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POLYNOMIALS IN LINEAR RELATIONS II

MICHAEL J. KASCIC, JR.

The study of linear relations is continued in the setting of the theory of locally convex linear topological spaces. The investigation is limited to the polynomials in one fixed closed linear relation. Conditions both on the relation and on the locally convex space are discussed that are sufficient or necessary and sufficient for all the polynomials in this relation to be also closed.

The reader is referred to [1] for full details of the algebraic properties of linear relations, and to [4] for a summary. Since we are concerned here with a special class of linear relations in locally convex spaces, we present a compendium of definitions tailored to this case.

Let X be a locally convex space equipped with the Mackey-Arens topology. A *linear relation* in X is a linear subspace of $X \oplus X$. This concept generalizes that of an operator on X. If T is a linear relation in X, the definitions of the domain and range of $T, D(T)$ and $R(T)$ respectively, are obvious.

If S and T are linear relations in X,

$$S + T = \{(x, y + z): (x, y) \in S\ (x, z) \in T\}$$
$$S \cdot T = \{(x, z): \exists y \in X \ni (x, y) \in T, (y, z) \in S\}$$

are linear relations in X. If λ is a complex number, we may consider λ as the linear relation $\{(x, \lambda x): x \in X\}$ in X. We write λT for $\lambda \cdot T$.

Combining the three operations defined above, we can arrive at a well-defined notion of a polynomial in a linear relation. To avoid the complications treated in [1], we shall assume that the coefficient of the highest power of the linear relation is always nonzero.

Finally a linear relation is closed if it is a closed subspace of $X \oplus X$.

2. Polyclosed relations.

If T is a closed linear relation, and if for every polynomial $P, P(T)$ is a closed linear relation, then we shall say that T is *polyclosed*. There exist closed linear relations in Hilbert space that are not *polyclosed*. This is demonstrated in the following example, originally due to Arens.

EXAMPLE 2.1. Let l^2 be the Hilbert space of square summable sequences of complex numbers, and let $X = l^2 \oplus l^2$. Let A be the operator on l^2 such that $A(x_1, x_2, \cdots) = (x_1/1, x_2/2, x_3/3, \cdots)$. The domain of $A = l^2$, A is continuous, and hence A is a closed operator.

The range of A is a dense subspace which we shall call M. For every $x \in M$, $y \in l^2$, let $T(x, y) = (Ay, A^{-1}x)$. It is easily seen that T is a closed linear relation since A and A^{-1} are. Let us now consider T^2. $T^2(x, y) = T(T(x, y)) = T(Ay, A^{-1}x) = (AA^{-1}x, A^{-1}Ay)$. It is easily seen that for every $x \in M$ and $y \in l^2$, $AA^{-1}x = x$ and $A^{-1}Ay = y$. Thus T is the identity on the domain of T and is not defined elsewhere. The domain of T however, is $M \oplus l^2$ which is certainly not closed in X. Thus T^2 is not closed in $X \oplus X$. Thus T^2 is not a closed linear relation and T is not polyclosed.

In the following sections we will seek generalizations of [4, 3.16] which we present below as 2.2. In 2.2 we shall assume that all topologies are relativized Mackey-Arens topologies, a standardization we shall continue throughout the paper.

THEOREM 2.2. *If T is a closed linear relation on X such that for some $\lambda \in \boldsymbol{C}$, $R(T - \lambda)$ is closed and $(T - \lambda)^{-1}$ is single-valued and continuous, then T is polyclosed.*

3. Selective relations.

If X is a locally convex space, we may imbed it in a linearly homeomorphic manner in a complete locally convex space $\tilde{X}$. If T is a closed linear relation in X, then we may speak of $\tilde{T}$, i.e., the completion of T considered as a subspace of $X \oplus X$. It is obvious that $\tilde{T} \cap X \oplus X = T$ since T is closed. We shall call T *left selective* relative to X if $\tilde{T} \cap \tilde{X} \oplus X = T$, *right selective* relative to X if $\tilde{T} \cap X \oplus \tilde{X} = T$, and *selective* relative to X if it is *either* left or right selective relative to X. We omit the phrase "relative to X" whenever there is no danger of confusion.

First let us show that there is a large class of selective relations in noncomplete spaces.

PROPOSITION 3.1. *Let T be a continuous operator with closed domain. Then T is right selective.*

Proof. It is well known that if T is continuous then it has a continuous extension $\hat{T}$ to the completion of its domain. We shall first show that $\hat{T} \cap X \oplus \tilde{X} = T$. Let $(x_1, x_2) \in \hat{T} \cap X \oplus \tilde{X}$. Then

$$x_1 \in D(\hat{T}) \cap X = D(T)$$

since it is closed. Since $\hat{T}$ is an extension of T, $x_2 = Tx_1$ which is an element of X. Thus $\hat{T} \cap X \oplus \tilde{X} = T$. All that remains is to show that $\hat{T} = \tilde{T}$. First let $x_2 = \hat{T}x_1$. Since $\hat{T}$ is a continuous extension of T, $\hat{T}x_1 = \lim Tx_\gamma$ for any net $x_\gamma \to x_1$ where all $x_\gamma \in X$. For all x_γ, $(x_\gamma, Tx_\gamma) \in T$. Since T is closed, $(\lim x_\gamma, \lim Tx_\gamma) \in \tilde{T}$ because both limits exist. But $(\lim x_\gamma, \lim Tx_\gamma) = (x_1, x_2)$. This shows that $\hat{T} \subset \tilde{T}$. Almost the identical arguments in reverse show that $\hat{T} \supset \tilde{T}$. Thus $\hat{T} = \tilde{T}$, and T is right selective. Note that if T^{-1} is a continuous operator on a closed domain, then T is left selective since the definitions are symmetric.

However an operator need not be continuous to be selective.

EXAMPLE 3.11. Let $X = C^\infty([0, 1])$ with the sup norm topology. Then $\tilde{X} = C([0, 1])$ with the sup norm topology. Let D be the differentiation operator on $C^\infty([0, 1])$ and let $T = D^{-1}$. T is a many valued relation with a singlevalued inverse defined on all of $C^\infty([0, 1])$ and $\tilde{T} = \tilde{D}^{-1}$ where $\tilde{D}$ is the differentiation operator defined on $C^1([0, 1])$. It is well known that T^{-1} is not continuous. Nonetheless T is both left and right selective. For if $(x_1, x_2) \in \tilde{T} \cap \tilde{X} \oplus X$ then x_1 is the derivative of x_2 which is in $C^\infty([0, 1])$. Hence x_1 is in $C^\infty([0, 1])$ also. Thus $(x_1, x_2) \in T$. If $(x_1, x_2) \in T \cap X \oplus \tilde{X}$, then $x_2 = \int_0^t x_1(s)ds +$ constant and again $x_2 \in C^\infty([0, 1])$.

In essence, the important fact about selective relations is that if $\tilde{T}$ is polyclosed as a subspace of $\tilde{X} \oplus \tilde{X}$, then T is polyclosed as a subspace of $X \oplus X$. There is a possibility of confusion when we say T is selective, because it is a subspace of $\tilde{X} \oplus \tilde{X}$, $\tilde{X} \oplus X$ and $X \oplus \tilde{X}$ as well as $X \oplus X$. As a matter of convention, T shall always be considered as acting in X, and $\tilde{T}$ in $\tilde{X}$. Thus even if $T = \tilde{T}$, we shall use both symbols remembering which spaces we are considering. This convention shall also apply to the special linear relation λ for every $\lambda \in \boldsymbol{C}$.

LEMMA 3.2. *For every closed linear relation T and each $\lambda \in \boldsymbol{C}$, $\tilde{T} - \tilde{\lambda} = \widetilde{T - \lambda}$.*

Proof. Direct verification.

PROPOSITION 3.21. *Let T be a closed linear relation such that $T - \lambda$ is selective for some $\lambda \in \boldsymbol{C}$. Then for every polynomial P, $P(\tilde{T}) \cap X \oplus X = P(T)$.*

Proof. Let us first consider T to be selective. We will use induction. For polynomials of zero degree, we may apply 3.1. Now we assume the proposition true for every polynomial of degree $\leqq n$. Let P be a polynomial of degree $n + 1$. From [1,2.3] there exists a polynomial Q such that $P(\tilde{T}) - \tilde{\alpha} = \tilde{T} \circ Q(\tilde{T}) = Q(\tilde{T}) \circ \tilde{T}$. Thus

$$(P(\tilde{T}) - \tilde{\alpha}) \cap X \oplus X = \tilde{T} \circ Q(\tilde{T}) \cap X \oplus X = Q(\tilde{T}) \circ \tilde{T} \cap X \oplus X .$$

Suppose T is left selective. We will show

$$\tilde{T}\circ Q(\tilde{T})\cap X\oplus X=(\tilde{T}\cap X\oplus X)\circ(Q(\tilde{T})\cap X\oplus X)\,.$$

Right selectivity of T is treated mutatis mutandis using $Q(\tilde{T})\circ\tilde{T}$. Let $(x_1, x_2)\in\tilde{T}\circ Q(\tilde{T})\cap X\oplus X$. Then there exists x_3 such that $(x_1, x_3)\in Q(\tilde{T})$ and $(x_3, x_2)\in\tilde{T}$. Since T is left selective, $x_3\in X$. Thus $(x_1, x_3)\in Q(\tilde{T})\cap X\oplus X$ and $(x_3, x_2)\in\tilde{T}\cap X\oplus X$. Now let

$$(x_1, x_2)\in(\tilde{T}\cap X\oplus X)\circ(Q(\tilde{T})\cap X\oplus X)\,.$$

Then there exists $x_3\in X$ such that $(x_1, x_3)\in Q(\tilde{T})$ and $(x_3, x_2)\in\tilde{T}$. Then $(x_1, x_2)\in\tilde{T}\circ Q(\tilde{T})$ and certainly x_1 and $x_2\in X$. We are now reduced to the statement that

$$(P(\tilde{T})-\tilde{\alpha})\cap X\oplus X=(\tilde{T}\cap X\oplus X)\circ(Q(\tilde{T})\cap X\oplus X)\,.$$

We may rewrite the right hand side as $T\circ Q(T)$ by our induction assumption. Thus $(P(\tilde{T})-\tilde{\alpha})\cap X\oplus X=T\circ Q(T)$. To finish the proof we need only show $P(\tilde{T})\cap X\oplus X-\alpha=(P(\tilde{T})-\tilde{\alpha})\cap X\oplus X$. This is easy since the left hand side consists of pairs of the form $(x, y-\alpha x)$ such that $(x, y)\in P(T)$ and $x, y\in X$ while the right hand side consists of pairs of the form $(x, y-\alpha x)$ such that $(x, y)\in P(T)$ and $x, y-\alpha x\in X$. These classes obviously are the same. Now suppose $T-\lambda$ is selective. Given a polynomial P, there exist a unique polynomial P_1 of same degree such that $P_1(\tilde{T}-\tilde{\lambda})=P(\tilde{T})$. By 3.2

$$P_1(\tilde{T}-\tilde{\lambda})=P_1(\widetilde{T-\lambda})\,.$$

We have already proven

$$P_1(\widetilde{T-\lambda})\cap X\oplus X=P_1(T-\lambda)\,.$$

Thus $P(T)\cap X\oplus X=P(T)$. This completes the proof.

We are now prepared to prove the important theorem concerning selective relations.

THEOREM 3.22. *Let T be a closed linear relation in X such that $(T-\lambda)$ is selective for some $\lambda\in\boldsymbol{C}$. If P is a polynomial such that $P(\tilde{T})$ is closed in $\tilde{X}$, then $P(T)$ is closed in X.*

Proof. First assume T is selective. If $P(\tilde{T})$ is closed, then

$$\widetilde{P(T)}\subset\widetilde{P(\tilde{T})}=P(\tilde{T})\,.$$

Thus $\widetilde{P(T)}\cap X\oplus X\subset P(\tilde{T})\cap X\oplus X$. But $\widetilde{P(T)}\cap X\oplus X=\overline{P(T)}$, and by 3.21, $P(\tilde{T}) \cap X \oplus X = P(T)$. Thus $\overline{P(T)} \subset P(T)$. The other inclusion being obvious, we have that $\overline{P(T)} = P(T)$. If $T - \lambda$ is selective, choose Q such that $Q(T - \lambda) = P(T)$ and repeat the argument.

COROLLARY 3.23. *Let T be a closed linear relation in X such that $(T - \lambda)$ is selective for some $\lambda \in \boldsymbol{C}$. If $\tilde{T}$ is polyclosed in $\tilde{X}$, then T is polyclosed in X.*

We will finish this section by giving an example to show that the converse of 3.23 is false. Let us go back to the Example 2.1, redefining $X = M \oplus l^2$. T can indeed be considered as a closed linear relation in X. Since $T = \tilde{T}$, T is selective. $\tilde{T}$ is not polyclosed as was shown in 2.1. However, it can easily be shown that T is polyclosed. Since T^2 is nothing more than the identity on X, any polynomial in T equals a linear polynomial which is closed.

4. Polyclosed relations in complete spaces. Let us address ourselves to the question of proving polyclosedness of $\tilde{T}$. Thus in this section we shall be concerned with complete spaces. Before dealing with this case, let us introduce some new concepts that do not demand that X be complete.

Let us define the *generalized graph* of any relation. For $n \geqq 1$, $G_n(T) = \{(x_0, x_1, \cdots, x_n)$ such that for $k = 0, 1, \cdots, n-1, (x_k, x_{k+1}) \in T\}$. For $n = 0$, $G_n(T) = X$. Thus $G_n(T)$ is a subspace of X^{n+1}. We topologize it with the relativized product of the Mackey topology taken $(n+1)$ times. The name "generalized graph" is justified in that for $n = 1$, $G_n(T)$ is T, and for single-valued T would be the graph of T in the classical sense. The following proposition establishes the principal property enjoyed by the generalized graph.

PROPOSITION 4.1. *If T is a closed linear relation, then for every $n \geqq 0$, $G_n(T)$ is a closed subspace of X^{n+1}.*

Proof. The case for $n = 0$ being obvious, let $(x_0^\gamma, \cdots, x_n^\gamma)$ be a net in $G_n(T)$ such that $x_k^\gamma \rightarrow x_k$ for $k = 0, 1, \cdots, n$, or $(x_0^\gamma, x_1^\gamma, \cdots, x_n^\gamma) \rightarrow (x_0, x_1, \cdots, x_n)$. Then $(x_k^\gamma, x_{k+1}^\gamma) \rightarrow (x_k, x_{k+1})$ and since T is closed, $(x_k, x_{k+1}) \in T$ for $k = 0, 1, \cdots, n$. Thus by definition

$$(x_0, x_1, \cdots, x_n) \in G_n(T) .$$

COROLLARY 4.11. *If T is a closed linear relation in a complete space X, then for every $n \geqq 0$, $G_n(T)$ is a complete subspace of X^{n+1}.*

Because of the fact that closedness of T implies closedness of

$G_n(T)$, if we can connect $G_n(T)$ with $P(T)$, we then have a possible means of ascertaining whether $P(T)$ is closed. This connection is provided by the *realization map*, $\psi(P, T, \circ)$. For a fixed n^{th} degree polynomial P such that $P(t) = a_0 + a_1 t + \cdots + a_n t^n$, and a fixed T a closed linear relation

$$\psi(P, T, (x_0, x_1, \cdots, x_n)) = \left(x_0, \sum_{k=0}^{n} a_k x_k\right).$$

The reader can easily show that $\psi(P, T, \circ)$ is indeed a mapping from $G_n(T)$ into $P(T)$. The crucial fact about ψ is that it is *onto*. This fact is essentially due to Arens [1, 2.2]. However, Arens did not define the realization map explicitly, and stated his proposition directly in terms of elements of T. We present the state of affairs as a proposition.

PROPOSITION 4.2. *For a fixed polynomial, P, of degree n and a fixed closed linear relation T, $\psi(P, T, \circ)$ is a continuous linear map of $G_n(T)$ onto $P(T)$.*

Proof. See [1, 2.2].

It is easy to see that in general ψ is not a one-to-one map. For instance, let $T = X \oplus X$ and let $P(t) = t^2$. Then the null space of $\psi(P, T, \circ) = \{(0, x_1, x_2): x_1 + x_2 = 0\}$. We shall say that a linear relation has a *unique decomposition* or is a *unique decomposition relation* (abbreviated u.d.) if and only if for every polynomial P, $\psi(P, T, \circ)$ is a one-to-one map. By means of the next proposition, we see the usefulness of this concept as a generalization of an operator with a nonempty resolvent. First we prove the following lemma.

LEMMA 4.3. *Let T be a linear relation, and for $k = 0, 1, \cdots, n$, let $y_k = \sum_{i=0}^{k} (-\lambda)^i \binom{k}{i} x_{k-i}$. Then the map f such that*

$$f(x_0, x_1, \cdots, x_n) = (y_0, y_1, \cdots, y_n)$$

is a linear homeomorphism of $G_n(T)$ onto $G_n(T - \lambda)$.

Proof. We first show that $(y_0, y_1, \cdots, y_n) \in G_n(T - \lambda)$ for

$$(x_0, x_1, \cdots, x_n) \in G_n(T).$$

This is so if and only if $(y_k, y_{k+1}) \in T - \lambda$ for $k = 0, 1, \cdots, n - 1$, if and only if $(y_k, y_{k+1} + \lambda y_k) \in T$ for $k = 0, 1, \cdots, n - 1$. Substituting for y_k and y_{k+1} we get

$$\left(\sum_{i=0}^{k}(-\lambda)^i\binom{k}{i}x_{k-i}, \sum_{i=0}^{k+1}(-\lambda)^i\binom{k+1}{i}x_{k-i+1} + \lambda\sum_{i=0}^{k}(-\lambda)^i\binom{k}{i}x_{k-i}\right)$$
$$= \left(\sum_{i=0}^{k}(-\lambda)^i\binom{k}{i}x_{k-i}, \sum_{i=0}^{k+1}(-\lambda)^i\binom{k+1}{i}x_{k-i+1} - \sum_{i=1}^{k+1}(-\lambda)^i\binom{k}{i-1}x_{k-i+1}\right)$$
$$= (x_k, x_{k+1})$$
$$+ \left(\sum_{i=1}^{k}(-\lambda)^i\binom{k}{i}x_{k-i}, \sum_{i=1}^{k}\left\{(-\lambda)^i\binom{k+1}{i}x_{k-i+1} - (-\lambda)^i\binom{k}{i-1}x_{k-i+1}\right\}\right).$$

Since $(x_k, x_{k+1}) \in T$ we are reduced to showing

$$q_k = \sum_{i=1}^{k}\left((-\lambda)^i\binom{k}{i}x_{k-i}, (-\lambda)^i\left[\binom{k+1}{i} - \binom{k}{i-1}\right]x_{k-i+1}\right) \in T .$$

Let us consider each term in the sum

$$\left((-\lambda)^i\binom{k}{i}x_{k-1}, (-\lambda)^i\left[\binom{k+1}{i} - \binom{k}{i-1}\right]x_{k-i+1}\right)$$
$$= (-\lambda)^i\binom{k}{i}\left(x_{k-i}, \binom{k}{i}^{-1}\left[\binom{k+1}{i} - \binom{k}{i-1}\right]x_{k-i+1}\right).$$

Direct calculation shows $\left[\binom{k+1}{i} - \binom{k}{i-1}\right] = \binom{k}{i}$. Hence

$$q_k = \sum_{i=1}^{k}(-\lambda)^i\binom{k}{i}(x_{k-i}, x_{k-i+1}) .$$

Since each $(x_{k-i}, x_{k-i+1}) \in T$, $q_k \in T$. Thus f maps $G_n(T)$ into $G_n(T-\lambda)$. If we let g be the map on $G_n(T-\lambda)$ defined by

$$g(y_0, \cdots, y_n) = (x_0, \cdots, x_n)$$

where $x_k = \sum_{j=0}^{k}\lambda^j\binom{k}{j}y_{k-j}$ for $k = 0, \cdots, n$, the same argument used above shows that g maps $G_n(T-\lambda)$ into $G_n(T)$. If we can show $f \circ g = 1_Y$ and $g \circ f = 1_X$, then we will have proven that f is a bijection of $G_n(T)$ onto $G_n(T-\lambda)$ and $f^{-1} = g$. Let $(x_0, \cdots, x_n) \in G_n(T)$ and $(g \circ f)\ (x_0, \cdots, x_n) = (z_0, \cdots, z_n)$. Then

$$z_k = \sum_{j=0}^{k}\lambda^j\binom{k}{j}\,y_{k-j} = \sum_{j-0}^{k}\sum_{i=0}^{k-j}\lambda^j(-\lambda)^i\binom{k}{j}\binom{k-j}{i}x_{k-j-i} .$$

We may rearrange the sum as follows

$$z_k = \sum_{h=0}^{k}\sum_{j=0}^{h}\lambda^j(-\lambda)^{h-j}\binom{k}{j}\binom{k-j}{h-j}x_{k-h}$$

or

$$z_k = \sum_{h=1}^{k} \left\{ \sum_{j=0}^{h} (-\lambda)^j (-\lambda)^{h-j} \binom{k}{j} \binom{k-j}{n-j} \right\} x_{k-h} + x_k .$$

But

$$\binom{k}{j}\binom{k-j}{h-j} = \binom{k}{h}\binom{h}{j}$$

and

$$\sum_{j=0}^{h} (-1)^j \binom{h}{j} = \sum_{j=0}^{h} (-1)^j (+1)^{h-j} \binom{h}{j} = ((-1)+1)^h = 0 .$$

Thus

$$z_k - x_k = \sum_{h=1}^{k} \lambda^h \left\{ \sum_{j=1}^{h} (-1)^{h-j} \binom{k}{j} \binom{k-j}{h-j} \right\} x_{k-h} = 0 .$$

Similar arguments show that $f \circ g = 1_Y$. Thus f is a bijection and $f^{-1} = g$. That f is linear and a homeomorphism is now obvious from the form of f and f^{-1}. This completes the proof of the lemma.

PROPOSITION 4.31. *If T is a linear relation such that T is single-valued or such that $(T-\lambda)^{-1}$ is singlevalued for some $\lambda \in \boldsymbol{C}$, then T is a unique decomposition relation.*

Proof. We must show that for each polynomial P, the null space of $\psi(P, T, \circ)$ is $\{0\}$. The null space of

$$\psi(P, T, \circ) = \left\{ (0, x_1, \cdots, x_n) \colon \sum_{i=1}^{n} a_i x_i = 0 \right\}$$

where $P(t) = \sum_{i=0}^{n} a_i t^i$. If T is single-valued, it is obvious that $T(0) = x_1 = 0$. This argument may be continued to show that $x_k = 0, k = 0, 1, \cdots, n$.

Next let us suppose that T^{-1} is single-valued. Since

$$\sum_{i=1}^{n} a_i x_i = 0 , \qquad T^{-1}\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=2}^{n} a_i x_{i-1} = 0 .$$

Repeated application of T^{-1} shows $(T^{-1})^{n-1}(\sum_{i=1}^{n} a_i x_i) = a_n x_1 = 0$. Thus $x_1 = 0$. Since $(T^{-1})^{n-2}(\sum_{i=1}^{n} a_i x_i) = a_n x_2 + a_{n-1} x_1 = 0$, $x_2 = 0$. Continuing in this manner we get $x_k = 0, k = 1, \cdots, n$. If $(T-\lambda)^{-1}$ is single-valued, we can use Lemma 4.3. Since

$$\sum_{i=1}^{n} a_i x_i = 0, \sum_{i=1}^{n} \sum_{k=0}^{i} \lambda^i a_i \binom{i}{k} y_{i-k} = \sum_{h=0}^{n} \sum_{j=0}^{n-h} a_{j+h} \lambda^j \binom{j+h}{h} y_h = 0 .$$

Since $(T-\lambda)^{-1}$ is single-valued, we have already established that $y_h = 0, h = 1, \cdots, n$. Because the map f defined in 4.3 is a bijection, $x_k = 0, k = 1, \cdots, n$. Thus T has a unique decomposition.

The above results also show that if T is a unique decomposition relation, then for every polynomial P of degree n, there exists one and only one polynomial Q of degree n such that $P(T) = Q(T-\lambda)$ and then the following diagram is commutative.

(4.32)
$$\begin{array}{ccc} P(T) & \xleftrightarrow{\psi(P,\, T,\, \circ)} & G_n(T) \\ \| & & \updownarrow f \\ Q(T-\lambda) & \xleftrightarrow{\psi(Q,\, T-\lambda,\, \circ)} & G_n(T-\lambda)\,. \end{array}$$

5. Sufficient conditions for polyclosedness. We are now ready to state the main theorem of this paper. The proof will be quite direct since the foundation for it has been laid in previous sections.

THEOREM 5.1. *Let X be complete and T a closed linear relation with a unique decomposition. If $\psi(P, T, \circ)^{-1}$ is continuous, then $P(T)$ is a closed linear relation.*

Proof. Since $G_n(T)$ is complete and $\psi(P, T, \circ)$ is a linear homeomorphism, $P(T)$ is complete and hence closed as a subspace of $X \oplus X$.

COROLLARY 5.11. *Let X be complete and T a closed linear relation with unique decomposition such that for every polynomial P, $\psi(P, T, \circ)^{-1}$ is continuous. Then T is polyclosed.*

PROPOSITION 5.12. *Let T be a closed linear relation (X not necessarily complete) with a unique decomposition such that $(T-\lambda)$ is selective for some $\lambda \in C$. If $\psi(P, \tilde{T}, \circ)^{-1}$ is continuous for every polynomial P, then T and $\tilde{T}$ are polyclosed.*

Proof. Combine 5.11 with 3.23.

PROPOSITION 5.2. *If T is a closed linear relation such that $(T-\lambda)^{-1}$ is a single-valued continuous operator for some $\lambda \in C$ and X is complete, then $\psi(P, T, \circ)^{-1}$ is continuous for every polynomial P.*

Proof. First suppose $\lambda = 0$. Let $(x_0^\gamma, \sum_{i=0}^n a_i x_i^\gamma)$ be a Cauchy net in $P(T)$. Then x_0^γ is a Cauchy net in X as is $\sum_{i=0}^n a_i x_i^\gamma$. Since T^{-1} is continuous, $T^{-1}(a_1 x_1^\gamma + \cdots + a_n x_n^\gamma) = a_1 x_0^\gamma + \cdots + a_n x_{n-1}^\gamma$ is Cauchy. Since x_0^γ is Cauchy, $a_2 x_1^\gamma + \cdots + a_n x_{n-1}^\gamma$ is Cauchy. We repeatedly apply T^{-1} until we arrive at the fact that $a_n x_1^\gamma$ is Cauchy. We may then retrace our steps to find that $(x_0^\gamma, x_1^\gamma, \cdots, x_n^\gamma)$ is a Cauchy net in X^{n+1}. Since $\psi(P, T, °)^{-1}$ is a closed linear relation, and as an operator it takes Cauchy nets into Cauchy nets, it is continuous. To generalize the above proof to the case $\lambda \neq 0$, combine it with the fact that 4.32 is a commutative diagram.

We may now fit together the pieces to produce the proposed generalization of 2.2.

THEOREM 5.3. *Let T be a closed linear relation such that $R(T-\lambda)$ is closed and $(T-\lambda)^{-1}$ is continuous for some $\lambda \in \boldsymbol{C}$. Then $T-\lambda$ is selective and $\psi(P, \tilde{T}, °)$ is a linear homeomorphism for every polynomial P. Consequently T (and $\tilde{T}$) is polyclosed.*

Proof. Combine 3.1, 5.12 and 5.2

6. **Necessary and sufficient conditions for polyclosedness.** For u.d. closed linear relations in a complete space, we have found that a sufficient condition for $\psi(P, T, °)^{-1}$ to be continuous is for $P(T)$ to be closed. We now seek conditions on X such that the continuity of $\psi(P, T, °)^{-1}$ is necessary as well. The basic tool which we shall use is the generalized closed graph theorem. This theorem has undergone many improvements since the first version was proven by Banach [2]. For up to date information, the reader is referred to [3] and [5]. We shall avail ourselves of the following version.

CLOSED GRAPH THEOREM 6.1. [*Ptak, Robertson and Robertson*]. Let X be a tonnelé locally convex space and Y a B_r-complete (resp. B-complete) locally convex space. If f is a one-to-one linear map (resp. linear map) of X into Y such that the graph of f is closed in $X \oplus Y$, then f is continuous.

Proof. See [5, IV, 8.5].

We shall make the following definitions: X is a (PB)-space if and only if X^n for every $n \geqq 1$ is B-complete and, each closed subspace $X \oplus X$ is tonnelé. X is a (PB_r)-space if X^n for every $n \geqq 1$ is B_r-complete and each closed subspace of $X \oplus X$ is tonnelé.

That each of these classes is sufficiently large to be of interest is guaranteed by

PROPOSITION 6.11. *If X is a Frechet space, it is a (PB)-space. If X is a (PB)-space, it is a (PB_r)-space.*

Proof. See [5; IV, 6.4]

We may now state the following propositions that act as partial converses of 5.1 and 5.11.

PROPOSITION 6.2. *Let X be a (PB_r)-space and T a closed linear relation in X with a unique decomposition. For a fixed polynomial P, $P(T)$ is closed if and only if $\psi(P, T, ^\circ)$ is a linear homeomorphism.*

Proof. The sufficiency has already been proven. To show necessity, suppose that $P(T)$ is closed. Since X is a (PB_r)-space, $P(T)$ is tonnelé considered as a subspace of $X \oplus X$. Since $G_n(T)$ is closed, it is B_r-complete. Hence $\psi(P, T, ^\circ)^{-1}$ is a closed map from a tonnelé space into a B_r-complete space. By 6.1 $\psi(P, T, ^\circ)^{-1}$ is continuous. Hence $\psi(P, T, ^\circ)$ is a linear homeomorphism.

COROLLARY 6.21. *Let X be a (PB_r)-space and T a closed linear relation with a unique decomposition. T is polyclosed if and only if $\psi(P, T, ^\circ)$ is a linear homeomorphism for every polynomial P.*

Due to the fact that quotients of B-complete spaces by closed subspaces are again B-complete, in the class of (PB)-spaces we may strengthen 6.2 and 6.21 to closed linear relations that do not have a unique decomposition.

PROPOSITION 6.3. *Let X be a (PB)-space and T a closed linear relation. For a fixed polynomial, P, $P(T)$ is closed if and only if $\psi(P, T, ^\circ)$ is a topological homomorphism.*

Proof. Since $G_n(T)$ is closed, it is B-complete. Since $\psi(P, T, ^\circ)$ is continuous, $N(\psi(P, T, ^\circ))$ is a closed subspace of $G_n(T)$. Hence $G_n(T)/N(\psi(P, T, ^\circ))$ is also B-complete. If $\psi(P, T, ^\circ)$ is a topological homomorphism, then it induces a canonical linear homeomorphism of $G_n(T)/N(\psi(P, T, ^\circ))$ onto $P(T)$. Thus $P(T)$ is B-complete and obviously closed. Conversely, if $P(T)$ is closed, it is tonnelé and the canonical linear bijection induced by $\psi(P, T, ^\circ)$ is a linear homeomorphism and hence $\psi(P, T, ^\circ)$ is a topological homomorphism.

PROPOSITION 6.31. *Let X be a (PB)-space and T a closed linear relation. For T to be polyclosed, it is necessary and sufficient that $\psi(P, T, ^\circ)$ be a topological homomorphism for every polynomial P.*

Thus in (PB)-spaces, 6.31 characterizes the polyclosed linear relations as those whose realization maps are all topological homomorphisms.

7. Completing the generalized graph. Let us return to the case where X is not necessarily complete. We will say that a closed linear relation has a *uniformly completable* generalized graph if and only if $G_n(\tilde{T}) = \tilde{G}_n(T)$ for every $n \geqq 0$. An example of such a linear relation is afforded by Example 3.11. This can be easily seen by realizing that if $(x_0, x_1, \cdots, x_n) \in G_n(T)$ then $x_n \in C^n([0, 1])$ and $x_k = x_n{}^{(n-k)}$. The Weierstrass approximation theorem guarantees that there exists a sequence of polynomials p_i such that $p_i^{(k)} \longrightarrow x_n^{(k)}$ for $k = 0, 1, \cdots, n$. Since $(p_i^{(n)}, p_i^{(n-1)}, \cdots, p_i) \in G_n(T)$, we conclude that

$$(x_0, x_1, \cdots, x_n) \in \tilde{G}_n(T) .$$

The usefulness of this property can be seen in the light of the following simple proposition.

PROPOSITION 7.1. *For a fixed polynomial P, and a fixed closed linear relation T, $\tilde{\psi}(p, T, {}^\circ) = \psi(P, \tilde{T}, {}^\circ) \mid \tilde{G}_n(T)$.*

Proof. Both of these mappings are continuous linear extensions of $\psi(P, T, {}^\circ)$ to $\tilde{G}_n(T)$. Thus they must be equal.

COROLLARY 7.11. *Let T be a closed linear relation with a uniformly completable generalized graph, and P a fixed polynomial. Then $\tilde{\psi}(P, T, {}^\circ) = \psi(P, \tilde{T}, {}^\circ)$. Thus if $\psi(P, T, {}^\circ)$ is a linear homeomorphism, then $\psi(P, \tilde{T}, {}^\circ)$ is also.*

Proof. If $\psi(P, T, {}^\circ)$ is a linear homeomorphism, then $\tilde{\psi}(P, T, {}^\circ)$ is also. Hence the same is true for $\psi(P, \tilde{T}, {}^\circ)$.

COROLLARY 7.12. *Let T be a selective closed linear relation with a uniformly completable generalized graph. Then if $\psi(P, T, {}^\circ)$ is a linear homeomorphism for every polynomial P, T (and $\tilde{T}$) is polyclosed.*

Proof. By 7.11, $\psi(P, \tilde{T}, {}^\circ)$ is a linear homeomorphism. Hence $\tilde{T}$ is polyclosed. Since T is selective, by 3.23, T is polyclosed.

This result reflects the basic fact found in all our results, namely that to prove that T is polyclosed, we have had to prove that $\tilde{T}$ is polyclosed as well. Unfortunately the conjecture that this is always the case is false, even under the restricted hypotheses of selectivity and uniformly completable generalized graph. This can be seen by considering the example following 3.23. Since $\tilde{T} = T$, T is selective and has a uniformly completable generalized graph. However, we have already seen that T is polyclosed (as a subspace of $X \oplus X$) whereas $\tilde{T}$ is not. Note that $\psi(P, T, ^\circ)$ is not a linear homeomorphism in this case even for $P(t) = t^2$.

The result derived in 7.1 suggests that it might be profitable to consider $\psi(P, T, ^\circ)$ as a restriction. An instance of this is provided by the following proposition.

PROPOSITION 7.2. *Let X be a complete space and S a polyclosed linear relation such that $\psi(P, S, ^\circ)$ is a linear homeomorphism for every polynomial P. Then for every closed linear relation T such that $T \subset S$, T is polyclosed.*

Proof. $\psi(P, T, ^\circ)^{-1} = \psi(P, S, ^\circ)^{-1} \mid P(T)$ and thus is continuous.

COROLLARY 7.21. *Let X be a (PB_r)-space and S a polyclosed linear relation with a unique decomposition. Then for every closed linear relation T such that $T \subset S$, T is polyclosed.*

Proof. If S is polyclosed, then $\psi(P, S, ^\circ)$ is a linear homeomorphism for every polynomial P. An application of 7.2 completes the proof.

We have seen that for the case where T^{-1} is single-valued, it is not necessary for T^{-1} to be continuous for $\psi(P, T, ^\circ)$ to be a linear homeomorphism. For our final result we present a specialization under restricted hypotheses where a continuity condition is necessary and sufficient for $\psi(P, T, ^\circ)$ to be a linear homeomorphism.

PROPOSITION 7.3. *Let T be a closed linear relation such that for some $\lambda \in \boldsymbol{C}$, $(T - \lambda)^{-1}$ is single-valued, and such that there exists a continuous linear map $F\colon D(T - \lambda) \to X$ such that $F \subset (T - \lambda)$. Then $\psi(P, T, ^\circ)$ is a linear homeomorphism for every polynomial P, if and only if for every $n \geqq 1$, $(T - \lambda)^{-1}$ is continuous on $N((T - \lambda)^{-n})$.*

Proof. Let us consider the case $\lambda = 0$. First suppose T^{-1} is continuous on $N(T^{-n})$. Let $P(t) = \sum_{k=0}^{n} a_k t^k$. Let $(x_0^\gamma \sum_{k=0}^{n} a_k x_k^\gamma) \to (x_0, y)$. We may consider $x_0 = y = 0$ without loss of generality. Then $x_0^\gamma \to 0$ and $\sum_{k=1}^{n} a_k x_k^\gamma \to 0$. Let $m_1^\gamma = F(x_0^\gamma) - x_1^\gamma$. Then $m_1^\gamma \in N(T^{-1})$ since $(x_0^\gamma, F(x_0^\gamma)) \in T$. Next let $\hat{m}_2^\gamma = F(x_1^\gamma) - x_2^\gamma$. Again $\hat{m}_2^\gamma \in N(T^{-1})$. Thus $F^2(x_0^\gamma) = x_2^\gamma + \hat{m}_2^\gamma + F(m_1^\gamma)$. Let $m_2^\gamma = \hat{m}_2^\gamma + F(m_1^\gamma)$. Then $m_2^\gamma \in N(T^{-2})$ since $(0, m_1^\gamma) \in T$ and $(m_1^\gamma, m_2^\gamma) \in T$. Also $F^{-1}(m_2^\gamma) = m_1^\gamma + F^{-1}(m_2^\gamma) = m_1^\gamma$. Continuing in this manner we get $F^k(x_0^\gamma) = x_k^\gamma + m_k^\gamma$ where $m_k^\gamma \in N(T^{-k})$

and $F^{-1}(m_k^\gamma) = m_{k-1}^\gamma$ for $k = 1, \cdots, n$. Thus

$$\sum_{k=1}^{n} a_k x_k^\gamma = \sum_{k=1}^{n} a_k F^k(x_0^\gamma) + \sum_{k=1}^{n} a_k m_k^\gamma \longrightarrow 0 .$$

Thus $\sum_{k=1}^{n} a_k m_k^\gamma \longrightarrow \sum_{k=1}^{n} - a_k F^k(x_0^\gamma) = 0$. Since

$$\sum_{k=1}^{n} a_k m_k^\gamma \in N(T^{-n}) , \qquad T^{-1}\Big(\sum_{k=1}^{n} a_k m_k^\gamma\Big) = \sum_{k=1}^{n} a_{k+1} m_k^\gamma$$

is convergent. Repeated application of T^{-1} implies that $a_n m_1^\gamma$ is convergent and hence m_1^γ is convergent. Proceeding back up the line, we get m_k^γ convergent for $k = 1, \cdots, n$. Thus $x_k^\gamma = F^k(x_0^\gamma) - m_k$ is convergent. Thus $(x_0^\gamma, \cdots, x_n^\gamma)$ is convergent. Since $\psi(P, T, ^\circ)^{-1}$ is closed, it is continuous.

Conversely if $\psi(P, T, ^\circ)$ is a linear homemorphism for every polynomial P, let m_n^γ be a net in $N(T^{-n})$ such that $m_n^\gamma \longrightarrow 0$. Let $P(t) = t^n$. Then $(0, m_1^\gamma, \cdots, m_{n-1}^\gamma, m_n^\gamma) \longrightarrow (0, m_1, \cdots, m_{n-1}, 0)$. Since T^{-1} is single-valued, $m_k = 0$ for $k = 1, \cdots, n - 1$. Hence $T^{-1}(m_n^\gamma) = m_{n-1}^\gamma \longrightarrow 0$ and T^{-1} is continuous on $N(T^{-n})$.

To prove the proposition for $\lambda \neq 0$, use the commutativity of diagram 4.32.

COROLLARY 7.31. *Let the hypotheses of* 7.3 *be satisfied with* T *being selective and having a uniformly completable generalized graph. Then if* $(T - 1)^{-1}$ *is continuous on* $N((T - \lambda)^{-n})$ *for* $n \geqq 1$, T *as well as* $\tilde{T}$ *is polyclosed.*

Proof. Apply 7.12.

COROLLARY 7.32. *Let the hypotheses of* 7.3 *be satisfied with* X *being a* (PB_r)*-space. Then* T *is polyclosed if and only if* $(T - \lambda)^{-1}$ *is continuous on* $N((T - \lambda)^{-n})$ *for* $n \geqq 1$.

Proof. Apply 6.21.

We will conclude by proving that T as defined in Example 3.11 is polyclosed. This is done by application of 7.31 for $\lambda = 0$. The only facts we must verify are that there exists the proper continuous F and that T^{-1} is continuous on $N(T^{-n})$ for $n \geqq 1$. Let $F(x) = \int_0^t x(s)ds$. Certainly F is continuous, is defined on $C^\infty([0, 1])$ and $\left(x, \int_0^t x(s)ds\right) \in T$ for every $x \in C^\infty([0, 1])$. $N(T^{-n})$ consists of polynomials of degree $<n$. But it is easily seen that T^{-1}, namely differentiation, is continuous on such a space. Let P_k be a sequence of polynomials of degree $<n$,

such that $P_k \to 0$. Then the coefficients of each t^i, $i = 0, \cdots, n-1$ converge to 0. Thus $P'_k \to 0$ also.

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STEVENS INSTITUTE OF TECHNOLOGY
HOBOKEN, NEW JERSEY