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G-SPACES, H-SPACES AND W-SPACES

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The notions of G-space, W-space, H-space, and higher order Whitehead product are differentiated through example.

In [3], [4] and [5] D. H. Gottlieb introduces certain subgroups, $G_n(X, x_0)$, of the homotopy groups of a space. These groups are related to the problem of sectioning fibrations with fibre X. Related to the groups $G_n(X, x_0)$ is the notion of a G-space. A G-space is a space with $G_n(X, x_0) = \pi_n(X, x_0)$ for all n. It is a simple matter to show that every H-space is a G-space (see below). However, till recently the status of the converse remained undecided. Recently, Gottlieb produced an example of a two-stage Postnikov system that is a G-space but not an H-space (unpublished). The purpose of this note is to clarify the situation further. We produce a 3-dimensional manifold that is a G-space but not an H-space. Incidently, the theory of G-spaces tells us that our example is also a W-space, that is, a space whose Whitehead products all vanish.

Finally we would like to resolve a question of G. Porter [6]. Namely, our example is also an example of a space whose higher order Whitehead products all vanish but, again, is not an H-space.

We would like to acknowledge the priority of D. H. Gottlieb's example mentioned above and thank him for his help in the preparation of this paper.

1. Preliminaries.

In this section we review the elementary theory of G-spaces presented in [4] and [5].

NOTATION 1.1. We assume all our spaces X are path connected C. W. complexes with base point x_0. We let X^X be the space of maps X to X. We let $M(X)$ be the component of the identity map $1: X \to X$ in X^X. Consider the evaluation map $e: M(X) \to X$ given by $e(f) = f(x_0)$. This map gives a fibration with fibre $M(X)_0$, the space of maps in $M(X)$ with $f(x_0) = x_0$.

DEFINITION 1.2. We define

$$G_n(X, x_0) = e_*(\pi_n(M(X), 1)) \subseteq \pi_n(X, x_0) .$$

THEOREM 1.3. *The groups $G_n(X, x_0)$ are invariant with respect to base point and homotopy type but not natural with respect to maps.*

Proof. [5].

THEOREM. 1.4.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \cdot F: X \times S^n \longrightarrow X \text{ with } F/X \vee S^n = 1 \vee f\} .$$

Proof. [5].

THEOREM 1.5.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \text{ a fibration } X \subseteqq E \xrightarrow{p} S^{n+1} \text{ with } [f] = \partial_*[1]\} ,$$

where $1: S^{n+1} \rightarrow S^{n+1}$ *is the identity map.*

Proof. [4].

DEFINITION 1.6. $P_n(X, x_0)$ is the subgroup of elements $[f]$ in $\pi_n(X, x_0)$ with $[[f], [g]] = 0$ (Whitehead product) for all m and all $[g] \in \pi_m(X, x_0)$.

THEOREM 1.7. $G_n(X, x_0) \subseteqq P_n(X, x_0)$.

Proof. [5] (see 1.4 above).

REMARK. Ganea [1] has shown that in general $G_n(X, x_0) \neq P_n(X, x_0)$. (see 3.4 below).

DEFINITION 1.8. (a) A *G-space* is a space X with $G_n(X, x_0) = \pi_n(X, x_0)$, all n.

(b) A *W-space* is a space X with $P_n(X, x_0) = \pi_n(X, x_0)$ for all n.

THEOREM 1.9. (a) *Every H-space is a G-space.*
(b) *Every G-space is a W-space.*

Proof. [5]. (a) Follows from 1.4.
(b) Follows from 1.7.

2. A G-space that is not an H-space. As mentioned in 1.3 the groups $G_n(X, x_0)$ are not natural with respect to maps. However, we can prove the following.

LEMMA 2.1. *Suppose we are given a map* $F: Y \times X \rightarrow Y$ *with* $F/Y \vee X = 1 \vee f$ *then* $f_*: \pi_n(X, x_0) \rightarrow G_n(Y, y_0)$.

Proof. For $g: S^n \rightarrow X$ consider the composition

$$Y \times S^n \xrightarrow{1 \times g} Y \times X \xrightarrow{F} Y .$$

Now apply 1.4.

EXAMPLE 2.2. Let H be a closed subgroup of a Lie group G. Let $(H\backslash G)$ be the left coset space. Let $p: G \to (H\backslash G)$ be the projection. We have the usual pairing $(H/G) \times G \xrightarrow{F} (H/G)$ with $F/(H/G) \vee G = 1 \vee p$.

THEOREM 2.3. *In the situation of 2.2 assume $i_*: \pi_n(H, e) \to \pi_n(G, e)$ is an inclusion for all n, then $(H\backslash G)$ is a G-space hence a W-space.*

Proof. Since i_* is an inclusion $p_*: \pi_n(G, e) \to \pi_n(H\backslash G, [e])$ is an epimorphism. On the other hand, by 2.1 $p_*\pi_n(G, e) \subseteqq G_n(H\backslash G, [e])$ hence $G_n(H\backslash G, [e]) = \pi_n(H\backslash G, [e])$ or $H\backslash G$ is a G-space.

We are now prepared to produce our example. We represent S^1 by the complex numbers $e^{i\theta}$ $0 \leqq \theta \leqq 2\pi$. In $SO(3)$ we let the symbol (θ) denote the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

2.4. *Example of a G-space that is not an H-space.* Embed $S^1 \subseteqq SO(3) \times S^1$ as a subgroup by the following map $i(e^{i\theta}) = (2\theta) \times e^{i3\theta}$. We let

$$T \overset{\text{def}}{\equiv} i(S^1)\backslash SO(3) \times S^1 .$$

LEMMA 2.5. *T is a G-space, hence a W-space.*

Proof. By 2.3 we need only check

$$\begin{array}{ccc} i_*: \pi_1(S^1) & \longrightarrow & \pi_1(SO(3) \times S^1) \\ \| & & \| \\ Z & & Z_2 \oplus Z \end{array}$$

is an inclusion, but it is easy to check $i_*(1) = 0 \oplus 3$. Note this implies $\pi_1(T) = Z_2 \oplus Z_3$.

LEMMA 2.6. *T is not an H-space.*

Proof. (a) T is a 3-dimensional manifold hence $H^n(T, Z_3) = 0$,

$n > 3$.

(b) $H'(T, Z_3) = Z_3$, generated say by α. This is by remarks at the end of 2.5.

(c) From the universal coefficient theorem we know there is $\beta \neq 0$ in $H^2(T, Z_3)$ β indecomposable ($\alpha^2 = 0$).

(a), (b) and (c) implies that $H^*(T, Z_3)$ does not support a Hopf algebra structure, hence, T is not an H space. In particular if $T \times T \xrightarrow{h} T$ is a Hopf map.

$$0 = h^*(\beta^2) = (1 \otimes \beta + \beta \otimes 1 + r(\alpha \otimes \alpha))^2 = 2\beta \otimes \beta + \cdots \neq 0 .$$

We could also note that $T = Z_3 \backslash SO(3)$ where Z_3 is the group (0), ($2/3\pi$), ($4/3\pi$). Then, using the spectral sequence of a covering we have

$$H^n(T, Z_3) = \begin{cases} Z_3 & n = 0, 1, 2, 3 \\ 0 & n > 3 . \end{cases}$$

This does not support a Hopf algebra structure.

3. Higher order Whitehead products.

The purpose of this section is to point out that our example also answers a question of G. Porter [6].

DEFINITION 3.1. A space X is said to have *trivial higher order Whitehead products.* If given any set of homotopy elements

$$[f_i] \in \pi_{p_i}(X, x_0) \ 1 \leqq i \leqq n .$$

The map $\bigvee_{i=1}^{n} f_i \colon \bigvee S^{p_i} \to X$ extends to some $f \colon \bigtimes_{i=1}^{n} S^{p_i} \to X$. (see [6]).

THEOREM 3.2. *Any G-space has trivial spherical Whitehead products.*

Proof.

LEMMA. *Given any* $n-1$ *elements* $[f_i] \in \pi_{p_i}(X, x_0)$ $1 \leqq i \leqq n-1$ *we can find a map*

$$h \colon \left(\bigtimes_{i=1}^{n-1} S^{p_i}\right) \times X \longrightarrow X \qquad \textit{with}$$

$$h \Big/ \left(\bigvee_{i=1}^{n-1} S^{p_i}\right) \vee X = \left(\bigvee_{i=1}^{n-1} f_i\right) \vee 1 .$$

This is proved by induction. For $n = 2$ this is 1.4. Suppose we have a map

$$\bar{h}\colon \Big(\mathop{\times}_{i=1}^{n-2} S^{p_i}\Big) \times X \longrightarrow X$$

with the required property.

Consider $\tilde{h}\colon S^{p_{n-1}} \times X \to X$ an extension of $f_{n-1} \vee 1$ (1.4). Finally, consider the composition

$$S^{p_{n-1}} \times \Big(\Big(\mathop{\times}_{i=1}^{n-2} S^{p_i}\Big) \times X\Big) \xrightarrow{1 \times \bar{h}} S^{p_{n-1}} \times X \xrightarrow{\tilde{h}} X\,.$$

Set $h = \tilde{h}(1 \times \bar{h})$.

We now finish the proof by noting that the composition

$$\Big(\mathop{\times}_{i=1}^{n-1} S^{p_i}\Big) \times S^{p_n} \xrightarrow{1 \times f_n} \Big(\mathop{\times}_{i=1}^{n-1} S^{p_i}\Big) \times X \xrightarrow{h} X$$

is the required extension of $\bigvee_{i=1}^{n} f_i$.

THEOREM 3.3. *There exists finite dimensional spaces with trivial higher order Whitehead products that are not H-spaces.*

Proof. The space T of 2.4 is such an example.

FINAL REMARKS 3.4. Ganea [2] has constructed an infinite dimensional example of a W-space that is not a G-space. G. Lang (unpublished) points out that using recent results of Gottlieb [5] one can show that $CP(3)$ is a finite dimensional example of such a space. In [1] it is shown that $CP(3)$ is a W-space, but in [5] it is shown that every finite dimensional G-space has Euler-Poinare characteristic 0 hence $CP(3)$ is not a G-space.

Porter [7] shows that $CP(3)$ has nontrivial higher order Whitehead products. It would be interesting to have examples of spaces with vanishing higher order Whitehead products that are not G-spaces.

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