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ON LIAPUNOV FUNCTIONS WITH A SINGLE CRITICAL POINT

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In this paper we discuss the geometry of the level surfaces of functions $f(x) = f(x_1, x_2, \cdots, x_n)$ of class C'' in E^n that possess an isolated relative minimum point at the origin, and no other critical points, finite or infinite. Our principal result is that such a function satisfies the condition $f(x) > f(0)$ for all $(x) \neq (0)$. The levels sets $f(x) = c$ and the domains they bound are discussed. The results are useful in Liapunov stability theory.

A finite critical point of $f(x)$ is a point of E^n at which $f_{x_i} = 0$ $(i = 1, 2, \cdots, n)$. We shall say that $f(x)$ possesses an *infinite critical point* if there is some sequence of point $\{x_n\}$, $(x^n) \rightarrow \infty$, for which the function

$$F(x) = f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2 \tag{1}$$

tends to zero. To say that $f(x)$ has no infinite critical point means then that there exists a positive constant ε and a sphere $\|x\| = r^2$ such that $F(x) \geqq \varepsilon$ outside the sphere.

Let $f(x) = f(x_1, x_2, \cdots, x_n)$ be a function of class C'' in E^n, and suppose that $f(x)$ is *positive definite* neighboring the origin; that is $f(0) = 0$, and $f(x) > 0$ near the origin. We term such a function $f(x)$ *admissible*. The point $x = 0$ is then a relative minimum point (possibly degenerate) of $f(x)$. Suppose that $f(x)$ is admissible and that it has no critical point, finite or infinite, except at the origin. It follows that $F(x)$ is bounded away from zero in the complement of every spherical ball with center at the origin. There is then a sphere S with center at 0 on which $f > 0$. Let m be the minimum value assumed by f on S, and consider the set of points M inside S for which $f = a (0 < a < m)$. This set clearly exists, for consider the continuous function F along any continuous arc joining the origin to a point on S where $f = m$. M has the following properties. At each point of M the implicit-function theorem guarantees that the equation

$$f(x) = a$$

can be solved for one of the variables x_i, inasmuch as at least one of the functions $f_{x_i} \neq 0$ at each point of M. This solution will be locally of class C' in the remaining variables. It follows that an open neighborhood of each point of M is a homeomorphic image of an open disk in E^{n-1} and, consequently, that M is a closed manifold.[1] Next, we shall show that M bounds a domain containing the origin. Consider any continuous arc joining the origin to an arbitrary point of S. At some point of this arc f must assume the value a. This point belongs to M. It should be observed that M is connected, for if it were not, it would be composed of a set of bounded, complementary domains in each of which f would have a minimum point. This would contradict the assumption of the existence of only one critical point.

The curves orthogonal to the family of level surfaces $f(x) = c$ are solutions of the system [see 2]

$$\frac{dx_i}{d\tau} = \frac{f_{x_i}}{f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2} \qquad (i = 1, 2, \cdots, n) . \tag{2}$$

Further, Morse [3] shows that writing the differential equations for the orthogonal trajectories in the form (2) permits a parametrization $x = x(\tau)$ of each trajectory with the property

$$f[x(\tau)] \equiv \tau . \tag{3}$$

It follows that $f \not\equiv$ *constant* on any subarc of the trajectory. Because of our assumption of the absence of critical points, except the origin, fundamental existence theorems guarantee that there is a unique solution without multiple points of (2) through each point of $E^n - \{0\}$, and that this solution can be extended (in both directions along the curve) to the boundary of $E^n - \{0\}$. It follows that each trajectory goes from the origin to infinity as τ increases steadily from the value zero.

Further, if τ ranges on a finite interval $0 < \tau_0 \leqq \tau \leqq \tau_1$, we shall see that the functions $x_i(\tau)$ remain bounded. It will follow (since the trajectories go from the origin to infinity) that τ, and hence f, increases steadily from 0 to $+\infty$ along each trajectory. To prove this note that (1) implies that each of the n functions

$$\frac{f_{x_i}}{f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2} = \frac{f_{x_i}}{F}$$

is bounded outside a sufficiently small sphere S_0 having the origin as its center. If M denotes a common bound of these quotients, we have from (2) that

$$x_i(\tau) = c_i + \int_{\tau_0}^{\tau} \frac{f_{x_i}[x(\tau)]}{F[x(\tau)]} d\tau \qquad (i = 1, 2, \cdots, n) ,$$

[1] In addition to being locally euclidean M is clearly bounded. Further, since f is continuous, $f^{-1}(a)$, the map of a closed set (single point) of R, is also closed, accordingly, M is compact.

and

$$|x_i(\tau)| \leqq |c_i| + M(\tau - \tau_0) .$$

Here, c_i is a constant, and τ_0 is the value assumed by f at the point where the trajectory pierces S_0.

The set of points $M: f = a$ has been shown to be a closed bounded manifold. If the set of points $f = c_0 (c_0 > a)$ is a closed bounded manifold bounding an open domain D containing the manifold M, and if D contains no critical point except the origin, Morse's program in [3] is readily extended to show that the sets $f \leqq c$ and $f \leqq a (a \leqq c \leqq c_0)$ are homeomorphic. The question arises as to how large c_0 may be taken in the present analysis. To answer this consider the family of trajectories orthogonal to M each parametrized so that (3) holds. Let $c_0 (> a)$ be any value assumed by f and extend each trajectory from $\tau = a$ to $\tau = c_0$. Each such endpoint $\tau = c_0$ of a trajectory clearly lies on the level surface $f = c_0$ We shall show that these "ends" constitute a closed bounded manifold.

To accomplish this, note that we may show, as above, that the functions $x(\tau)$ are bounded for $a \leqq \tau \leqq c_0$. Next, let P be any point where $f = c_0$. There is a unique solution of (2) through P, and it can be parametrized so that (3) holds. Extend that trajectory in the direction of decreasing τ to $\tau = a$. This point clearly lies on M, and the trajectory is the unique trajectory orthogonal to M, at this point. Thus, the set of points $f = c_0$ are bounded, and the trajectories provide a one-to-one continuous mapping of the set $f \leqq a$ into the set $f \leqq c_0$, precisely as in Morse's analysis.

Now let M_1 be any bounded closed manifold determined by the equation $f(x) = c_1$ that bounds a domain D_1 containing the origin, and let P be any point of D_1 inside M_1. We shall show that $f(P) < c_1$. For, consider the trajectory $T: x = x(\tau)$ through P orthogonal to M_1, and suppose that $f(P) \geqq c_1$. The function $f[x(\tau)]$ is of class C' on T. As one continues T from M_1 through P, the arc A must either go to the origin or go off to infinity. In the latter case, the arc would have to intersect M_1 a second time, and $f[x(\tau)]$ would attain on T either a relative maximum or a relative minimum value at a point $x = \xi \notin M_1$; that is, at an interior point of a subarc of T within M_1. At $x = \xi$, we would then have

$$f_{x_1}\frac{dx_1}{d\tau} + f_{x_2}\frac{dx_2}{d\tau} + \cdots + f_{x_n}\frac{dx_n}{d\tau} = 0 .$$

But since $x = \xi$ lies on T, equations (2) must be satisfied, and it follows that

$$f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2 = 0$$

at $x = \xi$; that is, $x = \xi$ is a critical point of $f(x)$, contrary to hypothesis. Accordingly, the arc A that starts at M_1 and passes through P goes to the origin. If $f(P) \geqq c_1$, it would follow that $f[x(\tau)]$ would possess an extremum at an interior point of A, and the argument employed above would show that this extremum would actually be a critical point of f. From this contradiction we infer that $f(P) < c_1$.

Suppose now that P_1 is any point of $E^n \notin D_1 + M_1$. We shall show that $f(P_1) > c_1$. For, suppose $f(P_1) \leqq c_1$. Then we continue the trajectory T_1 through P_1 orthogonal to M_1 from P_1 to the origin. On this arc there would again be an extremum of the function $f[x(\tau)]$ that can be shown, as above, to be a critical point of f.

We combine the foregoing results in the following statement.

THEOREM. *Let $f(x)$ be admissible and have no critical point, finite or infinite, except the origin. Then, $f(x) > 0$, $(x) \neq (0)$, throughout $E^n - \{0\}$. The set of points $f(x) = c$, where c is any (positive) value assumed by f, is a bounded closed manifold M that bounds an (open) domain D containing the origin. Further, $f(x) < c$ throughout D and $f(x) > c$ exterior to M. Finally, if $0 < c_1 < c$, the closed manifold $f = c_1$ lies wholly in D.*

The following corollary[2] is an immediate consequence of the theorem.

COROLLARY 1. *If $f(x)$ is admissible and if $f(x_0) \leqq 0$ for some point $(x_0) \neq (0)$, $f(x)$ has a critical point, finite or infinite, in addition to that at the origin.*

We continue with a definition. A solution curve of (2) joining the origin to a point P on which the only critical point of f is the origin will be called an *α-arc* joining these two points. We have then the following result.

COROLLARY 2. *If $f(x)$ is admissible and $f(x_0) \leqq 0$, $(x_0) \neq (0)$, there can be no α-arc joining the origin to the point $(x) = (x_0)$.*

For, the assumption of the existence of such an arc would lead, as above, to the existence of a critical point of f on the arc.

The following examples will illuminate the theory.

EXAMPLE. The function $f(x, y) = y^2 + x^4$ has precisely one critical point, the (degenerate) relative minimum point at the origin. The

[2] The question answered by this corollary was put to the writer by Professor George Szegö.

level lines $y^2 + x^4 = c(0 < c < \infty)$ are closed ovals about the origin, and their orthogonal trajectories are the curves $x = 0$ and

$$y = k \exp(-1/4x^2) ,$$

k constant. The trajectory through each point in the plane, except the origin, is clearly an α-arc.

EXAMPLE. Let f be the function

$$f(x, y) = 6x^2 + y^2 + 2x^3 .$$

Clearly, f is positive definite at the origin and vanishes along a curve that passes through the point $(-5, 10)$. Accordingly, f must possess a critical point in addition to that at $(0, 0)$. It is readily seen that this is the point $(-2, 0)$. The equations $f(x, y) = c(0 < c \leqq 8)$ determine closed curves around the origin. The trajectories orthogonal to these level lines are the curves

$$y^6 = k\frac{x}{x+2} .$$

It will be observed, for example, that the trajectories orthogonal to the level curves at points (x_0, y_0) for which $-2 < x_0 < 0, y_0 \neq 0$, start at the origin and go off to infinity asymptotic to the line $x = -2$, the abscissa of the second critical point. On the other hand, the line $y = 0$ joins every point $P_0(x_0, 0), x_0 < -2$, to the origin and is the trajectory through P_0 orthogonal to the given level lines. It clearly passes through the critical point $(-2, 0)$. There is clearly no α-arc passing through any point to the left of the line $x = -2$. All points except the origin for which $f \leqq 0$ lie to the left of this line.

Some of the preceding analysis can be recast as follows. Let $f(x)$ be a function of class C'' in E^n with a relative minimum point at $x = a$, and suppose that $x = a$ is an isolated critical point of $f(x)$. If $f(a) = k$, the equation $f(x) = k + \varepsilon$, where ε is a sufficiently small positive number, represents an $(n-1)$-manifold M in a neighborhood of $x = a$. Through each point of M there exists a unique trajectory orthogonal to M. We extend each such trajectory in both directions from M terminating the extension only when we reach a critical point of f. Let k be the point set union of all such trajectories with all critical points deleted. Finally, let B be set of all points in E^n for which $f(x) \leqq k$. It follows that $K \cap B = \emptyset$.

An analogous result may, of course, be stated when $x = a$ is a relative maximum point of f.

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