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TRIVIALLY EXTENDING DECOMPOSITIONS OF E^n

JOSEPH ZAKS

Let G be a monotone decomposition of E^n, then G can be extended in a trivial way, to the monotone decomposition G^1 of E^{n+1}, where $E^n = \{(x_1, \cdots, x_n, 0) \in E^{n+1}\}$, by adding to G all points of $E^{n+1} - E^n$. If the decomposition space E^n/G of G is homeomorphic to E^n, E^n/G is said to be obtained by a pseudo-isotopy if there exists a map $F: E^n \times I \rightarrow E^n \times I$, such that $F_t(=F \mid E^n \times t)$ is homeomorphism onto $E^n \times t$, for all $0 \leqq t < 1$, F_0 is the identity and F_1 is equivalent to the projection $E^n \rightarrow E^n/G$.

The purpose of this paper is to present a relation between these two notions. It will then follow, that if G is the decomposition of E^3 to points, circles and figure-eights, due to R. H. Bing, for which E^3/G is homeomorphic to E^3, then E^4/G^1 is not homeomorphic to E^4.

Moreover, we will present a direct, geometric proof to this particular property.

For definitions, see [1]. See also [2].

THEOREM 1. *If G is a monotone decomposition of E^n, such that E^n/G is homeomorphic to E^n, then the following are equivalent:*

(1) *E^{n+1}/G^1 is homeomorphic to E^{n+1}.*

(2) *E^n/G can be obtained by a pseudo-isotopy.*

Proof. (1) $\Rightarrow$ (2). Let $h: E^{n+1}/G^1 \rightarrow E^{n+1}$ be a homeomorphism and let $p: E^{n+1} \rightarrow E^{n+1}/G^1$ be the projection map.

The map $H: E^n \times I \rightarrow E^{n+1}$, defined by $H(x, t) = hp(x, 1 - t)$ for all $x \in E^n, t \in I$, is such that H_t is a homeomorphism into for all $0 \leqq t < 1$, H_1 is equivalent to the projection map $E^n \rightarrow E^n/G$, and $H(E^n \times I)$ is homeomorphic to $E^n \times I$, hence, up to a homeomorphism of $E^n \times I$ onto itself, H is the required pseudo-isotopy.

(2) $\Rightarrow$ (1). Let $F: E^n \times I \rightarrow E^n \times I$ be the pseudo-isotopy for E^n/G. The map $H: E^{n+1} \rightarrow E^{n+1}$, where

$$H(x, t) = \begin{cases} F(x, 1 + t) & -1 \leqq t \leqq 0 \\ F(x, 1 - t) & 0 \leqq t \leqq 1 \\ (x, t) & t \geqq 1 \text{ or } t \leqq -1 \end{cases} \qquad \text{where } x \in E^n .$$

is well defined, $H(E^{n+1}) = E^{n+1}$, and $H(E^{n+1})$ is homeomorphic to E^{n+1}/G^1, because $H_0 = F_1$ and it is equivalent to the projection map E^n onto E^n/G. The proof is completed.

Using Theorem 1 of [4], we have the following

COROLLARY. *If G is a monotone decomposition of E^2, such that E^2/G is homeomorphic to E^2, then E^3/G^1 is homeomorphic to E^3.*

It is well known that the decomposition G of E^3 to points, circles and figure-eights, as described in § 4 of [3], is such that E^3/G is homeomorphic to E^3 but E^3/G cannot be obtained by a pseudo-isotopy, see [1] and [2]. Therefore, it follows from Theorem 1 that this G has the property that E^4/G^1 is not homeomorphic to E^4; see our remark at the end of this paper.

However, we would like to present a direct proof for

THEOREM 2. *Let G be the decomposition of E^3, as described in § 4 of [3], then E^4/G^1 is not homeomorphic to E^4.*

Proof. Suppose it is not true, then let $h: E^4/G^1 \rightarrow E^4$ be a homeomorphism, and $p: E^4 \rightarrow E^4/G^1$ be the projection map.

Let f be the map of the complete 2-complex, C_7^2, with 7 vertices, into E^4, which is affine on each triangle of C_7^2, and is almost an embedding, except for its effect $f(P) = f(Q)$ for two points P and Q of C_7^2, where P and Q are points in the relative interior of two disjoint (in C_7^2) triangles A and B, respectively. f is described in [6], see also [7].

Without loss of generality we may assume, as we do, that $f(A) \subset E^2 \subset E^4$, and $f(P) = f(Q) =$ the origin. Therefore $f(B)$ has in E^3 only an edge l, passing through the origin, as described in Figure 1, where we also describe the two disks, which are the union of all the circles and figure-eights of G. In order that the disk of G, which is perpendicular to $f(A)$, will not meet $f(A)$ except in the common radius of the two disks of G, we push, continuously and without touching the rest of $f(C_7^2)$, the interior of the disk D, which is contained in $f(A)$, so that it will have small positive values in the 4-th coordinate.

By doing this, we defined the two disks to lie in E^3, therefore we get an equivalent decomposition to that of § 4 of [3], which we denote again by G, and we let G^1 be its extension to E^4.

The set $pf(A \cup B)$ in E^4/G^1 is homeomorphic to the union of two disjoint disks, together with a simple arc α joining an interior point of one disk to an interior point of the other. Therefore, $[pf(C_7^2)$-interior $\alpha]$ is homeomorphic to C_7^2 in E^4/G^1, and since h is supposed to be a homeomorphism, $h[pf(C_7^2)$-interior $\alpha]$ is a subset of E^4, homeomorphic to C_7^2.

This contradicts a well known result of A. Flores, [5], therefore

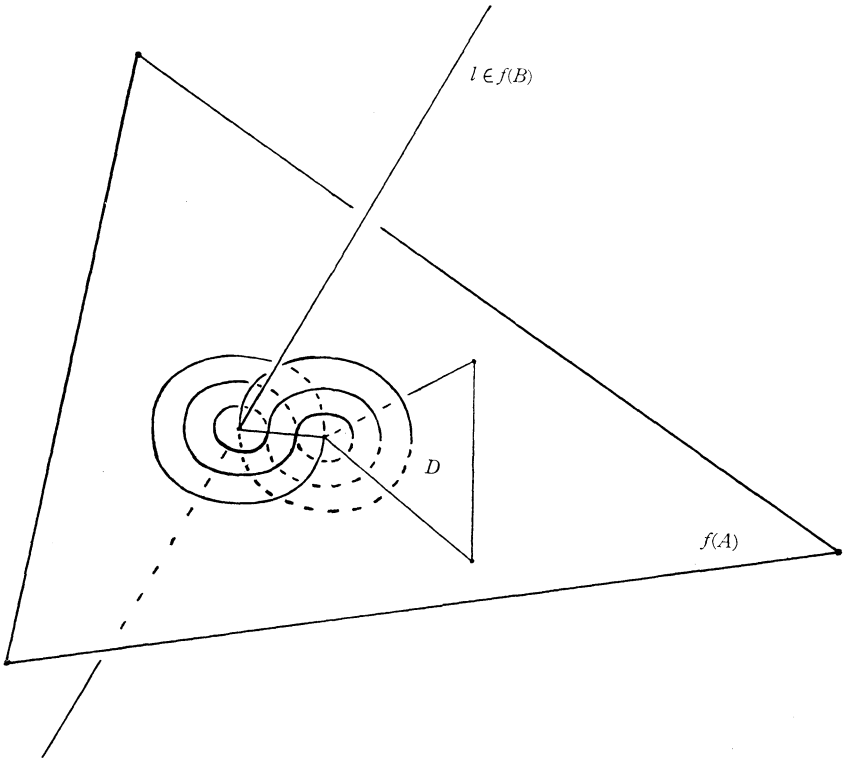

FIGURE 1

the proof is completed.

In fact, E^4/G^1 is even not embeddable in E^4, (same proof).

REMARK. Theorem 2 was proved by M. M. Cohen in his "Simplicial structures and transverse cellularity", Ann. of Math. 85 (1967) 218–245.

REFERENCES

1. S. Armentrout, *Monotone decompositions of* E^3, Topology Seminar, Wisconsin, 1965, edited by R. H. Bing and R. J. Bean, Ann. of Math. Studies, No. 60.
2. R. J. Bean, *Decompositions of* E^3 *which yield* E^3, Pacific J. Math. **20** (1967), 411–413.
3. R. H. Bing, *Decompositions of* E^3, Topology of 3-manifolds and Related Topics, edited by M. K. Fort, Jr., Prentice-Hall, 1962.
4. M. L. Curtis, *An embedding theorem,* Duke Math. J. **24** (1957), 349–351.
5. A. Flores, *Über die Existenz n-dimensionaler komplexe, die nicht in den* R_{2n} *topologisch einbettbar sind,* Ergeb. Math. Kolloq. **5** (1932/33), 17–24.
6. B. Grünbaum, *Graphs and complexes,* lecture notes. University of Washington, 1967.
7. J. Zaks, *On a minimality property of complexes,* Proc. Amer. Math. Soc. **20** (1969), 439–444.

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UNIVERSITY OE WASHINGTON
SEATTLE, WASHINGTON