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STABILITY THEOREMS FOR LIE ALGEBRAS OF DERIVATIONS

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Let A be a finite dimensional algebra over a field F of characteristic zero and let L be a completely reducible Lie algebra of derivations of A. If A is associative, then there exists an L-invariant Wedderburn factor of A. If A is a Lie algebra, there exists an L-invariant Levi factor of A. If A is a solvable Lie algebra, there exists an L-invariant Cartan subalgebra of A. This paper deals with the uniqueness of such L-invariant subalgebras. For the associative case the assumption of characteristic zero can be dropped if we assume that the radical of A is L-invariant.

2. Preliminaries. If A is a finite dimensional associative algebra over a field F with radical R such that A/R is separable (that is, semisimple and remains so under every field extension of F), then the Wedderburn principal theorem states that there exists a separable subalgebra S such that $A = S + R$, $S \cap R = \{0\}$. S is called a Wedderburn factor of A. Since R is nilpotent, for r in R, $(1 - r)^{-1} = 1 + r + \cdots + r^{n-1}$, where $r^n = 0$. Let C_{1-r} be the inner automorphism of A defined by conjugation by the invertible element $1 - r$. The Malcev Theorem states that if S is any separable subalgebra of A and T is a Wedderburn factor of A, then there exists r in R such that $C_{1-r}(S) \subseteqq T$. Thus, the Wedderburn factors of A are just the maximal separable subalgebras. See [4] for the above information. In § 3 it is shown that if L is completely reducible (every L-invariant subspace of A has a complementary L-invariant subspace), F arbitrary, R L-invariant, and S, T two L-invariant Wedderburn factors of A, then there exists an element r in R such that $C_{1-r}(S) = T$ and $D(r) = 0$ for all D in L. Such an element r is called an L-constant.

If A is a Lie algebra over a field F of characteristic zero and R is the radical (maximal solvable ideal) of A, then the Levi theorem states that $A = S + R$, $S \cap R = \{0\}$, where S is a semisimple subalgebra of A isomorphic to A/R. S is called a Levi factor of A. The Malcev-Hanish-Chandra theorem states that any two Levi factors of A are conjugate by an automorphism $\exp(Adx)$, where x is in N, the nil radical (maximal nilpotent ideal) of A. In § 4 it is shown that for L completely reducible and S, T L-invariant Levi factors of A, then there is an L-constant x in N such that $\exp(Adx)(S) = T$.

If A is a solvable Lie algebra over a field F of characteristic zero, then any two Cartan subalgebras are conjugate by an automorphism of the form $\exp(Adx)$, for $x \in A^\infty = \bigcap_{n=1}^{\infty} A^n$, see [2]. In § 5, we show that for L completely reducible and S, T L-invariant Cartan subalgebras of A, then there is a L-constant x in A^∞ such that $\exp(Adx)(S) = T$.

In [8] Mostow considered the situation where G, a completely reducible group of algebra automorphisms, acts on a finite dimensional algebra A over a field F of characteristic zero. For each of the three cases for A mentioned above, Mostow shows that there exists the corresponding kind of G-invariant subalgebra. One can use an algebraic group argument, see [1], to conclude the corresponding existence of L-invariant subalgebras. The problem of relating G-invariant subalgebras has been studied by Taft [9], and uniqueness in that case is given via automorphisms defined by fixed points of G. The uniqueness results for L-invariant subalgebras (in terms of L-constants) can be shown directly, and also, for characteristic zero, can be shown to follow from the results of Taft. It should be noted that if x is an L-constant (G-fixed) then C_{1-x} centralizes L (or G) so that if S is an L (or G) invariant subalgebra, so is $C_{1-x}(S)$.

Let F have characteristic zero. The relationship between the situations of L acting on A and that of G acting on A is given by the correspondence between a linear algebraic group and its associated Lie algebra, see Chevalley [3]. In particular, if G is an algebraic group of algebra automorphisms of A, then its associated Lie algebra will consist of derivations of A. Also, complete reducibility is preserved in the algebraic group-Lie algebra correspondence. The following lemma follows easily from the definition of the Lie algebra of an algebraic group. We state it for reference.

LEMMA 2.1. *Let V be a finite dimensional vector space over a field F. Let G be an algebraic group of automorphisms of V and g its associated Lie algebra. If x in V is a fixed point of G, then $X(x) = 0$ for all X in g.*

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3. The associative algebra case.

THEOREM 3.1. *Let A be a finite dimensional associative algebra over a field F of characteristic zero and let L be a completely reducible Lie algebra of derivations of A. If S is an L-invariant semisimple subalgebra of A and T an L-invariant maximal semisimple subalgebra of A, then there exists an L-constant r in R, the ra-*

dical of A, such that C_{1-r} carries S into T.

Proof. Given L, let $\bar{L}$ be its algebraic hull, i.e., the smallest algebraic Lie algebra containing L, and let G be the unique connected algebraic group of algebra automorphisms with Lie algebra $\bar{L}$. Then G is also completely reducible. We can apply Theorem 2 of Taft [9] to get r in R such that $C_{1-r}(S) \subseteqq T$ and r is a fixed point of G. By Lemma 2.1 we have that $X(r) = 0$ for all X in $\bar{L}$, and $L \subseteqq \bar{L}$ implies that r is an L-constant.

COROLLARY 1. *Let A and L be as in Theorem 3.1. Then any two L-invariant Wedderburn factors of A are conjugate under an inner automorphism of the form C_{1-r}, where r is an L-constant in R. Also, we may write C_{1-r} in the form* $\exp(Ady)$, *where y is an L-constant in R.*

Proof. The first statement follows immediately from Theorem 3.1. Let $y = \log(1 - r) = -r - r^2/2 - r^3/3 - \cdots$. Then $X(y) = 0$ for all $x \in L$ and $C_{1-r} = C_{\exp(\log(1-r))} = \exp(Ad(\log(1 - r))) = \exp(Ady)$.

COROLLARY 2. *Let A and L be as in Theorem 3.1. Then any L-invariant semisimple subalgebra S of A is contained in an L-invariant Wedderburn factor.*

Proof. Let T be any L-invariant Wedderburn factor. By Theorem 3.1 there exists an L-constant r in R such that $C_{1-r}(S) \subseteqq T$. Thus, $S \subseteqq (C_{1-r})^{-1}(T) = C_{1-y}(T)$, where $y = -r - r^2 - r^3 - \cdots$. Thus y is an L-constant in R. If $t \in T$, then $C_{1-y}(t) = (1 + y + \cdots + y^n)t(1 - y)$, where $y^{n+1} = 0$. For D in L, $DC_{1-y}(t) = C_{1-y}(D(t))$ since y is an L-constant. Thus, $C_{1-y}(T)$ is L-invariant.

If we drop the assumption of characteristic zero in Theorem 3.1, then the uniqueness result can be proven directly with the additional hypothesis that R be L-invariant. (This is always true for characteristic zero.) The technique used in Theorem 3.1 whereby the situation involving derivations of A is carried over to the situation involving algebra automorphisms of A does not, in general, carry over to the case when F has characteristic $p \neq 0$. It is possible to have an algebraic Lie algebra of derivations of a finite dimensional associative algebra A over a field F of characteristic $p > 0$ which is not the Lie algebra of an algebraic group of algebra automorphisms of A. This cannot occur in characteristic zero. For example, let G be a cyclic group of order p and F an algebraically closed field of characteristic p. Let $A = F(G)$, the group algebra of G over F. Then $\{1, g, \cdots, g^{p-1}\}$ is a basis for A over F and $\{g - 1, \cdots, g^{p-1} - 1\}$ is a basis for the radical R of A. Define a map D of A by $D: g \to 1$ and extend D to a derivation of A. The smallest restricted Lie algebra L of linear transformations of A containing D is algebraic, see [5]. Since the Lie algebra of all derivations of A is restricted, L consists of derivations of A. If G is any algebraic group of automorphisms of A with Lie algebra L, then G cannot consist of algebra automorphisms of A. If so, then R would be G-invariant, and, hence, L-invariant, which is not the case.

THEOREM 3.2. *Let A be a finite dimensional associative algebra over a field F of arbitrary characteristic. Let R be the radical of A and assume A/R is separable. Let L be a completely reducible Lie algebra of derivations of A and assume R is L-invariant. If S is an L-invariant separable subalgebra of A and T is an L-invariant Wedderburn factor of A, then there exists an L-constant x in R such that C_{1-x} carries S into T.*

Proof. We consider two cases:

Case 1. $R^2 = \{0\}$. Let z in R be such that $C_{1-z}(S) \subseteqq T$. z exists by the Malcev theorem. We claim that $D(z) \in R \cap C$, for all $D \in L$, where C is the centralizer of S in A. Given $D \in L$, define $AdD(z)$, a linear map of A, by $AdD(z): a \to D(z)a - aD(z)$, for $a \in A$. Using the facts that $R^2 = \{0\}$ and R is L-invariant, we have that

$$AdD(z) = DC_{1-z} - C_{1-z}D .$$

For $s \in S$, $AdD(z)(s) = DC_{1-z}(s) - C_{1-z}D(s) \in T$ since S and T are L-invariant and $C_{1-z}(S) \subseteqq T$. By assumption, $D(z) \in R$, so $AdD(z)(S) \in R$. Hence, $AdD(z): S \to T \cap R = \{0\}$. Thus, $D(z) \in R \cap C$. $R \cap C$ is an L-invariant subspace of R, so by complete reducibility we have $R = (R \cap C) \oplus U$, where U is an L-invariant subspace of R. Write $z = y + x$, where $y \in R \cap C$ and $x \in U$. Thus $x = z - y$ and for $D \in L$, $D(x) = D(z) - D(y) \in (R \cap C) \cap U = \{0\}$. Hence, x is an L-constant, and $x = z - y$ where $y \in C$ implies that $C_{1-x}(S) = C_{1-z}(S) \subseteqq T$.

If $R^2 \neq \{0\}$, we proceed by induction on the dimension of A. Since R is L-invariant, we have that L is a completely reducible Lie algebra of derivations of R, $T + R^2$, and A/R^2, all of which have dimension less than that of A. Let $a \to \bar{a} = a + R^2$ denote the natural homomorphism of A onto $\bar{A} = A/R^2$. Then $\bar{A}$ has radical $\bar{R}$ and $\bar{S}$ is an L-invariant separable subalgebra of $\bar{A}$ while $\bar{T}$ is an L-invariant Wedderburn factor of $\bar{A}$. By induction, there exists $\bar{v} \in \bar{R}$ such that $C_{\underset{1-v}{\longrightarrow}}(\bar{S}) \subseteqq \bar{T}$ and $D(v) \in R^2$ for all D in L. R^2 is an I-invariant subspace of R, so by complete reducibility, we have $R = R^2 \oplus U$, where U is L-invariant. Let $v = z + u$, $z \in R^2$, $u \in U$. Then u is an L-constant and $\bar{u} = \bar{v}$. Consider the algebra $T + R^2$. It has dimension less than that of A, has radical R^2, $C_{1-u}(S)$ is an L-invariant separable subalgebra of it (since u is an L-constant and S is L-invariant) and T is an L-invariant Wedderburn factor of $T+R^2$. By induction, there exists r in R^2 such that $D(r)=0$ for all $D \in L$ and $C_{1-r}C_{1-u}(S) \subseteqq T$. Let $x = u+r-ur$. Then for $D \in L$, $D(x) = D(u) + D(r) - D(u)r - uD(r) = 0$. So x is an L-constant and $C_{1-x}(S) = C_{1-r}C_{1-u}(S) \subseteqq T$.

COROLLARY. *Let A and L be as in Theorem 3.2. Then every L-invariant separable subalgebra of A is contained in an L-invariant Wedderburn factor of A.*

The assumption that R be L-invariant is needed in the above theorem. An example can be given of a semisimple derivation D of an associative algebra A over a field of characteristic 3 such that D leaves invariant more than one Wedderburn factor of A and $D(r)=0$ for $r \in R$, the radical of A, implies that $r=0$. Let F be any field of characteristic 3 containing roots of the polynomial x^3+x+1. Let G be a cyclic group of order 3, $G = \langle g \rangle$, $g^3 = 1$, and form the group algebra $F(G)$ of G over F. Let Q be the quaternion algebra over F, i.e., Q has basis $\{1, i, j, k\}$ over F and $i^2 = j^2 = k^2 = -1$, and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Let $A = F(G) \otimes_F Q$. Then A is an associative algebra over F of dimension 12. A can also be thought of as the algebra of 2×2 – matrices with entries from $F(G)$. If we write for example, gi for the element $g \otimes i$ of A, then A has basis $\{1, g1, g^21, i, gi, g^2i, j, gj, g^2j, k, gk, g^2k\}$. $\{1, i, j, k\}$ forms a basis for a Wedderburn factor W of A and $\{g1-1, g^21-1, gi-i, g^2i-i, gj-j, g^2j-j, gk-k, g^2k-k\}$ forms a basis for the radical R of A. Then $R^3 = \{0\}$. Let $r \in R$ where $r = \alpha(g1-1) + \beta(g^21-1) + \gamma(g^2k-k)$ and $\beta\gamma - \alpha\gamma = \gamma - 1$, $\alpha, \beta, \gamma \in F$. Consider the Wedderburn factors of A obtained by applying C_{1-r} to W. We get the following bases for the resulting Wedderburn factors:

$$\begin{aligned}\{1, (1+\gamma^2)i + \gamma^2 gi + \gamma^2 g^2 i + j + (1-\gamma)gj \\ + (1+\gamma)g^2 j, -i + (\gamma-1)gi + (-\gamma-1)g^2 i \\ + (1+\gamma^2)j + \gamma^2 gj + \gamma^2 g^2 j, k\} = \{1, b_1, b_2, k\} .\end{aligned}$$

The polynomial X^3+X+1 has three distinct roots in F and for each distinct root γ we define a distinct Wedderburn factor of A by the above. Define a map D of A as follows:

$$\begin{aligned}&D(1) = 0, D(g1) = g1, D(g^21) = -g^21, D(i) = gj , \\ &D(gi) = gi + g^2j, D(g^2i) = -g^2i + j, D(j) = -gi , \\ &D(gj) = -g^2i + gj, D(g^2j) = -i - g^2j, D(k) = 0 , \\ &D(gk) = gk, D(g^2k) = -g^2k\end{aligned}$$

and extend linearly to all of A. Then D defines a derivation of A, and it is easy to check that for $r \in R$, $D(r) = 0$ implies that $r = 0$. Also, R is not D-invariant since $D(g1 - 1) = g1$ and $(g1)^3 = 1 \notin R$. Also D is semisimple. Consider the Wedderburn factors with bases $\{1, b_1, b_2, k\}$ obtained before, where $\gamma^3 + \gamma + 1 = 0$. Then a direct check shows that $D(b_1) = (\gamma + 1)b_2$ ahd $D(b_2) = -(\gamma + 1)b_1$. So all three Wedderburn factors of A are D-invariant, and they cannot be conjugate by a D-constant in R since the only such constant is 0.

4. The Lie algebra case.

THEOREM 4.1. *Let A be a finite dimensional Lie algebra over a field of characteristic zero and N its nil radical. Let L be a completely reducible Lie algebra of derivations of A. If S is an L-invariant semisimple subalgebra of A and T is an L-invariant Levi factor of A, then there exists an L-constant x in N such that* $\exp(Adx)$ *carries S into T.*

Proof. The proof is similar to that of Theorem 3.2, and the theorem also follows by using Lemma 2.1 and Theorem 4 of [9], where uniqueness is given in this situation in terms of fixed points of a group of automorphisms of A.

5. Solvable Lie algebras.

THEOREM 5.1. *Let A be a finite dimensional solvable Lie algebra over a field of characteristic zero. Let L be a completely reducible Lie algebra of derivations of A. If S and T are L-invariant Cartan subalgebras of A, then there exists x in A^∞ such that x is an L-constant, and* $\exp(Adx)(S) = T$.

Proof. An analogous proof to the theorem for groups in [9] can be given. Also the result follows by Lemma 2.1 and Theorem 6 of [9].

If F has characteristic $p \neq 0$, there are examples of solvable Lie algebras with Cartan subalgebras of different dimensions. For arbitrary characteristic Winter [10] has shown that if G is a completely reducible group of automorphisms of a solvable Lie algebra A and G has no nonzero fixed points, then A has at most one G-invariant Cartan subalgebra. If L is a completely reducible Lie algebra of derivations of a solvable Lie algebra A over a field of arbitrary characteristic, then one can adapt Winter's proof to show that if A has no nonzero L-constants, then A has at most one L-invariant Cartan subalgebra.

6. A counter-example. Let A be a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero and let s be a semisimple automorphism of A. Jacobson [6] shows that there exists an s-invariant Cartan subalgebra in this situation. The question arises as to whether or not a uniqueness result holds in the sense dealt with previously, i.e., given two s-invariant Cartan subalgebras of A, are they conjugate by an automorphism t of A such that t commutes with s? An example will be given to show that uniqueness in this sense need not hold. Let A and s be as above. Recall that s is an invariant automorphism if it is a product

$$\exp(Adx_1)\cdots\exp(Adx_m) ,$$

where each Adx_i is a nilpotent derivation of A. By a result in Borel-Mostow [2] there exists a Cartan subalgebra H of A which is pointwise fixed by s when s is also an invariant automorphism. This follows from the fact that if a regular element is left fixed by S, then the Cartan subalgebra it determines is left pointwise fixed. So let s be an invariant automorphism of A such that H is a Cartan subalgebra of A left pointwise fixed by s. Given any other s-stable Cartan subalgebra T of A, if uniqueness held we would have an automorphism t of A such that $t: H \to T$ and $st = ts$. Then it follows that T is also pointwise fixed by s. However, the following example shows that a semisimple invariant automorphism s of a semisimple Lie algebra A need not leave every s-stable Cartan subalgebra pointwise fixed. Let A be the simple Lie algebra of $n \times n$-matrices of trace 0 over an algebraically closed field of characteristic zero. Then A has dimension $n^2 - 1$ with Cartan subalgebras of dimension $n - 1$. Let H denote the diagonal matrices of trace 0. Then H has dimension $n - 1$ with basis X_i, $2 \leqq i \leqq n$, where X_i has 1 in the (1, 1)-position and -1 in the (i, i)-position with zeros elsewhere. Let M be the invertible $n \times n$-matrix with 1's in the $(i, i+1)$-position, $1 \leqq i \leqq n-1$, 1 in the $(n, 1)$-position, and zero elsewhere. Define an automorphism s of A by $s: N \to M^{-1}NM$ for $n \in A$. Then s is an invariant automorphism of A, Jacobson [7, p. 283]. Since $M^n = I$, s has order at most n, and so s is semisimple. Thus by the result of Borel-Mostow we know that there exists a Cartan subalgebra of A left pointwise fixed by s. One checks directly that s acts on H as follows: $s(X_i) = X_{i+1} - X_2$ for $2 \leqq i \leqq n-1$ and $s(X_n) = -X_2$. Thus, H is not pointwise fixed by s, and it also follows that s has order exactly n.

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