, p. 433]: Let $f: Q^2 \to E^2$ where both coordinate functions of f are ACT and Jf is L_2 integrable. Then, is

$$\mathfrak{L}(f) = \int_{Q^2} | Jf(x) | dL_2(x) \ ?$$

By using techniques employed in this paper, one can show that if the additional hypothesis is made that v satisfies condition N_1 on $W_s = u^{-1}(s) \cap \{x: \dim [u^{-1}(s), x] > 0\}$ for L_1 almost all $s \in E^1$, then the question can be settled in the affirmative.

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Received August 30, 1965. This work was supported in part by a research grant from the National Science Foundation.

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