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STABILITY IN TOPOLOGICAL DYNAMICS

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This paper is concerned with two types of stability in transformation groups. The first is a generalization of Lyapunow stability. In the past this notion has been discussed in a setting where the phase group was either the integers or the one-parameter group of reals. In this paper it is defined for replete subsets of a more general phase group in a transformation group. Some connections between this type of stability and almost periodicity are given. In particular, it is shown that a type of uniform Lyapunov stability will imply Bohr almost periodicity. The second type of stability in this paper is a limit stability. This gives a condition which is necessary and sufficient for the limit set to be a minimal set. Finally, these two types of stability are combined to provide a sufficient condition for a limit set to be the closure of a Bohr almost periodic orbit.

Throughout this paper X will be assumed to be a uniform space. It will be implicity assumed that the Hausdorff topology of X is the one induced by the uniformity. T will denote a topological group and the triple (X, T, π) will be called a transformation group provided X and T are as above and $\pi: X \times T \to X$ such that if e is the identity of T then:

(1) $\pi(x, e) = x$ for all x in X,

(2) $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 t_2)$ for all x in X and t_1, t_2 in T,

(3) π is continuous. Henceforth we shall write $\pi(x, t) = xt$; and if $A \subset T$ then $xA = \{xt: t \varepsilon A\}$.

DEFINITION 1. A subset A of T is called $\{$*left*$\}\{$*right*$\}$ *syndetic* [6] in T provided there exists a compact set $K \subset T$ such that $\{AK = T\}$ $\{KA = T\}$. It is clear that if A is left syndetic in T then A^{-1} is right syndetic in T.

DEFINITION 2. A point $x \in X$ is called *S-Lyapunov stable* ($S \subset T$) with respect to a set $B \subset X$ provided that for each index α of X there exists an index β of X such that if $y \in B \cap x\beta$ then $yt \in xt\alpha$ for all t in S.

THEOREM 1. *If S is left syndetic in T and* Cl (xT) ($=$ *closure* xT) *is compact then a necessary and sufficient condition that $x \in X$ be T-Lyapunov stable with respect to xT is that x be S-Lyapunov stable with respect to xT.*

Proof. The necessity is clear. To prove the sufficiency let α, an index of X, be given. Since S is left syndetic there exists a compact set $K \subset T$ such that $SK = T$. Since both $\mathrm{Cl}\,(xT)$ and K are compact the mapping π: $\mathrm{Cl}\,(xT) \times K \rightarrow \mathrm{Cl}\,(xT)$ is uniformly continuous. Hence there exists an index β of X such that if $p \in \mathrm{Cl}(xT)$ and $q \in xT$ such that $q \in p\beta$ then $qk \in (pk)\ \alpha$ for all $k \in K$. The assumption that x is S-Lyapunov stable with respect to xT implies that there exists an index γ of X such that if $y \in x\gamma \cap xT$ then $ys \in\ (xs)\ \beta$ for all $s \in S$. Let $t \in T$ and $y \in x\gamma \cap xT$ be given. There exists an $s \in S$ and $k \in K$ such that $sk = t$. Thus $ys \in (xs)\beta$ which implies that $y(sk) \in (x(sk))\ \alpha$ or $yt \in xt\alpha$. Since t is arbitrary the theorem is proved.

Simple examples show that in one sense this is about as strong an inheritance theorem that one can prove. For example, if $\mathrm{Cl}\,(xT)$ is not compact then S being syndetic in T and x being S-Lyapunov stable is not sufficient for x to be T-Lyapunov stable.

We now consider some connections between S-Lyapunov stability and almost periodicity. To do this we need the following lemma which provides a characterization of repleteness.

DEFINITION 3. A subset M of T is said to be *replete* [6] in T provided M contains some bilateral translate of each compact subset of T.

LEMMA 1. *In order that a subset S of T be replete it is sufficient that S intersect each translate of each left syndetic subset of T and if T is commutative this condition is also necessary.*

Proof. In order to show that this condition is sufficient we assume that S is not replete in T. That is, there exists a compact set $K \subset T$ such that for all $t_1, t_2 \in T$ we have $t_1 K t_2 \not\subset S$. Let $A(t_1, t_2) = t_1 K t_2 - S$ and $A = \bigcup A(t_1, t_2)$, $((t_1, t_2) \in T \times T)$. It follows that $A = T - S$. Clearly we can assume that $e \in K$. We now show that A is left syndetic. Since $e \in K^{-1}$ it follows that $T - S \subset AK^{-1}$. Let $s \in S$ be given. If $sK \cap A = \varnothing$ then $sKe \subset S$ since $A = T - S$, which is impossible. Therefore $sK \cap A \neq \varnothing$. Hence there exists a $k \in K$ and $a \in A$ such that $sk = a$. Therefore $s = ak^{-1}$ and $s \in AK^{-1}$ which, since s was arbitrary, implies that $S \subset AK^{-1}$. Therefore $AK^{-1} = T$ and A is left syndetic in T. However $Ae \cap S = \varnothing$ and this contradiction proves the sufficiency.

To show that this condition is also necessary if T is commutative we assume that S is replete and that there exists a syndetic set $A \subset T$ and a $t' \in T$ such that $t'A \cap S = \varnothing$. Let K be a compact set which contains the identity and has the property that $AK = T$. Since

S is replete there exists a $t_1 \in T$ such that $K^{-1} t_1 \subset S$. (Repleteness reduces to this property when T is commutative.) Since $e \in K^{-1}$ it follows that $t_1 \in S$. Also, $AK = T$ which implies that $t'AK = T$. Hence there exists an $a \in A$ and $k \in K$ such that $t'ak = t_1$. This implies that $t'a = t_1 k^{-1}$. Hence $t'a \in S$ and $t'A \cap S \neq \emptyset$ which is a contradiction.

DEFINITION 4. T is said to be *almost periodic at* x [6] provided that for each index α of X there exists a left syndetic set $A \subset T$ such that $xA \subset x\alpha$.

DEFINITION 5. T is said to be *Bohr almost periodic* at $x \in X$ (x is *Bohr almost periodic*) provided that corresponding to each index α of X there exists a left syndetic set A in T such that $xtA \subset xt\alpha$ for all $t \in T$.

It is clear that if T is commutative and x is both almost periodic and T-Lyapunov stable with respect to xT then x is Bohr almost periodic. However, it is possible to weaken these conditions and still obtain Bohr almost periodicity. Throughout the rest of this paper it will be assumed that T is commutative.

THEOREM 2. *Let S be a replete subset of T. If $x \in X$ is S-Lyapunov stable with respect to xT and x is almost periodic then x is Bohr almost periodic.*

Proof. Let α, an index of X, be given and let β be a symmetric index of X such that $\beta^2 \subset \alpha$. Let S be a replete subset of T such that x is S-Lyapunov stable with respect to xT. Then there exists an index γ of X such that if $y \in x\gamma \cap xT$ then $yt \in (xt)\beta$ for all t in S. Let δ be a symmetric index of X such that $\delta^2 \subset \gamma$. Since x is almost periodic under T there exists a syndetic set $A \subset T$ such that $xA \subset x\delta$. Let $t' \in T$ and $a \in A$ be given. We will now show that $x(t'a) \in xt'\alpha$ which will complete the proof. Since π is continuous there exists an index σ of X such that if $y \in x\sigma$ then $ya \in xa\delta$. Let η be an index of X with the property that $\eta \subset \sigma \cap \delta$. Once again there exists a syndetic set $B \subset T$ such that $xB \subset x\eta$ since x is almost periodic. Since S is a replete subset of T, S^{-1} is also replete in T. Also, since Bt'^{-1} is syndetic it follows from Lemma 1 that $Bt'^{-1} \cap S^{-1} \neq \emptyset$. That is, there exists an $s \in S$ and $t_1 \in B$ such that $t_1 t'^{-1} = s^{-1}$ which implies that $xt_1 \in x\eta$ thus $xt_1 \in x\sigma$. Hence $xt_1 a \in xa\delta$. The fact that $xa \in x\delta$ implies that $xt_1 a \in x\gamma$. Since $t't_1^{-1} = s \in S$ it follows that

$$xt_1at't_1^{-1} \in (xt't_1^{-1})\beta$$

or

$$x(t'a) \in x(t't_1^{-1})\beta$$

However since $xt_1 \in x\eta$ it follows that $xt_1 \in \delta$. Therefore

$$xt_1(t't_1^{-1}) \in (xt't_1^{-1})\beta$$

or $xt' \in (xt't_1^{-1})\beta$. Since β is symmetric and $\beta^2 \subset \alpha$ it follows that $xt'a \in xt'\alpha$ which completes the proof.

In [6, 6.34] the *P-limit set of* x for $P \subset T$ and $x \in X$ is defined by $P_x = \bigcap \mathrm{Cl}\,(xtP)(t \in T)$. In this same reference it is stated that if P is a replete semi-group in T then P_x is closed and invariant and if $\mathrm{Cl}\,(xP)$ is compact then $P_x \neq \emptyset$. Using this notion it is possible to give another set of conditions which are sufficient for x to be Bohr almost periodic. This theorem generalizes a theorem of A. A. Markov [7, p. 390].

DEFINITION 6. The orbit of a point $x \in X$ is said to be (*uniformly*) *S-Lyapunov stable* with respect to a set $B \subset X$ provided that for each index α of X there exists an index β of X such that if $y \in xT$ and $z \in B$ with $y \in z\beta$ then $yt \in zt\alpha$ for all t in S.

THEOREM 3. *Let S be a replete semi-group in T. If the orbit of x is S-Lyapunov stable with respect to xT and Cl (xS^{-1}) is compact then x is Bohr almost periodic.*

Proof. If we can show under these hypotheses that x is almost periodic under T then by using Theorem 2 we can deduce that x is Bohr almost periodic.

Let S be a replete semi-group of T, let the orbit of x be S-Lyapunov stable with respect to xT and let $\mathrm{Cl}\,(xS^{-1})$ be compact. It is clear that S^{-1} is also a replete semi-group of T. Therefore, by the above remarks it follows that S_x^{-1} is nonempty. Since S_x^{-1} is closed it is compact. It follows from [6, 4.06] that there exists a $y \in S_x^{-1}$ such that y is almost periodic. It follows from [6, 4.07] that $\mathrm{Cl}\,(yT)$ is a compact minimal set. If $x \in \mathrm{Cl}\,(yT)$ then x is almost periodic and the theorem is proved.

Assume $x \notin \mathrm{Cl}\,(yT)$. Then there exists an index α of X with the property that $x \notin (\mathrm{Cl}\,(yT))\alpha$. Let β be an index of X with the property that $\beta^2 \subset \alpha$. Since the orbit of x is S-Lyapunov stable with respect to xT there exists an index γ of X with the property that if $p \in q\gamma$ and $p, q \in xT$ then $pt \in qt\beta$ for all $t \in S$. Let δ be a symmetric index of X with the property that $\delta^2 \subset \gamma$. Since $y \in S_x^{-1} - xT$ and $S_x^{-1} = \bigcap_{t \in T} \mathrm{Cl}\,(xtS^{-1})$ we have $y \in S_x^{-1} \subset \mathrm{Cl}\,(xS^{-1})$. Hence there exists an $s_1 \in S$ such that $xs_1^{-1} \in y\delta$. Since π is continuous there exists an index σ of X such that if $p \in y\sigma$ then $ps_1 \in ys_1\beta$. There exists an $s \in S$ such that $xs^{-1} \in y\sigma \bigcap y\delta$. Thus $xs^{-1}s_1 \in ys_1\beta$. Also $xs \in y\delta$ which implies that $xs_1^{-1} \in xs^{-1}\gamma$. Thus $xs_1^{-1}s_1 \in xs^{-1}s_1\beta$. These two statements imply that

$$x = xs_1^{-1}s_1 \in xs_1\beta^2 \subset (\mathrm{Cl}\,(xT))\,\alpha$$

which is a contradiction. It follows that $x \in \mathrm{Cl}\,(yT)$ and the theorem is proved.

A subset E of T is said to be *P-extensive* (P is a replete semigroup not equal to T) [2, p. 1146] provided that $pP \cap E \neq \varnothing$ for all p in P. A point $x \in X$ is said to be P-recurrent provided that for each index α of X there exists a P-extensive set E such that $xE \subset x\alpha$. Using these concepts and the previous theorem we are able to give a set of necessary and sufficient conditions in order for a point to be Bohr almost periodic.

THEOREM 4. *If S_x is compact for some replete semigroup S of T then the following statements are equivalent*:

(1) *x is Bohr almost periodic,*

(2) *x is S-recurrent and the orbit of x is S-Lyapunov stable with respect to S_x.*

Proof. If x is Bohr almost periodic and α is any index of X then there exists a syndetic set $A \subset T$ such that $xtA \subset (xt)\,\alpha$ for all t in T. It follows from Lemma 1 that A is P-extensive for each replete semi-group of T. Hence x is S-recurrent. From [1] it follows that $\mathrm{Cl}\,(xT) = S_x$ and hence $\mathrm{Cl}\,(xT)$ is compact. By [6, 4.37] it follows that the orbit of x is T-Lyapunov stable with respect to xT. It follows in the same manner as in [7, p. 385] that the orbit of x is T-Lyapunov stable with respect to $\mathrm{Cl}\,(xt)$. Since $S_x = \mathrm{Cl}\,(xT)$ the orbit of x is T-Lyapunov stable with respect to S_x which completes the proof of this half of the theorem.

If x is S-recurrent then it follows that $\mathrm{Cl}\,(xT) = S_x$. Hence $\mathrm{Cl}\,(xT)$ is compact. Since the orbit of x is S-Lyapunov stable with respect to S_x it follows from Theorem 3 that x is Bohr almost periodic.

An alternate proof of this theorem can be given using the main theorem in [3] and the theorem of Gottschalk [5] relating uniform almost periodicity and equi-continuity.

We now introduce the concept of S-orbital stability which is generalized from the notion of a final point being asymptotically stable which was discussed by Friedlander [4].

DEFINITION 6. A point $y \in X$ is said to be *S-orbitaly stable* (SO-stable) with respect to a set $B \subset X$ provided there exists an open set U containing y such that if $x \in B \cap U$ then $S_x = S_y$. When X is a uniform space then the orbit of y is *uniformly SO-stable* with respect to $B \subset X$ provided there exists an index α of X such that if $x \in B \cap yt\alpha$ then $S_x = S_y$.

LEMMA 2. *If S is a replete semi-group and S_y is compact and nonempty then a necessary and sufficient condition that S_y be a minimal set is that the orbit of y be uniformly SO-stable with respect to S_y.*

Proof. If S_y is a minimal set then it follows immediately that the orbit of y is uniformly SO-stable with respect to S_y.

Let the orbit of y be uniformly SO-stable with respect to S_y and $x \in S_y$. There exists an index δ of X such that if $z \in S_y \cap yt\delta$ then $S_z = S_y$. Since $x \in S_y$ there exists a $t' \in T$ such that $x \in yt'\delta$ which implies $S_x = S_{yt'}$. But, since S is a replete semi-group of T, $S_{yt'} = S_y$ hence $S_x = S_y$. Therefore, $S_x \subset \mathrm{Cl}\,(xS) \subset \mathrm{Cl}\,(xT)$ implies $S_y = \mathrm{Cl}\,(xT)$ for all $x \in S_y$. Thus S_y is a minimal set.

THEOREM 5. *Let S be a replete semi-group in T and let S_y be compact and nonempty. If the orbit of y is uniformly SO-stable and S-Lyapunov stable with respect to S_y then S_y is the closure of a Bohr almost periodic orbit.*

Proof. It follows from the previous lemma that S_y is a minimal set. By Theorem 4 it is sufficient to show that if $x \in S_y$ then the orbit of x is S-Lyapunov stable with respect to $S_y = S_x$. Let δ be an index of X and β be a symmetric index of X such that $\beta^2 \subset \delta$. Since the orbit of y is S-Lyapunov stable with respect to S_y there exists an index γ of X such that if $z \in S_y \cap (yt')\gamma$ then $zt \in (yt't)\beta$ for all $t \in S$. Let α be a symmetric index of X with the property that $\alpha^2 \subset \gamma$. Let $z \in S_y \cap (xt')\alpha$. There exists a $\bar{t} \in T$ such that $y\bar{t} \in (xt')\alpha$ which implies $y\bar{t}t \in (xt't)\beta$ and $zt \in (y\bar{t}t)\beta$ for all $t \in S$. Hence $zt \in (xt't)\beta^2 \subset (xt't)\delta$ for all $t \in S$. This implies that x is S-Lyapunov stable with respect to S_y and completes the proof of the theorem.

The question of necessary and sufficient conditions on y in order that S_y be the closure of a Bohr almost periodic point is still an open question. Theorem 5 shows that a necessary condition must be found on y which will imply that $x \in S_y$ is uniformly S-Lyapunov stable with respect to S_y.

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